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MATH 4565 **TOPOLOGY** **Fall 2021**

Homotopy

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1. Homotopic functions

Two continuous functions from one topological space to another are called *homotopic* if one can be "continuously deformed" into the other, such a deformation being called a *homotopy* between the two functions. More precisely, we have the following definition.

Definition 1.1. Let X, Y be topological spaces, and $f, g\colon X \to Y$ continuous maps. A *homotopy* from f to g is a continuous function $F\colon X \times [0,1] \to Y$ satisfying

$$F(x,0) = f(x) \text{ and } F(x,1) = g(x), \text{ for all } x \in X.$$

If such a homotopy exists, we say that f is *homotopic* to g, and denote this by $f \simeq g$.

If f is homotopic to a constant map, i.e., if $f \simeq \text{const}_y$, for some $y \in Y$, then we say that f is *nullhomotopic*.

Example 1.2. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ any two continuous, real functions. Then $f \simeq g$.

To see why this is the case, define a function $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(x,t) = (1-t) \cdot f(x) + t \cdot g(x).$$

Clearly, F is continuous, being a composite of continuous functions. Moreover, $F(x,0) = (1-0) \cdot f(x) + 0 \cdot g(x) = f(x)$, and $F(x,1) = (1-1) \cdot f(x) + 1 \cdot g(x) = g(x)$. Thus, F is a homotopy between f and g.

In particular, this shows that any continuous map $f\colon \mathbb{R} \to \mathbb{R}$ is nullhomotopic.

This example can be generalized. First, we need a definition.

Definition 1.3. A subset $A \subset \mathbb{R}^n$ is said to be *convex* if, given any two points $x, y \in A$, the straight line segment from x to y is contained in A. In other words,

$$(1-t)x + ty \in A, \text{ for every } t \in [0,1].$$

Proposition 1.4. *Let A be a convex subset of $\mathbb{R}^n$, endowed with the subspace topology, and let X be any topological space. Then any two continuous maps $f, g\colon X \to A$ are homotopic.*

Proof. Use the same homotopy as in Example 1.2. Things work out, due to the convexity assumption. □

Let X, Y be two topological spaces, and let $\text{Map}(X, Y)$ be the set of all continuous maps from X to Y.

Theorem 1.5. *Homotopy is an equivalence relation on* $\mathrm{Map}(X, Y)$*.*

Proof. We need to verify that $\simeq$ is reflexive, symmetric, and transitive.

Reflexivity ($f \simeq f$). The map $F\colon X \times I \to X$, $F(x,t) = f(x)$ is a homotopy from f to f.

Symmetry ($f \simeq g \Rightarrow g \simeq f$). Suppose $F\colon X \times I \to X$ is a homotopy from f to g. Then the map $G\colon X \times I \to X$,

$$G(x,t) = F(x, 1-t)$$

is a homotopy from g to f.

Transitivity ($f \simeq g$ & $g \simeq h \Rightarrow f \simeq h$). Suppose $F\colon X \times I \to X$ is a homotopy from f to g and $G\colon X \times I \to X$ is a homotopy from g to h. Then the map $H\colon X \times I \to X$,

$$H(x,t) = \begin{cases} F(x, 2t) & \text{if } 0 \le t \le 1/2, \\ G(x, 2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

is a homotopy from f to h, as can be verified, using the Pasting Lemma. □

We shall denote the homotopy class of a continuous map $f\colon X \to Y$ by $[f]$. That is to say:

$$[f] = \{g \in \mathrm{Map}(X,Y) \mid g \simeq f\}.$$

Moreover, we shall denote set of homotopy classes of continuous maps from X to Y as

$$[X,Y] = \mathrm{Map}(X,Y)/\simeq .$$

Example 1.6. From Example 1.2, we deduce that $[\mathbb{R}, \mathbb{R}] = \{[\mathrm{const}_0]\}$. More generally, let X be any topological space, and let A be a (non-empty) convex subset of $\mathbb{R}^n$. We then deduce from Proposition 1.4 that

$$[X, A] = \{[\mathrm{const}_a]\}, \quad \text{for some } a \in A.$$

Proposition 1.7. *Let $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ be continuous maps, and let $g \circ f, g' \circ f'\colon X \to Z$ be the respective composite maps. If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$.*

Proof. Let $F\colon X \times I \to Y$ be a homotopy between f and f' and $G\colon Y \times I \to Z$ be a homotopy between g and g'. Define a map $H\colon X \times I \to Z$ by

$$H(x,t) = G(F(x,t),t).$$

Clearly, H is continuous. Moreover,

$$H(x,0) = G(F(x,0),0) = G(f(x),0) = g(f(x))$$
$$H(x,1) = G(F(x,1),1) = G(f'(x),1) = g'(f'(x)).$$

Thus, H is a homotopy between $g \circ f$ and $g' \circ f'$. □

As a consequence, composition of continuous maps defines a function

$$[X, Y] \times [Y, Z] \to [X, Z], \quad ([f], [g]) \mapsto [g \circ f].$$

2. Homotopy equivalences

Definition 2.1. Let $f\colon X \to Y$ be a continuous map. Then f is said to be *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y \quad \text{and} \quad g \circ f \simeq \mathrm{id}_X\,.$$

The map g in the above definition is said to be a *homotopy inverse* to f.

Remark 2.2. Every homeomorphism $f\colon X \to Y$ is a homotopy equivalence: simply take $g = f^{-1}$. The converse is far from true, in general.

The previous definition leads to a basic notion in algebraic topology.

Definition 2.3. Two spaces X and Y are said to be *homotopy equivalent* (written $X \simeq Y$) if there is a homotopy equivalence $f\colon X \to Y$.

Remark 2.4. By Remark 2.2,

$$X \cong Y \Longrightarrow X \simeq Y.$$

But the converse is far from being true. For instance, $\mathbb{R} \simeq \{0\}$, but of course $\mathbb{R} \not\cong \{0\}$ (since $\mathbb{R}$ is infinite, so there is not even a bijection from $\mathbb{R}$ to $\{0\}$).

Proposition 2.5. *Homotopy equivalence is an equivalence relation (on topological spaces).*

Proof. We need to verify that $\simeq$ is reflexive, symmetric, and transitive.

Reflexivity ($X \simeq X$). The identity map $\mathrm{id}_X\colon X \to X$ is a homeomorphism, and thus a homotopy equivalence.

Symmetry ($X \simeq Y \Rightarrow Y \simeq X$). Suppose $f\colon X \to Y$ is a homotopy equivalence, with homotopy inverse g. Then $g\colon Y \to X$ is a homotopy equivalence, with homotopy inverse f.

Transitivity ($X \simeq Y$ & $Y \simeq Z \Rightarrow X \simeq Z$). Suppose $f\colon X \to Y$ is a homotopy equivalence, with homotopy inverse g, and $h\colon Y \to Z$ is a homotopy equivalence, with homotopy inverse k. Using Proposition 1.7 (and the associativity of compositions) the following assertion is readily verified: $h \circ f\colon X \to Z$ is a homotopy equivalence, with homotopy inverse $g \circ k$. ☐

Equivalence classes under $\simeq$ are called *homotopy types*. The simplest homotopy type is that of a singleton. This merits a definition.

Definition 2.6. A topological space X is said to be *contractible* if X is homotopy equivalent to a point, i.e., $X \simeq \{x_0\}$.