Assignment 2

Due Thursday July 16

Problem 1 (Continuing problem 8 from homework 1). Suppose that G is Abelian/commutative.

- Prove that the subset $H = \{a \in G : a^3 = e\}$ is a subgroup of G.
- Find a group G that is NOT Abelian/commutative such that H is NOT a subgroup of G. This shows you that the Abelian/commutative condition cannot be removed if you want the statement in part 1 to be true.

Problem 2 Let G be a group.

- Let H and K be subgroups of G. Is $H \cap K$ a subgroup of G? Prove or disprove.
- Let H and K be subgroups of G. Is $H \cup K$ a subgroup of G? Prove or disprove.
- Let $S = \{\text{subgroups of } G\}$ (S could be finite or infinite). Prove that $\bigcap_{H \in S} H$ is a subgroup of G.

Definition: A group homomorphism is a function $f : (G, *) \to (H, \star)$ between two groups $(G, *)$ and $(H, \star)$ satisfying $f(a * b) = f(a) \star f(b)$ for all elements $a, b \in G$. Note that $*$ and $\star$ refer to the group operation of G and H respectively.

Problem 3 Let G and H be groups.

- Let $f : (G, \cdot) \to (H, \cdot)$ be a homomorphism. Show that, if $a \in G$ has order n, then $f(a)$ has order at most n. Hint: first prove that $f(a^n) = f(a)^n$ if $f : (G, \cdot) \to (H, \cdot)$ is a homomorphism.
- Let $n > 1$. Using part 1, prove that there are no homomorphisms from the cyclic group $C_n = \{e, a, a^2, ..., a^{n-1}\}$ to the integers $\mathbb{Z}$ (Hint: use proof by contradiction). What about the case $n = 1$?

Problem 4 This problem revisits the proof of Lagrange that you did in class in more detail. Let G be a group and H a subgroup of G.

- For this part (and this part only) suppose G is Abelian/commutative. Let $a, b \in G$. Show that $(aH)(bH) = (ab)H$.
- Consider $f : \{aH | a \in G\} \to \{Ha | a \in G\}$ defined by $f(aH) = Ha^{-1}$. Prove that this function is well-defined, i.e. show that if $a_1H = a_2H$ then $f(a_1H) = f(a_2H)$. If this looks foreign to you, go back and review the definition of a function. Careful: the elements of the set $\{aH | a \in G\}$ are sets themselves!

  What you're doing here is constructing a function between the left and right cosets of H in G.
- Prove that this function is injective/one to one. I.e. show that if $f(a_1H) = f(a_2H)$ then $a_1H = a_2H$. You can show that $a_1H = a_2H$ by showing that $a_1H \subset a_2H$ and $a_1H \supset a_2H$.
- Prove that this function is surjective/onto. I.e. show that if $Hb \in \{Ha | a \in G\}$, then there exists $x \in G$ such that $f(xH) = Hb$.

You now have a bijection between the left and right cosets of H in G. This tells you that the number of left cosets is the same as the number of right cosets.

- Let $a \in G$. Then aH is an arbitrary left coset. Consider $g : H \to aH$ where $g(h) = ah$. Prove that this is a well-defined function, and show that it is bijective. This shows you that the order of every left coset is equal to the order of H.
- Conclusion: Recall that two elements of G being in the same coset is an equivalence relation on the set G. Here, the equivalence classes are the left/right cosets. Since equivalence classes partition the underlying set, we have that the left/right cosets partition the set G, and the union of all the left/right cosets gives us G. So, by summing up all the elements in all the equivalence classes, we obtain the total number of elements in G. Since $[G : H]$ denotes the number of equivalence classes, we get that $|G| = [G : H]|H|$. You just proved Lagrange's theorem!

Problem 5 Give the subgroup diagrams of the following groups:

- $(\mathbb{Z}_{24}, +)$
- $(\mathbb{Z}_{36}, +)$

Hint: Are these cyclic groups? What is true about all subgroups of cyclic groups?

Problem 6 Find all cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_3$.