Prof. Alexandru Suciu

MATH 4565 **TOPOLOGY** **Fall 2023**

FINAL EXAM

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1. Recall that a space X is *locally compact* if, for every $x \in X$, there exists a compact subspace which contains an open neighborhood of x.

(a) Show that any set X endowed with the discrete topology is locally compact.

(b) Show that the space of rational numbers $\mathbb{Q}$ (with the subspace topology inherited from $\mathbb{R}$) is not locally compact.

(c) Give an example of a locally compact space X and a continuous map $f\colon X \to Y$ such that $f(X)$ is not locally compact.

(d) Now assume $f\colon X \to Y$ is both continuous and open. Show that $f(X)$ is locally compact.

2. A subspace $A \subset X$ is called a *deformation retract* of X if there is a retraction $r\colon X \to A$ with the property that $i \circ r \simeq \mathrm{id}_X$. Prove the following:

(a) Let $B \subset A \subset X$. If A is a deformation retract of X and B is a deformation retract of A, then B is a deformation retract of X.

(b) If A is a retract of X and X is contractible, then A is also contractible, and A is a deformation retraction of X.

3. Let X be a topological space, let $A \subset X$ be a subspace, and let $i\colon A \hookrightarrow X$ the inclusion map. Fix a basepoint $a_0 \in A$, and consider the induced homomorphism on fundamental groups, $i_\sharp\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$.

(a) Suppose A is a retract of X. Show that $i_\sharp$ is injective.

(b) Give an example of an inclusion $i\colon A \hookrightarrow X$ where $i_\sharp$ is *not* injective.

(c) Suppose A is a deformation-retract of X. Show that $i_\sharp$ is an isomorphism.

(d) Give an example of an inclusion $i\colon A \hookrightarrow X$ that admits a retraction $r\colon X \to A$ for which $i_\sharp$ is *not* an isomorphism.

4. Let $f\colon X \to Y$ be a continuous map.

(a) Show that if X is contractible and Y is path connected, then f is null-homotopic.

(b) Show that if $Y = S^n$ and f is not surjective, then f is null-homotopic.

5. Let $g\colon [0,1] \to X$ be a path with $g(0) = x_0$ and $g(1) = x_1$, and let $\Phi_g\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ be the "change of basepoint" homomorphism determined by g.

(a) If $h\colon [0,1] \to X$ is a path with $h(0) = x_1$, show that $\Phi_{g*h} = \Phi_h \circ \Phi_g$.

(b) Let $f\colon X \to Y$ be a map. Show that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_\sharp} & \pi_1(Y, f(x_0)) \\ \Big\downarrow \Phi_g & & \Big\downarrow \Phi_{f\circ g} \\ \pi_1(X, x_1) & \xrightarrow{f_\sharp} & \pi_1(Y, f(x_1)) \end{array}$$

6. Let X be a path-connected space, with basepoint $x_0 \in X$. Show that the following are equivalent.

(a) If g and h are any two paths from x_0 to some $x_1 \in X$, then $\Phi_g = \Phi_h$.

(b) $\pi_1(X, x_0)$ is abelian.

7. Let $p\colon E \to B$ be a covering map. Suppose E is path-connected, and $\pi_1(B, b_0) = 0$, for some $b_0 \in B$.

(a) Show that $\pi_1(B, b) = 0$, for all $b \in B$.

(b) Show that p is a homeomorphism.

8. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and let $p\colon \mathbb{R} \to S^1$ be the standard covering map given by $p(t) = e^{2\pi i t}$. Consider the product covering map $p \times p\colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, and let $f\colon [0,1] \to S^1 \times S^1$ be the loop given by $f(t) = (e^{4\pi i t}, e^{6\pi i t})$. Find the lift $\tilde{f}\colon [0,1] \to \mathbb{R}^2$ of f at $(0,0)$, and sketch both f and $\tilde{f}$.