Assignment 3

Due Thursday July 23

On this homework, 4 problems will be picked at random and graded. So make sure you complete all problems.

IMPORTANT: Always prove your answers! I expect full proofs justifying any yes or no answers!

Problem 1 Let $H = \{(x, y) \in \mathbb{R}^2 : y = 0\}$.

1. Sketch H in the plane.
2. Consider $\mathbb{R}^2$ as a group under vector addition. Is H a subgroup?
3. Describe the cosets of H in geometric terms and make a sketch of a few of the cosets.

Problem 2

1. Let G be a group and H a subgroup of G. Prove that there exists an injective homomorphism $f : H \to G$.
2. Prove that D_3 (symmetry group of equilateral triangle) is a subgroup of S_3. Hint: What do elements of D_3 do to the vertices $\{1, 2, 3\}$ of the triangle? Does this tell you anything about how the elements in D_3 are related to the elements in S_3?
3. Prove that S_3 is isomorphic to D_3.
4. Use part 2 as inspiration to prove that D_n is a subgroup of S_n.

Problem 3

1. Let G be an Abelian group, and let n by any positive integer. Prove that the function $\phi : G \to G$ defined by $\phi(x) = x^n$ is a homomorphism.
2. Is $\phi : G \to G$ (as above) still a homomorphism if G is not Abelian?
3. Let $G = (\mathbb{Z}_{15}^{\times} = \{1, 2, 4, 7, 8, 11, 13, 14\}, \cdot)$ where $\cdot$ is multiplication modulo 15. Let $n = 2$, and find the kernel and image of ϕ.

Problem 4 Let G be a group.

1. Show that, if G is Abelian, then any subgroup of G is normal.
2. Is the intersection of a collection of normal subgroups of G normal?
3. Let $K < H < G$ be subgroups of G, and suppose that K is normal in G. Is K normal in H?
4. Let $K < H < G$ be subgroups of G, and suppose that K is normal in H. Is K normal in G?

Problem 5 Compute/find $\mathrm{Aut}(S_3)$.

Problem 6 Show that $S_3 \times \mathbb{Z}_2$ is isomorphic to D_6. How many subgroups, and how many normal subgroups does this group have?