For each problem be sure to explain the steps in your argument and fully justify your conclusions.

1. For the power series $\sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} x^n$,
   (a) (9 pts) Find the radius of convergence.

   Solution: Use the ratio test with $a_n = \frac{2^{2n}}{(3^{3n})\sqrt{n}} x^n$

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{2^{2n+2}|x|^{n+1}(3^{3n})\sqrt{n}}{2^{2n}|x|^n(3^{3n+3})\sqrt{n+1}} \\
&= \lim_{n\to\infty} \frac{2^2|x|}{3^3} \sqrt{\frac{n}{n+1}} \\
&= \lim_{n\to\infty} \frac{4|x|}{27} \sqrt{\frac{1}{1+1/n}} \\
&= \frac{4|x|}{27}
\end{aligned}$$

   From the ratio test, it follows that the series converges if $4|x|/27 < 1$ and diverges if $4|x|/27 > 1$. That is, the series converges if $|x| < 27/4$ and diverges if $|x| > 27/4$ so the raduius of convergence is $27/4$.

   (b) (9 pts) Find the exact interval of convergence.

   Solution: The endpoints of the interval of convergence are $x = 27/4$ and $-27/4$. For $x = 27/4$ the series is

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} x^n &= \sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} (27/4)^n \\
&= \sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} (3^3/2^2)^n \\
&= \sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} (3^{3n}/2^{2n}) \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad p\text{-series, } p = 1/2 \le 1 \text{, series diverges}
\end{aligned}$$

For $x = -27/4$ the series is

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} x^n &= \sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} (-27/4)^n \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} (-3^3/2^2)^n \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}}{(3^{3n})\sqrt{n}} (-1)^n (3^{3n}/2^{2n}) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\end{aligned}$$

$1/\sqrt{n+1} < 1/\sqrt{n}$, and $\lim_{n\to\infty} 1/\sqrt{n} = 0$ so the series $\sum (-1)^n/\sqrt{n}$ converges by the alternating series test. Thus, the interval of convergence is $-27/4 \le x < 27/4$; equivalently $[-27/4, 27/4)$.

2. Let $f_n(x) = \dfrac{3n + 1 - \sin(x)}{2n + \cos(x)}$.

(a) (9 pts) Show that (f_n) converges uniformly on $\mathbb{R}$. *Hint*: First decide what the limit function is and then show that convergence is uniform.

Solution: Since $\lim 1/n = 0$, $-1/n \le \sin(x) \le 1/n$, and $-1/n \le \cos(x) \le 1/n$, it follows from the squeeze theorem that for each $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \sin(x)/n = \lim_{n\to\infty} \cos(x)/n = 0$$

From the results in the text that the sum, product, and quotient of a limit is the the limit of the respective sum, product, and quotient we have that for each $x \in R$

$$\lim_{n\to\infty} \frac{3n + 1 - \sin(x)}{2n + \cos(x)} = \lim_{n\to\infty} \frac{3 + 1/n - \sin(x)/n}{2 + \cos(x)/n} = \frac{3 + 0 - 0}{2 + 0} = \frac{3}{2}$$

Hence the sequence $f_n(x)$ converges pointwise to $3/2$ for all $x \in \mathbb{R}$. To show that convergence is uniform, it suffices to show that given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\left| \frac{3n + 1 - \sin(x)}{2n + \cos(x)} - \frac{3}{2} \right| < \epsilon \quad \text{for all } n \ge N \text{ and all } x \in \mathbb{R} \tag{1}$$

Note that

$$\begin{aligned}\left| \frac{3n + 1 - \sin(x)}{2n + \cos(x)} - \frac{3}{2} \right| &= \left| \frac{6n + 2 - 2\sin(x) - 6n - 3\cos(x)}{4n + 2\cos(x)} \right| \\ &= \left| \frac{2 - 2\sin(x) - 3\cos(x)}{4n + 2\cos(x)} \right|\end{aligned}$$

Since $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$, it follows that

$$\begin{gathered}-3 \le 2 - 2 - 3 \le 2 - 2\sin(x) - 3\cos(x) \le 2 + 2 + 3 \le 7 \\ 0 \le 4n - 2 \le 4n + 2\cos(x) \le 4n + 2 \\ \frac{1}{4n - 2} \ge \frac{1}{4n + 2\cos(x)} \ge \frac{1}{4n + 2}\end{gathered}$$

and hence,

(2) $$\left|\frac{2-2\sin(x)-3\cos(x)}{4n+2\cos(x)}\right| \leq \frac{7}{4n-2} \quad \text{for all } n \in \mathbb{N} \text{ and all } x \in \mathbb{R}$$

Note that for any $\epsilon > 0$ the follow inequalities are equivalent to each other.

(3) $$\frac{7}{4n-2} < \epsilon$$
$$7 < 4n\epsilon - 2\epsilon$$
$$7 + 2\epsilon < 4n\epsilon$$
(4) $$\frac{7+2\epsilon}{4\epsilon} < n$$

Now given any $\epsilon > 0$, we have that $(7 + 2\epsilon)/(4\epsilon) > 0$, so by the Archimedean property there is an $N \in N$ such that $(7+2\epsilon)/(4\epsilon) < N$. It then follows that for all $n \geq N$, inequality (4) is true; hence, inequality (3) is true. Combining the inqualities (3) and (2) then gives inequality (1), and the proof is complete.

An **alternate approach** is to use 24.4 Remark in the text that a squence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on S if and only if

(5) $$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| \colon x \in S\} = 0$$

From this perspective, the result follows by showing (as was done above), that for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$

$$\left|\frac{3n+1-\sin(x)}{2n+\cos(x)} - \frac{3}{2}\right| = \left|\frac{2-2\sin(x)-3\cos(x)}{4n+2\cos(x)}\right| \leq \frac{7}{4n-2}$$

Thus, $0 \leq \sup\{|f(x)-f_n(x)| \colon x \in S\} \leq 7/(4n-2)$, for each $n \geq 1$. Equation (5) then follows using the squeeze theorem and the property that $\lim_{n\to\infty} 7/(4n-2) = 0$.

(b) (9 pts) Using your result in part (a) and results in the text, determine $\lim_{n\to\infty} \int_a^b f_n(x)\,dx$ for $a < b$. Be sure to cite any results you use to justify your answer.

Solution: By 25.2 Theorem in the text, if a sequence (f_n) of continuous functions converges uniformly on a closed interval $[a, b]$ to a function f, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Thus, for

$$f_n(x) = \frac{3n+1-\sin(x)}{2n+\cos(x)} \quad \text{and} \quad f(x) = \frac{3}{2}$$

and any closed interval $[a, b]$, we have

$$\lim_{n\to\infty} \int_a^b \frac{3n+1-\sin(x)}{2n+\cos(x)}\,dx = \int_a^b \frac{3}{2}\,dx = \frac{3}{2}x\Big|_a^b = \frac{3(b-a)}{2}$$

3. Let $f_n(x) = n^2 x e^{-nx^2}$.

(a) (9 pts) Show that the sequence (f_n) converges pointwise on $\mathbb{R}$ and determine the function $f = \lim_{n\to\infty} f_n$.

Solution: We claim that $f_n \to 0$ pointwise on $\mathbb{R}$. This is clear for $x = 0$ (since $f_n(0) = 0$), while for $x \neq 0$, l'Hospital's rule (applied twice) gives

$$\lim_{n\to\infty} \frac{n^2 x}{e^{nx^2}} = \lim_{n\to\infty} \frac{2nx}{x^2 e^{nx^2}} = \frac{2}{x} \lim_{n\to\infty} \frac{n}{e^{nx^2}} = \frac{2}{x} \lim_{n\to\infty} \frac{1}{x^2 e^{nx^2}} = \frac{2}{x^3} \lim_{n\to\infty} \frac{1}{(e^{x^2})^n} = 0,$$

where at the last step we used $e^{x^2} > 1$.

(b) (9 pts) Show that (f_n) does *not* converge uniformly on any interval containing 0.

Solution: Let S be an interval containing 0. Note that

(6) $$f_n(1/\sqrt{n}) = n^2 \cdot (1/\sqrt{n}) \cdot e^{-1} = n^{3/2}/e \xrightarrow{n\to\infty} \infty.$$

Since 0 belongs to the interval S, and since $\lim_{n\to\infty} 1/\sqrt{n} = 0$, there is an $N \in \mathbb{N}$ such that $1/\sqrt{n} \in S$ for $n > N$. Using now (6), we infer that

$$\sup\{|f_n(x)| : x \in S, n > N\} = \infty,$$

and thus

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in S\} = \infty.$$

Therefore, the sequence (f_n) does *not* converge uniformly on S.

(c) (9 pts) Show that (f_n) does converge uniformly on any interval of the form $[a, \infty)$ with $a > 0$.

Solution: Recall the Taylor series expansion (at 0) for the exponential function: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$, for all $x \in \mathbb{R}$. Therefore, $e^{nx^2} = 1 + nx^2 + \frac{1}{2}n^2x^4 + \frac{1}{6}n^3x^6 + \cdots$. Now let $x \geq a > 0$; then,

$$|f_n(x)| = \frac{n^2 x}{e^{nx^2}} \leq \frac{n^2 x}{\frac{1}{6}n^3 x^6} = \frac{6}{nx^5} \leq \frac{6}{na^5}.$$

Since $\lim_{n\to\infty} \frac{6}{na^5} = \frac{6}{a^5} \lim_{n\to\infty} \frac{1}{n} = 0$, it follows that

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in [a, \infty)\} = 0.$$

Therefore, the sequence (f_n) converges uniformly to 0 on the interval $[a, \infty)$.

4. Let $f_n(x) = \sqrt{x} + \frac{1}{\sqrt{n}}$ and $f(x) = \sqrt{x}$, for $x \in [0, \infty)$.

(a) (9 pts) Show that (f_n) converges to f uniformly on $[0, \infty)$.

Solution: Note that $|f_n(x) - f(x)| = \frac{1}{\sqrt{n}}$. Since $\lim_{n\to\infty} 1/\sqrt{n} = 0$, we conclude that

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in [0, \infty)\} = 0.$$

Therefore, the sequence (f_n) converges to f uniformly on $[0, \infty)$

(b) (9 pts) Show that (f_n^2) converges to f^2 pointwise on $[0, \infty)$.

Solution: In general, if $f_n \to f$ and $g_n \to g$ (pointwise) on a set $S \subset \mathbb{R}$, then $f_n g_n \to fg$ (pointwise) on S, by properties of limits of convergent sequences.

In our situation, we showed in part (a) that $f_n \to f$ (pointwise) on $[0, \infty)$. Therefore, $f_n^2 = f_n \cdot f_n$ converges pointwise to $f^2 = f \cdot f$ on $[0, \infty)$.

(c) (9 pts) Show that (f_n^2) does *not* converge uniformly to f^2 on $[0, \infty)$.

Solution: Note that

$$f_n^2(x) - f^2(x) = \left(\sqrt{x} + \frac{1}{\sqrt{n}}\right)^2 - \left(\frac{1}{\sqrt{n}}\right)^2 = x + 2\frac{\sqrt{x}}{\sqrt{n}} + \frac{1}{n} - \frac{1}{n} = x + 2\frac{\sqrt{x}}{\sqrt{n}}.$$

Therefore, for $x \geq 0$,

$$\left|f_n^2(x) - f^2(x)\right| \geq x,$$

and thus,

$$\lim_{n\to\infty} \sup \left\{\left|f_n^2(x) - f^2(x)\right| : x \in [0, \infty)\right\} = \infty.$$

Hence, (f_n^2) does not converge uniformly to f^2 on $[0, \infty)$.

5. (10 pts) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}x)}{n^{3/2}}$ converges uniformly on $\mathbb{R}$ to a continuous function.

Solution: Note that

$$\left|\frac{\sin(\sqrt{n}x)}{n^{3/2}}\right| \leq \frac{1}{n^{3/2}}.$$

Moreover, note that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is a p-series with $p = 3/2 > 1$, and thus it is a convergent series. Hence, by the Weierstrass M-test (with $M_n = 1/n^{3/2}$), the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}x)}{n^{3/2}}$ converges uniformly on $\mathbb{R}$.

Let $g(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}x)}{n^{3/2}}$ be the sum of this series. Note that each of the functions $g_n : \mathbb{R} \to \mathbb{R}$, $g_n(x) = \dfrac{\sin(\sqrt{n}x)}{n^{3/2}}$ is a continuous function (since the sine function is continuous). Therefore, since the sequence of *continuous* functions (g_n) converges *uniformly* to the function $g : \mathbb{R} \to \mathbb{R}$, we conclude that g is also continuous.