Assignment 5

Due Thursday August 13

Problem 1 Let $G = \mathbb{Z}_{32}^{\times}$. Find cyclic subgroups H of order 2 and K of order 8 with $HK = G$ and $H \cap K = \{e\}$. Write G as a product of cyclic groups up to isomorphism, prove your answer.

Problem 2 Let G be a finite group.

1. Show that if $|G| = 20$, then G is not simple.
2. Show that if $|G| = 10 \cdot 11^5$, then G is not simple.
3. Show that if $|G| = pq^r$ with $p < q$ both prime and $r > 0$, then G is not simple.

Problem 3 Let G be a group with $|G| = 30$.

1. How many Sylow 5-subgroups can G have? How many Sylow 3-subgroups?
2. Show that, if P_1 and P_2 are Sylow p-subgroups of G, then either $P_1 = P_2$ or $P_1 \cap P_2 = \{e\}$ (hint: this fact is not true in general, so think again if you didn't use something special to G).
3. If G has the largest possible number of Sylow 5-subgroups, then how many elements of order 5 does G contain? Answer the same question for $p = 3$.
4. Using 3.3, show that G is not simple.

Problem 4 Let p be a prime.

1. What is the order of a Sylow p-subgroup of S_p?
2. Find a Sylow p-subgroup $H \leq S_p$.
3. Use part 1 and 2 to show that H is not normal for $p > 3$.

Problem 5 Let $G = GL_3(\mathbb{Z}_2)$ be the group of invertible 3×3 matrices with entries in $\mathbb{Z}_2$.

1. Find a Sylow 2-subgroup of G explicitly. Identify this subgroup to a previously seen group.
2. Find out how many other Sylow 2-subgroups there are in G.

Problem 6 Let $G = D_6$ be the dihedral group of symmetries of the hexagon generated by a rotation r and a reflection s. Consider the normal subgroups $N_1 \cong \mathbb{Z}_2$ generated by s and $N_2 \cong \mathbb{Z}_3$ generated by r^2. Explain how the correspondence theorem works in each case and draw the correspondence diagram for the respective subgroup lattices.