Assignment 1

Due Thursday July 9

IMPORTANT: All homework submissions must be prepared using LaTeX and submitted in pdf form. Your homework will not be graded unless you follow this rule.

Grading Policy: 5 problems will be selected on this homework at random and will be graded. Make sure to complete all the problems, as you don't know which will be chosen. 50 percent of your grade will depend on the correctness of your solutions, while the other 50 percent will depend on the clarity and structure of your proofs. Make sure to familiarize yourself with the Introduction to Proofs document under Files → Other Course Materials.

Tip: If you find yourself struggling or not knowing where to start on a proof, it is usually a good idea to go back to your notes and make sure you understand the definitions and theorems that are relevant to the problem.

Problem 1 Find integers m and n such that $\gcd(a, b)$ is expressed in the form $ma + nb$ where

1. $a = 35$, $b = 14$
2. $a = 15$, $b = 11$

Problem 2 Let $a, b, c \in \mathbb{Z}$. Prove the following statements:

1. If $b \mid a$, then $b \mid ac$.
2. If $b \mid a$ and $c \mid b$, then $c \mid a$.
3. If $c \mid a$ and $c \mid b$, then $c \mid (ma + nb)$ for any integers m, n.

Problem 3

- Make addition and multiplication tables for $\mathbb{Z}_4$.
- For S_3 write the multiplication table (where multiplication is composition of functions). Here $S_3 = \{(1), (12), (23), (13), (123), (132)\}$ is the symmetric group on three elements. Note that (12) is the bijection which sends element 1 to element 2, element 2 to element 1, and element 3 to element 3. (1) is the identity, which sends element i to element i for $i = 1, 2, 3$. You can read inside the parenthesis from left to right to figure out what gets sent where. If an element is omitted, that means it gets sent to itself.

  How to multiply permutations: https://www.youtube.com/watch?v=gGir2mwNrbo

Problem 4 Find the multiplicative inverses of the given elements (if possible):

- $[14]_{15}$ in $\mathbb{Z}_{15}$
- $[38]_{83}$ in $\mathbb{Z}_{83}$

Problem 5 Determine (with proof) which of the following functions are one-to-one and which are onto:

- $f : \mathbb{R} \to \mathbb{R};\ f(x) = x + 3$
- $f : \mathbb{C} \to \mathbb{C}; f(x) = x^2 + 2x + 1$

- $f : \mathbb{Z}_5 \to \mathbb{Z}_5; f([x]_5) = [3x+8]_5$

Problem 6 Consider the binary operations $*$ and $\star$ on the set $S = \{a, b, c\}$ given by the following multiplication tables.

$*$	a	b	c
a	a	b	c
b	b	c	b
c	c	b	a

$\star$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Which (if either) of these binary operations gives S the structure of a group? Prove your answer.

Definition: Let G be a group. We say that G is Abelian/Commutative iff $ab = ba$ for all $a, b \in G$.

Problem 7 Consider the following weird multiplication:

a). Let $*$ denote the binary operation on $\mathbb{Z}$ given by $a * b = a + b + 1$. Is $(\mathbb{Z}, *)$ a group? If it is, is it Abelian/Commutative?

b). Let $\star$ denote the binary operation on $\mathbb{Q}$ (rational numbers) given by the formula $a \star b = a+b+ab$. Prove that $(\mathbb{Q}, \star)$ is NOT a group.

c). (BONUS) Find a rational number x such that $(\mathbb{Q} \setminus \{x\}, \star)$ is a group.

Problem 8 Let G be a group. Prove that if $a^2 = e$ for every $a \in G$, then G is Abelian/Commutative. Is the converse true? I.e. is it true that if G is Abelian/Commutative, then $a^2 = e$ for every $a \in G$? Prove your answer or give a counter-example.