Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Spring 2024**

Homework 1

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1. Write down all the possible multiplication tables on the set $S = \{0, 1\}$. In each case, determine whether the resulting magma $(S, *)$ has (one or more or none) of the following properties:

(i) The operation $*$ is associative (so that $(S, *)$ is a *semigroup*).

(ii) The operation $*$ has a (two-sided) identity element (so that $(S, *)$ is a *unital magma*).

(iii) The operation $*$ has the cancellation property (so that the multiplication table is a *Latin square*).

(iv) $(S, *)$ is a *group*.

2. Consider the two binary operations on the set $S = \{1, \ldots, 6\}$ given by the following multiplication tables (which are, in fact, reduced Latin squares):

1	2	3	4	5	6
2	3	4	5	6	1
3	6	1	2	4	5
4	1	5	6	2	3
5	4	6	3	1	2
6	5	2	1	3	4

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	5	6	1	2	3
5	6	4	2	3	1
6	4	5	3	1	2

Which (if any) of these binary operations gives S the structure of a group? Prove your answer.

3. Let G be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and $a \neq 0$.

(i) Show that G forms a group under matrix multiplication.

(ii) Find all elements of G that commute with $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

4. Let $G = \{x \in \mathbb{R} \mid x > 0 \text{ and } x \neq 1\}$. Define an operation $*$ on G by $x * y = x^{\ln y}$ for all $x, y \in G$. Show that $(G, *)$ is an abelian group.

5. Let G be a finite group with an even number of elements and with identity e. Show that there must exist an element $a \in G$ such that $a \neq e$ and yet $a^2 = e$.