Braids and Configuration Spaces

1. Ordered configurations

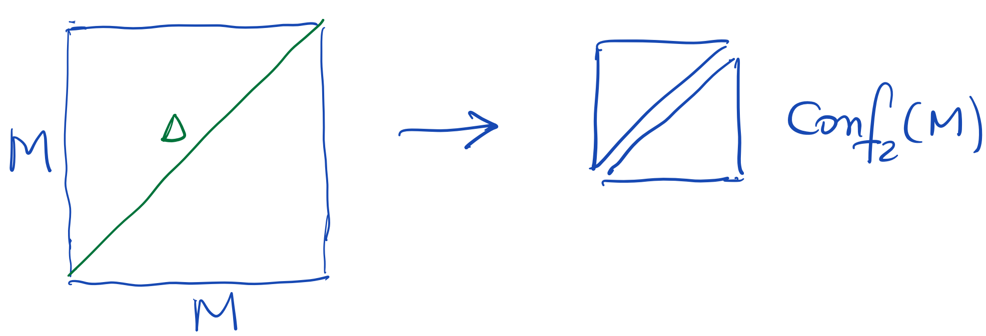
- Let M be a topological space.
- For each $n \geq 1$ let $M^n = \underbrace{M \times \cdots \times M}_{n \text{ times}}$ (with product topology).

Definition The *ordered configuration space* of n points in M is the space

$$\mathrm{Conf}_n(M) = \{(m_1, \ldots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\}.$$

- If M is a manifold of dimension d, then $\mathrm{Conf}_n(M)$ is a manifold of dimension dn.

Eg:
- $\mathrm{Conf}_1(M) = M$
- $\mathrm{Conf}_2(M) = M \times M \setminus \Delta$ $\quad (\Delta = \text{diagonal})$



- Suppose M is connected. Then
  - If $d = 1$ and $n \geq 2$, then $\mathrm{Conf}_n(M)$ is disconnected.
  - If $d \geq 2$, then $\mathrm{Conf}_n(M)$ is connected.

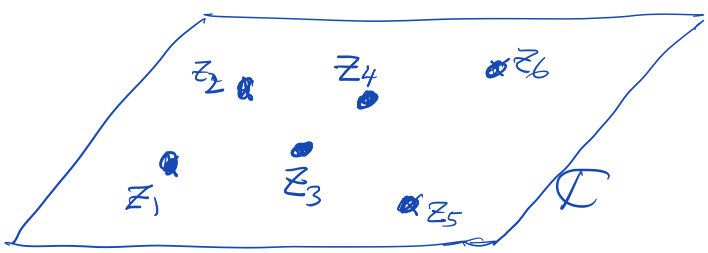
- Suppose now $M = G$ is a topological group (e.g., $G = \mathbb{R}^n$, $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, S^1, etc)

Then

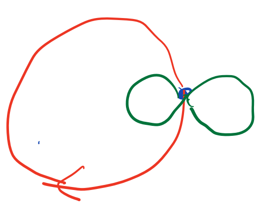
$$\mathrm{Conf}_n(G) \xrightarrow{\cong} G \times \mathrm{Conf}_{n-1}(G\setminus\{e\})$$
$$(g_1, \dots, g_n) \longrightarrow (g_1, (g_1^{-1}g_2, \dots, g_1^{-1}g_n))$$

- Most important case is when $M = \mathbb{R}^2$, i.e., $M = \mathbb{C}$.

- A point in $\mathrm{Conf}_n(\mathbb{C})$ can be viewed as an n-tuple of distinct points in $\mathbb{C}$



- $\mathrm{Conf}_1(\mathbb{C}) = \mathbb{C} \simeq \{0\}$
- $\mathrm{Conf}_2(\mathbb{C}) \cong \mathbb{C} \times (\mathbb{C}\setminus\{0\}) \simeq \{0\} \times S^1 \simeq S^1$
- $\mathrm{Conf}_3(\mathbb{C}) \cong \mathbb{C} \times \mathrm{Conf}_2(\mathbb{C}\setminus\{0\})$



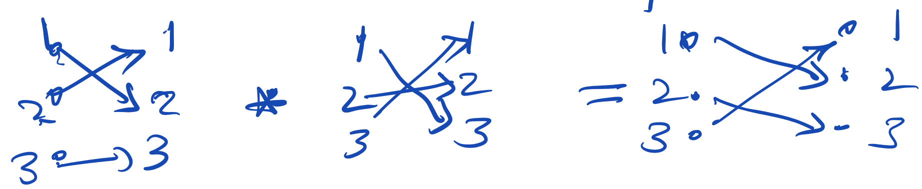
$$\cong \mathbb{C} \times (\mathbb{C}\setminus\{0\}) \times (\mathbb{C}\setminus\{0,1\})$$
$$\simeq S^1 \times (S^1 \vee S^1)$$

2. Unordered configurations

- A *permutation* of a set S is a bijection from S to S.

- For instance, if $S = \{1, 2\}$, its permutations are
$$\begin{matrix} 1 \to 1 \\ 2 \to 2 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 \mapsto 2 \\ 2 \mapsto 1 \end{matrix}$$

- Permutations can be composed. E.g.:



- The *symmetric group* S_n is the group of all permutations of the set $\{1, \dots, n\}$. Its order is $n!$

- Given a space M, the group acts on M^n by permuting the coordinates:
$$S_n \times M^n \longrightarrow M^n$$
$$(\sigma, (m_1, \dots, m_n)) \longmapsto (m_{\sigma(1)}, \dots, m_{\sigma(n)})$$

- This action restricts to a *free* action of S_n on $\mathrm{Conf}_n(M)$.

<u>Definition</u> The <u>unordered configuration space</u> of n points in a space M is

$$UConf_n(M) = Conf_n(M)/S_n$$

(the quotient space of the ordered configuration space by the above free action).

- $UConf_n(M)$ may be viewed as the space of all subsets $\{m_1, \dots, m_n\} \subset M$ of size n.

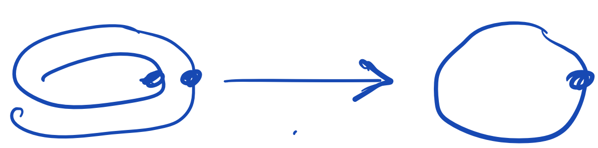
- The quotient map,

$$Conf_n(M) \longrightarrow UConf_n(M),$$

is a (regular) cover, with group of deck transformations S_n.

- E.g., $Conf_2(\mathbb{C}) \longrightarrow UConf_2(\mathbb{C})$ is, up to homotopy, the 2-fold cover

$$S^1 \longrightarrow S^1$$
$$z \longmapsto z^2$$



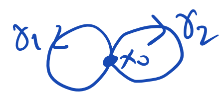
3. Fundamental groups of configuration spaces

Review:

- Fundamental group of M, based at x_0:

  $$\pi_1(M, x_0) = \left\{ \begin{array}{l} \text{loops } \gamma \text{ in } M \text{ based at } x_0 \\ \text{modulo homotopy rel } x_0 \end{array} \right\}$$

  with
  - $\gamma_1 \cdot \gamma_2 =$ concatenation of the two loop

  - $\gamma^{-1} =$ loop traversed in opposite direction
  - identity $=$ constant loop at x_0

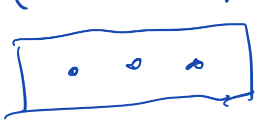
- If M is path-connected, then

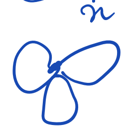
  $$\pi_1(M, x_0) \cong \pi_1(M, x_1) \quad \text{for all } x_0, x_1 \in M$$

  so write it simply as $\pi_1(M)$.

- $\pi_1(M_1 \times M_2) \cong \pi_1(M_1) \times \pi_1(M_2)$

- Examples:
  - $\pi_1(\text{contractible space}) = \{e\}$
    e.g., $\mathbb{R}^n$
  - $\pi_1(\mathbb{C} \setminus \{0\}) \cong \pi_1(S^1) = \mathbb{Z}$
  - $\pi_1(\mathbb{C} \setminus \{n \text{ points}\}) \cong \pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{n}) = F_n$

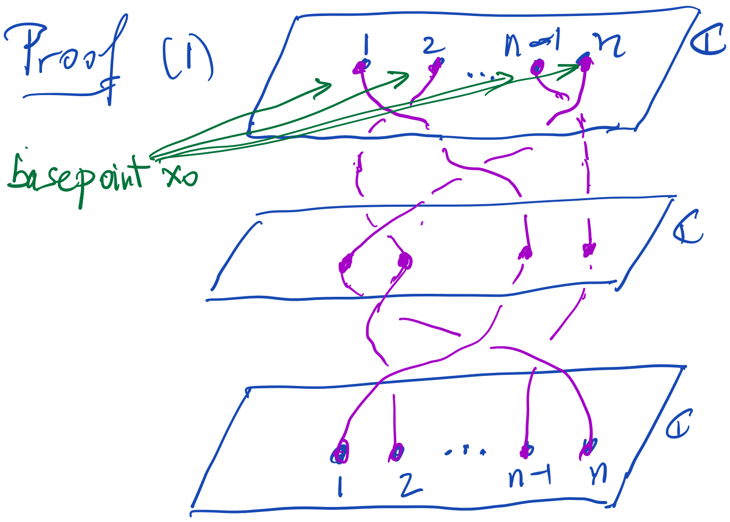
    F_n: free group on n letters

Theorem (Fox & Fadell – 1962)

(1) $\pi_1(\mathrm{Conf}_n(\mathbb{C})) \cong P_n$ — pure braid group on n strings

(2) $\pi_1(\mathrm{UConf}_n(\mathbb{C})) \cong B_n$ — braid group on n strings

Proof (1)



$t=0$

$t=1/2$

$t=1$

(2) similar

QED

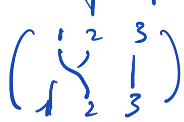
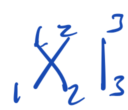
- Now recall the S_n-cover

$$\mathrm{Conf}_n(\mathbb{C}) \longrightarrow \mathrm{UConf}_n(\mathbb{C})$$

Using the relationship between covers and fundamental group, we get exact sequence

$$1 \longrightarrow P_n \overset{=\ker(q)}{\longrightarrow} B_n \xrightarrow{q} S_n \longrightarrow 1$$

where q sends a braid β to the induced permutation of the strands, e.g.,

$q($  $) =$ 

- Examples:

$n=1$
$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_1 & \xrightarrow{\cong} & B_1 & \longrightarrow & S_1 & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} & & \mathbb{Z} & & \{1\} & & \end{array}$$

$n=2$
$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_2 & \longrightarrow & B_2 & \longrightarrow & S_2 & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \end{array}$$

$n=3$
$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_3 & \longrightarrow & B_3 & \longrightarrow & S_3 & \longrightarrow & 1 \\ & & \cong & & & & & & \\ & & \mathbb{Z}\times F_2 & & & & & & \end{array}$$

- Remark In turn, the braid groups completely determine the homotopy types of the respective configuration spaces:
$$\mathrm{Conf}_n(\mathbb{C}) \simeq K(P_n, 1)$$
$$U\mathrm{Conf}_n(\mathbb{C}) \simeq K(B_n, 1)$$

———— ∘ ————

4. Braids and polynomials

- We may identify $\mathbb{C}^n$ with the space of all monic polynomials with coefficients in $\mathbb{C}$:
$$\boxed{\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\cong} & \mathrm{Poly}_n(\mathbb{C}) \\ (a_1, a_2, \dots, a_n) & \longrightarrow & P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n \end{array}}$$

- By the Fundamental Theorem of Algebra, every non-constant polynomial completely factors into a product of linear factors:

$$\boxed{P(x) = (x - z_1) \cdots (x - z_n)} \quad (*)$$

where $z_1, \dots, z_n$ are the roots of P.

- The coefficients of P may be recovered from its roots, as the elementary symmetric polynomials in those roots (up to sign):

$$\begin{cases} a_1 = z_1 + z_2 + \cdots + z_n \\ a_2 = \sum_{i<j} z_i z_j \\ \vdots \\ a_n = z_1 \cdots z_n \end{cases}$$

- These formulas were discovered by François Viète in the late 1500s. They provide an identification

$$\mathbb{C}^n / S_n \xrightarrow[\text{Vieta's map}]{\cong} \mathbb{C}^n$$

$$\begin{array}{ccc} (z_1, \dots, z_n) & \longrightarrow & (a_1, \dots, a_n) \\ \Updownarrow & & \Updownarrow \\ P(x) = (x - z_1) \cdots (x - z_n) & & P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \end{array}$$

- Some of the factors in (*) may be repeated. So let

$$SPoly_n(\mathbb{C}) = \left\{ \begin{array}{l} \text{space of polynomials} \\ \text{with no repeated linear factors} \end{array} \right\}$$

or, the space of square-free polynomials.

- There is then an identification

$$\begin{array}{ccc} SPoly_n(\mathbb{C}) & \xrightarrow{\cong} & UConf_n(\mathbb{C}) \\ (x-z_1)\cdots(x-z_n) & \longleftrightarrow & (z_1, \dots, z_n) \end{array}$$

- Therefore, $\pi_1(SPoly_n(\mathbb{C})) \cong B_n$.

- To conclude, let us describe in more concrete terms the space $SPoly_n(\mathbb{C})$

- Note that $P(x) = (x-z_1)\cdots(x-z_n)$ has a repeated root precisely when the polynomial (of degree n)

$$\Delta_n(z) := \prod_{1 \le i < j \le n} (z_i - z_j)^2$$

vanishes.

- This polynomial can be re-interpreted as a polynomial in the variables $a=(a_1,\dots,a_n)$
$$\Delta_n(a)$$
via the Vieta formulas.

- Therefore:
$$\boxed{SPoly_n(\mathbb{C}) = \{(a_1,\dots,a_n)\in\mathbb{C}^n \mid \Delta_n(a)\neq 0\}}$$
i.e., the complement in $\mathbb{C}^n$ of the discriminant hypersurface $\Delta_n=0$.

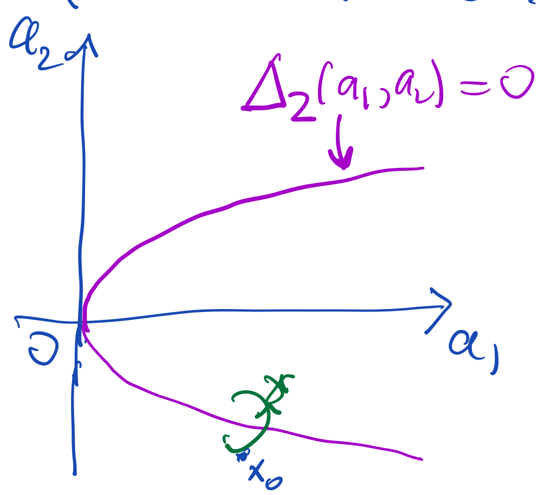
- Let us describe these hypersurfaces in low degrees $(n=2,3,4)$.

- $\boxed{n=2}$
$$\begin{aligned} P(x) &= (x-z_1)(x-z_2) \\ &= x^2-(z_1+z_2)x+z_1z_2 \\ &= x^2+a_1x+a_2 \end{aligned}$$
where $a_1=-(z_1+z_2)$, $a_2=z_1z_2$
$$\begin{aligned} \therefore\ \Delta_2 &= (z_1-z_2)^2 \\ &= z_1^2-2z_1z_2+z_2^2 \end{aligned}$$

$$= (z_1 + z_2)^2 - 4 z_1 z_2$$

$$\therefore \boxed{\Delta_2(a) = a_1^2 - 4a_2}$$

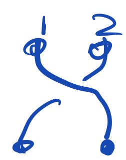
(more generally, if $P(x) = ax^2 + bx + c$, then $\Delta = b^2 - 4ac$, a formula that goes back to the Babylonians)



$$B_2 = \pi_1\left(\mathbb{C}^2 \setminus \{\Delta_2 = 0\}\right) \cong \mathbb{Z}$$

generated by the loop γ around the parabola

That is: $\gamma \longleftrightarrow$

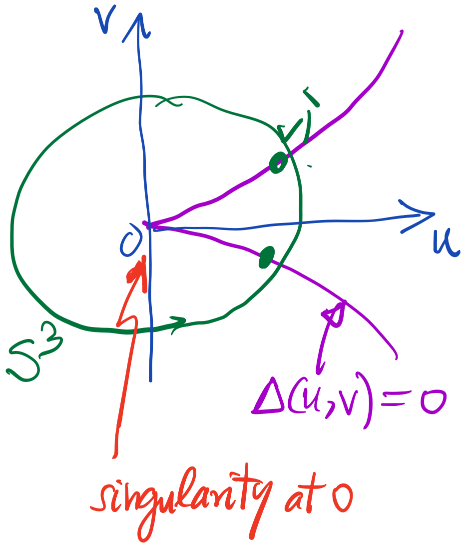


loop braid

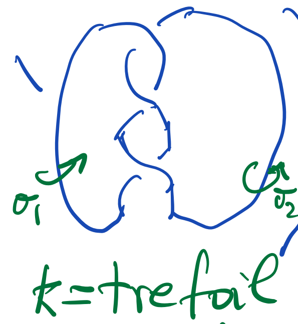
$\boxed{n=3}$

- Let $P(x) = x^3 + a_1 x + a_2 x^2 + a_3 x^3$ be a cubic polynomial.
- Changing variables via $x = t - \frac{a_1}{3}$ gives the simpler cubic
$$P(t) = t^3 + ut + v$$

- Up to sign, then, the discriminant is
$$\Delta(u,v) = 4u^3 + 27v^2$$



$$B_3 = \pi_1\left(\mathbb{C}^3 \setminus \{\Delta_3 = 0\}\right)$$
$$= \pi_1\left(\mathbb{C}^2 \setminus \{\Delta(u,v) = 0\}\right)$$
$$= \pi_1\left(S^3 \setminus \right.$$



$$\left.\right)$$

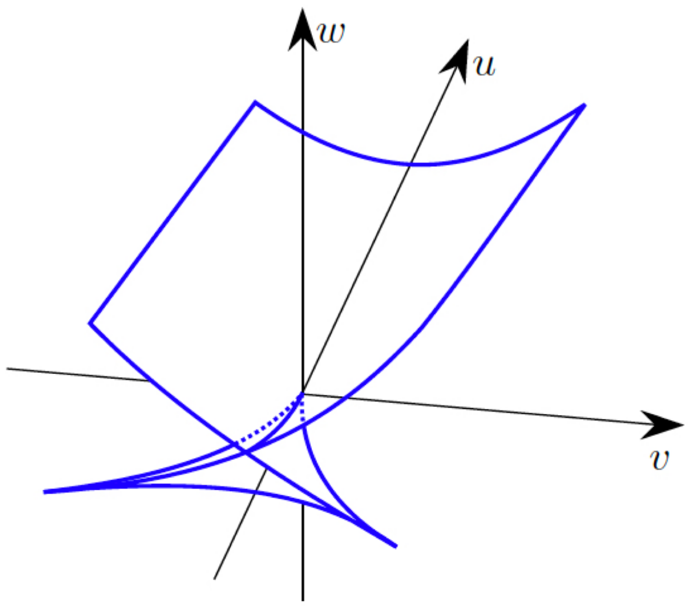
$$= \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

$\boxed{n=4}$

- Changing coordinates judiciously, we may write an arbitrary (monic) quartic as
$$P(x) = x^4 + ux^2 + vx + w$$

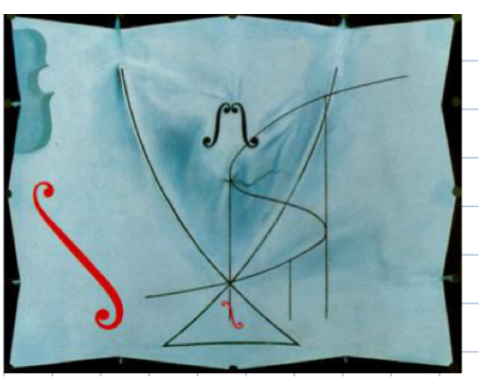
- The discriminant polynomial becomes
$$\Delta(u,v,w) = 8u^4w - 4u^3v^2 - 128u^2w^2 + 144uv^2w - 27v^4 + 256w^3.$$

- The corresponding hypersurface (in 3-space) is called the <u>swallowtail</u> singularity.



after this bird:

- Thus: $B_4 = \pi_1\left(\mathbb{C}^3 \setminus \{\Delta(u,v,w)=0\}\right)$



The Swallow's Tail
– by Salvador Dali
(1983)