Assignment 4

Due Thursday August 6

Definition: Let G be a group and X be a set. Let $f : G \times X \to X$ where $(g, x) \mapsto g * x$ be a group action.

1. We say that $a \in G$ fixes $x \in X$ if $a * x = x$.
2. We write X_a for the set of elements fixed by a and G_x for the set of the group elements fixing x.

   I.e. $X_a = \{x \in X | a * x = x\}$ (fixed set of a) and $G_x = \{a \in G | a * x = x\}$ (stabilizer subgroup of x).
3. The orbit of $x \in X$ is the set $Gx = \{a * x \in X | a \in G\} = \{y \in X | \exists a \in G \text{ s.t. } a * x = y\}$.
4. We say that the group action is faithful if the only group element fixing all of X is the unit e. I.e. $\bigcap_{x \in X} G_x = \{e\}$.
5. We say that the group action is free if $G_x = \{e\}$ for every $x \in X$.
6. We say it is transitive if for every $x, y \in X$, there exists $g \in G$ such that $g * x = y$. This is also equivalent to only having one orbit.

Problem 1 Which of the following group actions are faithful? Which are free? Which are transitive? What are the orbits and stabilizer subgroups in each case?

1. The symmetric group on a set X ($Sym(X)$) acts on X by $Sym(X) \times X \to X$ where $(\sigma, x) \mapsto \sigma(x)$.
2. The dihedral group D_n acts on the vertices of an n-gon, labelled by $\{1, 2, 3, ..., n\}$ in the usual fashion. For example: For D_3, the vertices of the triangle are labelled 1,2,3 in clockwise order. Then $r * 1 = 2$, $r^2 * 2 = 1$, $s * 1 = 1$ etc.
3. The general linear group $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication: $GL_n(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ where $(M, x) \mapsto Mx$.
4. (Cayley action) A group acts on itself by left multiplication: $G \times G \to G$ where $(a, b) \mapsto ab$.
5. A group acts on itself by conjugation: $G \times G \to G$ where $(a, b) \mapsto aba^{-1}$.
6. Let H be a subgroup of G. Then G acts on the set of left cosets G/H by left multiplication: $G \times G/H \to G/H$ where $(a, bH) \mapsto abH$.
7. Along with what you're asked in the question, first show that if H is a normal subgroup in G then G acts on G/H by conjugation: $G \times G/H \to G/H$ where $(a, bH) \mapsto aba^{-1}H$.
8. Along with what you're asked in the question, first show that if $f : G \to Sym(X)$ is a homomorphism, then G acts on X by $G \times X \to X$ where $(a, x) \mapsto f(a)(x)$ (this is evaluation of the function $f(a) : X \to X$ at x).

Problem 2

1. Prove that a group of order 1331 has non-trivial center. Hint: Write 1331 in its prime factorization, then use the class equation.

2. For every $n = 4k + 2$ for $k \in \mathbb{N}$, find a group of order n that has trivial center, explain your answer.

Problem 3 Let G be a group and H a subgroup of G.

- Show that if $H \subset Z(G)$, then H is normal in G.
- Show that if $H \subset Z(G)$ and G/H is cyclic, then G is Abelian.

Problem 4 Let p be a prime. A group G is a p-group if the order of G is a power of p.

1. Show that every p-group has non-trivial center (hint: use the class equation).
2. Use 3.2 and 4.1 to show that every group of order p^2 is Abelian.

Problem 5 List all the conjugacy classes in S_6 and list their orders. Justify your counting argument. Make the same type of table as in lecture 19.