Definition 1. A sequence (s_n) converges to the number S if given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_n - S| < \epsilon$ for all $n \geq N$.

Definition 2. A number U is called the *least upper bound or* sup of a set S if

(1) $s \leq U$ for all $s \in S$, and
(2) If L is any number with $L < U$, then there is an element $s \in S$ with $L < s$.

Completeness Axiom. Every nonempty subset S of the real numbers $\mathbb{R}$ that is bounded above has a least upper bound.

Definition 3. Given a sequence (x_n) and $N \in \mathbb{N}$, let $v_N = \sup\{x_n : n \geq N\}$. Then (v_N) is a decreasing sequence, and hence, has a limit. We define $\limsup x_n$ to be the limit of the sequence (v_N).
Set $u_N = \inf\{x_n : n \geq N\}$. Then (u_N) is an increasing sequence, and hence, has a limit. We define $\liminf x_n$ to be the limit of the sequence (u_N). Moreover, $u_N \leq v_N$ for all $N \in \mathbb{N}$.

Definition 4. A sequence (x_n) is called a *Cauchy sequence* if given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_m - s_n| < \epsilon$ for all $n \geq N$ and $m \geq N$.

Theorem 1. Every Cauchy sequence is bounded. A sequence of real numbers is a Cauchy sequence if and only if it is convergent.

Theorem 2. Every bounded monotone sequence of real numbers converges.

Theorem 3. Every subsequence (x_{n_k}) of a convergent sequence (x_n) converges, and $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$.

Theorem 4 (Bolzano–Weierstrass)**.** Every bounded sequence of real numbers has a convergent subsequence.

Definition 5 ($\delta - \epsilon$ definition of continuity)**.** Let f be a real-valued function whose domain, $\mathrm{dom}(f)$, is a subset of $\mathbb{R}$. The function f is *continuous* at a point $x_0 \in \mathrm{dom}(f)$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{for all } x \in \mathrm{dom}(f) \text{ with } |x - x_0| < \delta.$$

Theorem 5 (Extreme Value Theorem)**.** Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then f is bounded and it assumes its maximum and minimum values on $[a,b]$.

Theorem 6 (Intermediate Value Theorem)**.** Let $f\colon I \to \mathbb{R}$ be a continuous function defined on an interval I. Suppose a, b are two numbers in I such that $a < b$ and suppose y is a real number that lies between $f(a)$ and $f(b)$. Then there is an $x \in (a,b)$ such $f(x) = y$.

Definition 6 (Uniform continuity)**.** Let f be a real-valued function whose domain, $\mathrm{dom}(f)$, is a subset of $\mathbb{R}$. The function f is *uniformly continuous* on $\mathrm{dom}(f)$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in \mathrm{dom}(f) \text{ with } |x - y| < \delta.$$

Theorem 7. If $f\colon S \to \mathbb{R}$ is uniformly continuous of a set $S \subset \mathbb{R}$, and if (x_n) is a Cauchy sequence in S, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Theorem 8. If $f\colon [a,b] \to \mathbb{R}$ is a continuous function, then f is uniformly continuous on $[a,b]$.

Theorem 9 (Weierstrass M-test)**.** Let (M_k) be a sequence of nonnegative real numbers such that $\sum_{k=1}^{\infty} M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum_{k=1}^{\infty} g_k$ converges uniformly on S.

Theorem 10 (Mean Value Theorem)**.** Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists [at least one] $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Definition 7. Let $f\colon (a, b) \to \mathbb{R}$ be a function which admits derivatives of arbitrary order at a point $c \in (a, b)$. Then the *Taylor series of f about c* is the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{n!}(x - c)^k$, while the *n-th remainder* is the difference $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{n!}(x - c)^k$.

Theorem 11 (Taylor's Theorem)**.** For every $x \in (a, b)$, $x \neq c$, there is a y between c and x such that

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x - c)^n.$$

Corollary 1. *Suppose there is a constant $C > 0$ such that $\left|f^{(n)}(x)\right| < C$ for all $x \in (a, b)$ and all $n \geq 0$. Then $\lim_{n\to\infty} R_n(x) = 0$, for all $x \in (a, b)$.*

Definition 8. Let f be a bounded function on $[a, b]$, and let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. Set

$$U(f, P) = \sum\nolimits_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \quad \text{and} \quad L(f, P) = \sum\nolimits_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}),$$

where $M(f, [t_{k-1}, t_k]) = \sup\{f(t)\colon t \in [t_{k-1}, t_k]\}$ and $m(f, [t_{k-1}, t_k]) = \inf\{f(t)\colon t \in [t_{k-1}, t_k]\}$, and set

$$U(f) = \inf\{U(f, P)\colon P \text{ a partition of } [a, b]\} \quad \text{and} \quad L(f) = \sup\{L(f, P)\colon P \text{ a partition of } [a, b]\}.$$

We say that *f is integrable on* $[a, b]$ if $U(f) = L(f) \in \mathbb{R}$, in which case we set $\int_a^b f := U(f) = L(f)$.

Theorem 12. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function, and let P be any partition of $[a, b]$. Then $L(f, P) \leq L(f) \leq U(f) \leq U(f, P)$, and hence $0 \leq U(f) - L(f) \leq U(f, P) - L(f, P)$.

Theorem 13. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Then the following are equivalent.

(1) f is integrable on $[a, b]$
(2) For every $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.
(3) For every $\epsilon > 0$, there is a $\delta > 0$ such that: if P is a partition of $[a, b]$ with mesh$(P) < \delta$, then $U(f, P) - L(f, P) < \epsilon$.

Theorem 14 (Riemann's Theorem)**.** If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Theorem 15. If f is monotonic on $[a, b]$, then f is integrable on $[a, b]$.

Theorem 16 (First Fundamental Theorem of Calculus)**.** If g is a continuous function on $[a, b]$ that is differentiable on (a, b), and if g' is integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a)$$

Theorem 17 (Second Fundamental Theorem of Calculus)**.** Let f be an integrable function on $[a, b]$. For $x \in [a, b]$, let

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$. If f is continuous at $x_0 \in (a, b)$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$