Be sure to *fully justify your response* to each problem by citing any results in the text that you use and by writing out additional arguments as needed.

1. (10 pts) Determine the values of a for which $\lim_{n\to+\infty}\left(1+\frac{a}{n}\right)^n$ is finite, and give a formula for the limit for those values of a.

2. (15 pts) Let f be a function defined on $\mathbb{R}$. Suppose there exists $p > 1$ with the property that $|f(x) - f(y)| \le |x - y|^p$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

3. (10 pts) Let f be a function that is differentiable on an open interval (a, b). Show that if there is a number $M > 0$ such that
$$|f'(x)| \le M \quad \text{for all } x \in (a, b)$$
then f is uniformly continuous on (a, b).

4. (15 pts) Suppose f is a continuous function on $[a, b]$ and differentiable on the interior (a, b) with constant derivative $f'(x) = M$. Prove using the Mean Value Theorem that $f(x)$ is a linear function (i.e., there are constants A, B such that $f(x) = Ax + B$).

5. Let $f(x) = \ln(1 + x)$, let $\sum_{n=0}^{\infty} a_n x^n$ be the Taylor series for f, and let $R_n(x)$ be the remainder $R_n(x) = \ln(1 + x) - \sum_{k=0}^{n-1} a_k x^k$ for $x > -1$.
   (a) (10 pts) Using the formula for $R_n(x)$ in §31.3 Taylor's Theorem (p. 250), find an upper bound for $|R_n(x)|$.
   (b) (10 pts) Find all values of $x > -1$ for which it follows from your result in part (a) that $\lim_{n\to+\infty} R_n(x) = 0$.

6. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} x \sin\left(\frac{1}{2\pi x}\right), & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
   (a) (10 pts) Is f continuous?
   (b) (10 pts) Is the restriction of f to the interval $[-1, 1]$ uniformly continuous?
   (c) (10 pts) Is f differentiable?