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MATH 4565 **TOPOLOGY** **Fall 2023**

Homework 4

Definition 1. Let X and Y be two topological spaces. A map $f\colon X \to Y$ is *proper* if the preimage under f of any compact set is compact.

Definition 2. A topological space X is said to be *compactly generated* if the following condition is satisfied: A subspace A is closed in X if and only if $A \cap K$ is closed in K for all compact subspaces $K \subseteq X$.

Definition 3. A topological space X is *locally compact* if the following condition is satisfied: For every point $x \in X$, there is a compact subset $K \subseteq X$ that contains an (open) neighborhood of x.

1. Suppose X is compact, and Y is Hausdorff. Show that every continuous map $f\colon X \to Y$ is both closed and proper.

2. Let $X \times Y$ be the direct product of two topological spaces, and let $p\colon X \times Y \to X$ be the first-coordinate projection map. Show that p is proper if and only if Y is compact.

3. Show that every locally compact space is compactly generated.

4. Let $f\colon X \to Y$ be a proper, continuous map, and assume Y is a compactly generated, Hausdorff space. Show that f is a closed map.

5. Let $f\colon X \to Y$ be a continuous map from a compact space X to a Hausdorff space Y. Let C be a closed subspace of Y, and let U be an open neighborhood of $f^{-1}(C)$ in X. Show that there is an open neighborhood V of C in Y such that $f^{-1}(V)$ is contained in U.

6. Let (X, d) be a metric space. Given a point $x \in X$ and a subset $A \subset X$, define $d(x, A) := \inf\{d(x, a) : a \in A\}$. Similarly, given subsets $A, B \subset X$, define $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$.

(a) Show that $d(x, A) \leqslant d(x, y) + d(y, A)$, for all $x, y \in X$ and $A \subset X$.

(b) Show that the function $f\colon X \to \mathbb{R}$, $f(x) = d(x, A)$ is continuous, for all $\varnothing \neq A \subset X$.

(c) Suppose $B \subset X$ is closed. Show that $d(x, B) > 0$ for all $x \in X \setminus B$.

(d) Suppose $A \subset X$ is compact, $B \subset X$ is closed, and $A \cap B = \varnothing$. Show that $d(A, B) > 0$.