Prof. Alexandru Suciu

MATH 3175 | **Group Theory** | **Spring 2024**

Homework 2

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1. Given a commutative ring R and an element $a \in R$, we say that

- a is *invertible* (or, a *unit*) if there is an element $b \in R$ such that $ab = 1$.
- a is a *zero-divisor* if there is an element $b \in R$ such that $ab = 0$.
- a is an *idempotent* if $a^2 = a$.

For the ring $R = \mathbb{Z}_{18}$:

(i) List all the invertible elements, zero-divisors, and idempotents.

(ii) Are there any elements which are neither zero-divisors nor invertible?

(iii) Are there any zero-divisors which are not idempotent?

2. Let $\mathbb{Z}_n$ be the (additive) group of integers modulo n, and let $\mathbb{Z}_n^\times$ be the (multiplicative) group of invertible elements in $\mathbb{Z}_n$.

(i) Write down the addition and multiplication tables for $\mathbb{Z}_6$.

(ii) Write down the multiplication table for $\mathbb{Z}_8^\times$.

3. For each of the following groups, list all their elements, together with their orders.

(i) $\mathbb{Z}_{18}$.

(ii) $\mathbb{Z}_{18}^\times$.

(iii) $\mathbb{Z}_4 \times \mathbb{Z}_2$.

(iv) $\mathbb{Z}_8^\times \times \mathbb{Z}_3$.

4. For each of the following groups, find *all* the cyclic subgroups of the group.

(i) $\mathbb{Z}_8$.

(ii) $\mathbb{Z}_9^\times$.

(iii) The symmetric group S_3.

5. Let $G = \mathrm{GL}_2(\mathbb{R})$ be the (multiplicative) group of invertible 2×2 matrices with entries in $\mathbb{R}$. For each of the following sets of 2×2 matrices with real entries determine whether the set is a subgroup of G.

(i) $A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \, ab \neq 0 \right\}$.

(ii) $B = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| \, bc \neq 0 \right\}$.

(iii) $C = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| \, a \neq 0 \right\}$.