Homotopy Types of Complements of 2-Arrangements in $\mathbb{R}^4$

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A 2-*arrangement in* $\mathbb{R}^{2d}$ is a collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of codim 2 linear subspaces so that, $\forall I \subseteq \{1, \dots, n\}$, the space $\bigcap_{i \in I} H_i$ has even dimension.

$H_*(M; \mathbb{Z})$ computed by Goresky & MacPherson, Vassiliev, Jewell, Orlik & Shapiro.

$H^*(M; \mathbb{Z})$ computed by Björner & Ziegler.

These results generalize the work of Arnol'd, Brieskorn, and Orlik & Solomon on complex hyperplane arrangements. Unlike the Orlik-Solomon algebra, which is completely determined by $L(\mathcal{A})$, there are sign ambiguities in the relations defining $H^*(M; \mathbb{Z})$.

This ambiguity cannot be resolved. Ziegler found a pair of 2-arrangements in $\mathbb{R}^4$, $\mathcal{B} = \{H_1, \dots, H_4\}$, $\mathcal{B}' = \{H'_1, \dots, H'_4\}$, such that:

$$L(\mathcal{B}) \cong L(\mathcal{B}')$$
$$H^*(M(\mathcal{B})) \ncong H^*(M(\mathcal{B}'))$$

Question (Ziegler): Is the homotopy type of the complement of a 2-arrangement in $\mathbb{R}^4$, $M = M(\mathcal{A})$, determined by the cohomology ring $H^*(M)$?

Answer (M-S): No. There are 2-arrangements in $\mathbb{R}^4$ such that:

$$\begin{aligned} L(\mathcal{A}) &\cong L(\mathcal{A}') \\ H^*(M(\mathcal{A})) &\cong H^*(M(\mathcal{A}')) \\ \phi_k(G(\mathcal{A})) &= \phi_k(G(\mathcal{A}')) \\ \theta_k(G(\mathcal{A})) &= \theta_k(G(\mathcal{A}')) \\ M(\mathcal{A}) &\not\simeq M(\mathcal{A}') \end{aligned}$$

These arrangements are cones on Mazurovskiĭ's K and L configurations of 6 skew lines in $\mathbb{R}^3$.

The difference in homotopy types is picked up by the characteristic varieties associated to the Alexander module $A = A(\mathcal{A})$.

The method yields a complete homotopy classification of 2-arrangements in $\mathbb{R}^4$, for $n \le 6$.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a 2-arrangement in $\mathbb{R}^4$. Then $M \simeq K(G, 1)$, where

$$\begin{aligned} G &= F_{n-1} \rtimes_\alpha \mathbb{Z} \quad (\alpha \in P_{n-1}) \\ &= \langle t_1, \ldots, t_{n-1}, t_n \mid t_n^{-1} t_i t_n = \alpha(t_i) \rangle \end{aligned}$$

The Alexander module has presentation

$$C_1 \xrightarrow{\Omega = \begin{pmatrix} \mathrm{id} - \Theta(\alpha) & d_1 \end{pmatrix}} C_1 \oplus C_0 \to A \to 0$$

where $C_0 = \Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, $C_1 = \Lambda^{n-1}$.

Let $E_d(A)$ be the ideal generated by the $(n-d) \times (n-d)$ minors of Ω. It defines the d^{th} *characteristic variety*, $V_d = V_d(A)$, in the affine torus $(\mathbb{C}^*)^n$. Get a tower

$$(\mathbb{C}^*)^n \supset V_1 \supset \cdots \supset V_{n-2} \supset V_{n-1} = \{(1, \ldots, 1)\}$$

The characteristic varieties are well-defined up to an automorphism of $(\mathbb{C}^*)^n$ of the form $t_i \mapsto t_1^{a_{i1}} \cdots t_n^{a_{in}}$, where $\begin{pmatrix} a_{ij} \end{pmatrix} \in \mathrm{GL}(n; \mathbb{Z})$. As such, they depend only on the homotopy type of M—that is, only on G.

Example: The Ziegler arrangements

$$Q(\mathcal{B}) = xy(x-y)(x-2y)$$
$$Q(\mathcal{B}') = xy(x-y)(x-2\overline{y})$$

$$\begin{aligned}
V_1 = V_2 &= \{t \in (\mathbb{C}^*)^4 \mid t_4 - 1 = 0\} \\
V_1' &= \{t_4 - 1 = 0\} \cup \{t_4 - t_2^2 = 0\} \\
V_2' &= \{t_4 - 1 = t_2 + 1 = 0\} \cup \{t_3 - 1 = t_4 - 1 = t_2 - 1 = 0\} \\
&\quad \cup \{t_1 - 1 = t_4 - 1 = t_2 - 1 = 0\}
\end{aligned}$$

The characteristic varieties of $\mathcal{B}$ and $\mathcal{B}'$ may be distinguished by the number of irreducible components, or by codimension. Alternatively, by the number of p-torsion points. E.g.:

$$\mathrm{Tor}_3(V_1) = 27, \quad \mathrm{Tor}_3(V_1') = 45$$

Example: The Mazurovskiĭ configurations

May distinguish the characteristic varieties by the number of irreducible components, or by the number of torsion points. E.g.:

$$\mathrm{Tor}_2(V_1) = 32, \quad \mathrm{Tor}_2(V_1') = 31$$