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MATH 3175 **Group Theory** **Spring 2024**

Solutions for Homework 2

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1. Given a commutative ring R and an element $a \in R$, we say that
- a is *invertible* (or, a *unit*) if there is an element $b \in R$ such that $ab = 1$.
- a is a *zero-divisor* if there is an element $b \in R$ such that $ab = 0$.
- a is an *idempotent* if $a^2 = a$.

For the ring $R = \mathbb{Z}_{18}$:

(i) List all the invertible elements, zero-divisors, and idempotents.

- Invertible elements: 1, 5, 7, 11, 13, 17.
- Zero-divisors: 0, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16.
- Idempotents: 0, 1, 9, 10.

(ii) Are there any elements which are neither zero-divisors nor invertible?

No

(iii) Are there any zero-divisors which are not idempotent?

Yes: 2, 3, 4, 6, 8, 12, 14, 15, 16.

2. Let $\mathbb{Z}_n$ be the (additive) group of integers modulo n, and let $\mathbb{Z}_n^\times$ be the (multiplicative) group of invertible elements in $\mathbb{Z}_n$.

(i) Write down the addition and multiplication tables for $\mathbb{Z}_6$.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(ii) Write down the multiplication table for $\mathbb{Z}_8^\times$.

·	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

3. For each of the following groups, list all their elements, together with their orders.

(i) $\mathbb{Z}_{18}$.

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ord(g)	1	18	9	6	9	18	3	18	9	2	9	18	3	18	9	6	9	18

(ii) $\mathbb{Z}_{18}^{\times}$.

g	1	5	7	11	13	17
$\text{ord}(g)$	1	6	3	6	3	2

(iii) $\mathbb{Z}_4 \times \mathbb{Z}_2$.

g	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$
$\text{ord}(g)$	1	4	2	4	2	4	2	4

(iv) $\mathbb{Z}_8^{\times} \times \mathbb{Z}_3$.

g	$(1,0)$	$(3,0)$	$(5,0)$	$(7,0)$	$(1,1)$	$(3,1)$	$(5,1)$	$(7,1)$	$(1,2)$	$(3,2)$	$(5,2)$	$(7,2)$
$\text{ord}(g)$	1	2	2	2	3	6	6	6	3	6	6	6

4. For each of the following groups, find *all* the cyclic subgroups of the group.

(i) $\mathbb{Z}_8 = \{0,1,2,3,4,5,6,7,8\}$. This is a cyclic group of order 8. All subgroups are cyclic, and there are 4 of them: $\{0\}$, $\{0,4\}$, $\{0,2,4,6\}$, and $\mathbb{Z}_8$.

(ii) $\mathbb{Z}_9^{\times} = \{1,2,4,5,7,8\}$. This is a cyclic group of order 6, generated by 2 (or by 5). All subgroups are cyclic, and there are 4 of them: $\{1\}$, $\{1,8\}$, $\{1,4,7\}$, and $\mathbb{Z}_9^{\times}$.

(iii) The symmetric group S_3. This is a (non-abelian) group of order 6; in cycle notation, its elements are: $S_3 = \{(),(12),(13),(23),(123),(132)\}$. All its *proper* subgroups are cyclic, and there are 5 of them: $\{()\}$, $\{(),(12)\}$, $\{(),(13)\}$, $\{(),(23)\}$, and $\{(),(123),(132)\}$.

5. Let $G = \text{GL}_2(\mathbb{R})$ be the (multiplicative) group of invertible 2×2 matrices with entries in $\mathbb{R}$. For each of the following sets of 2×2 matrices with real entries determine whether the set is a subgroup of G.

(i) $A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \, ab \neq 0 \right\}$.

No, since this the matrices in A are not invertible, so A is not even a subset of G.

(ii) $B = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| \, bc \neq 0 \right\}$.

B is a subset of G, but it is not a subgroup of G, since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin B$, or, alternatively,

$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \bar{b} \\ \bar{c} & 0 \end{pmatrix} = \begin{pmatrix} b\bar{b} & 0 \\ 0 & c\bar{c} \end{pmatrix} \notin B.$

(iii) $C = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| \, a \neq 0 \right\}$.

Yes, since $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \bar{a} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \bar{a}^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\bar{a}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in C.$