Be sure to include your reasoning in your answers to the following questions.

1. (a) (10 pts) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < \frac{1}{n^{3/2}} \quad \text{for all } n \in \mathbb{N}$$

Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.

Solution: Let $\epsilon > 0$ be given. Since the p-series $\sum 1/n^{3/2}$ converges, it follows that the series $\sum 1/n^{3/2}$ satisfies the Cauchy criterion. Hence there is an $N \in \mathbb{N}$ such that

$$\sum_{i=0}^{\ell} \frac{1}{(n+i)^{3/2}} < \epsilon \quad \text{for all } n \geq N \text{ and all } \ell \geq 0 \tag{1}$$

For $k \in \mathbb{N}$, from the triangle inequality and equation (1) with $\ell = k - 1$, it follows that

$$\begin{aligned} |s_n - s_{n+k}| &= |s_n - s_{n+1} + s_{n+1} - s_{n+2} + s_{n+2} + \cdots - s_{n+k-1} + s_{n+k-1} - s_{n+k}| \\ &\leq |s_{n+1} - s_n| + |s_{n+2} - s_{n+1}| + \cdots + |s_{n+k} - s_{n+k-1}| \\ &< \frac{1}{(n)^{3/2}} + \frac{1}{(n+1)^{3/2}} + \frac{1}{(n+2)^{3/2}} + \cdots + \frac{1}{(n+k-1)^{3/2}} \\ &< \epsilon \end{aligned}$$

for all $n \geq N$, and hence, s_n is a Cauchy sequence.

(b) (10 pts) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < \frac{1}{n^{2/3}} \quad \text{for all } n \in \mathbb{N}.$$

Show by means of an example that the sequence (s_n) may **not** converge.

Solution: Consider the sequence (s_n) with

$$s_n = \sum_{k=1}^{n} \frac{1}{k^{2/3}}.$$

Then

$$|s_{n+1} - s_n| = \frac{1}{(n+1)^{2/3}} < \frac{1}{n^{2/3}}$$

for all $n \in \mathbb{N}$. On the other hand, the sequence (s_n) is the sequence of partial sums of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{2/3}},$$

which is a p-series with $p = 2/3 < 1$, and thus **not** a converging series. (This can also be shown by using the integral test for convergence/divergence.) By definition, this means that the sequence (s_n) does not converge, and we are done.

2. Consider the sequence (x_n) with terms $x_n = (1 - 1/n)\cos(n\pi/4)$.

(a) (10 pts) Write out the first 10 terms in this sequence

Solution:

$$\begin{aligned}
x_1 &= (1-1)\cos(\pi/4) = 0,\\
x_2 &= (1-1/2)\cos(\pi/2) = 0,\\
x_3 &= (1-1/3)\cos(3\pi/4) = -(2/3)(\sqrt{2}/2) = -\sqrt{2}/3,\\
x_4 &= -(1-1/4) = -3/4,\\
x_5 &= (1-1/5)\cos(5\pi/4) = (4/5)(-\sqrt{2}/2) = -(2/5)\sqrt{2},\\
x_6 &= (1-1/6)\cos(6\pi/4) = 0,\\
x_7 &= (1-1/7)\cos(7\pi/4) = (6/7)(\sqrt{2}/2) = (3/7)\sqrt{2},\\
x_8 &= (1-1/8)\cos(8\pi/4) = 7/8,\\
x_9 &= (1-1/9)(\sqrt{2}/2) = (4/9)\sqrt{2},\\
x_{10} &= (1-1/10)\cos(10\pi/4) = 0
\end{aligned}$$

(b) (10 pts) Give an example of a monotonic subsequence of (x_n).

Solution: Set $n_k = 8k$ for $k \in \mathbb{N}$, then the subsequence $s_k = x_{n_k}$ is given by

$$s_k = \left(1 - \frac{1}{8k}\right)\cos(8k\pi/4) = \left(1 - \frac{1}{8k}\right)\cos(2k\pi) = 1 - \frac{1}{8k}$$

$1/[8(k+1)] < 1/[8k]$ so

$$s_{k+1} = 1 - (1/[8(k+1)]) > 1 - 1/[8k] = s_k$$

and hence, (s_k) is an increasing subsequence of (x_n).

(c) (10 pts) Give the $\limsup x_n$ and $\liminf x_n$

Solution: We will show that $\liminf x_n = -1$ and $\limsup x_n = 1$.

By Theorem 11.8 we have for (s_n) be any sequence and S the set of subsequential limits of (s_n), that

$$\sup S = \limsup s_n \quad \text{and} \quad \inf S = \liminf s_n$$

If (t_n) is a convergent subsequence of (s_n) then $\lim t_n \in S$, and so from the result above we have that

(2) $$\liminf s_n \le \lim t_n \le \limsup s_n$$

Now for the subsequence s_k of (x_n) in the solution to part (b) given by $s_k = x_{8k} = 1-(1/8k)$, we have that $\lim s_k = 1$. Thus, from equation (2) we have that $1 \le \limsup x_n$.

Now recall that $\limsup x_n$ is the limit of the sequence $v_N = \sup\{x_n : n \ge N\}$. Also, note that $|x_n| = |(1-1/n)\cos(n\pi/4)| < 1$. So, $-1 < x_n < 1$ for all $n \in \mathbb{N}$ and it follows that $v_N \le 1$ for all n. Since (v_N) is a decreasing sequence, the sequence v_N converges to its inf; that is, $\lim v_N = \inf\{v_N : N \in \mathbb{N}\}$ which must be ≤ 1 since each v_N is less than or equal to 1. Thus,

$$\limsup x_n = \lim v_N \le 1$$

and the proof that $\limsup x_n = 1$ is complete.

The proof that $\liminf x_n = -1$ is similar. Set $n_k = 8k+4$, then the terms in the subsequence $s_k = x_{n_k}$ are

$$s_k = x_{8k+4} = \left(1 - \frac{1}{8k+4}\right)\cos([8k+4]\pi/4) = -\left(1 - \frac{1}{8k+4}\right) = -1 + \frac{1}{8k+4}$$

and it follows that $\lim s_k = -1$. From equation (2) it follows that $\liminf x_n \ge -1$.

$\liminf x_n = \lim u_N$ where $u_N = \inf\{x_n : n \geq \mathbb{N}\}$. u_N is an increasing sequence, and hence, converges to $\sup\{u_N\}$. Since $x_n > -1$ for all n, it follows that each $u_N \geq -1$ so $\sup\{u_N\} = \liminf x_n \geq -1$. This completes the proof that $\liminf x_n = -1$.

An alternate approach is to give an argument that for each $N \in \mathbb{N}$,

$$u_N = \inf\{x_n : n \geq N\} = -1 \quad \text{and} \quad v_N = \sup\{x_n : n \geq N\} = 1$$

The result then follows since by definition of lim sup and lim inf, we have $\limsup x_n = \lim v_N$ and $\liminf x_n = \lim u_N$.

3. (10 pts) Let (x_n) be a sequence with $\lim x_{2n} = 1$ and $\lim x_{2n+1} = 5$. Show that every convergent subsequence of x_n converges to either 1 or 5.

Solution: The proof is by contradiction.

Suppose that there is a convergent subsequence $s_k = x_{n_k}$ of x_n with $\lim s_k = a$ with $a \neq 1$ and $a \neq 5$. Then choose an $\epsilon > 0$ such that no two of the open intervals, $(1 - \epsilon, 1 + \epsilon)$, $(1 - a, 1 + a)$, and $(5 - \epsilon, 5 + \epsilon)$ have any any elements in common. For this one can choose $\epsilon = (1/3)\min\{|1 - a|, |5 - a|\}$. Then since $\lim s_k = a$, there is an $N_1 \in \mathbb{N}$ such that

$$|s_k - a| < \epsilon \quad \text{for all } k \geq N_1$$

Similarly, since $\lim x_{2n} = 1$ there is an $N_2 \in \mathbb{N}$ such that $|1 - x_{2n}| < \epsilon$ for all even numbers $2n$ with $2n \geq N_2$, and since $\lim x_{2n+1} = 5$, there is an $N_3 \in \mathbb{N}$ such that $|5 - x_{2n+1}| < \epsilon$ for all odd numbers $2n + 1$ with $2n + 1 \geq N_3$. Set $N = \max\{N_1, N_2, N_3\}$.

Now consider the term $s_N = x_{n_N}$. Since $N \geq N_1$, we have that $|x_{n_N} - a| < \epsilon$. Note that since $s_k = x_{n_k}$ is a subsequence of (x_n) it follows that $n_N \geq N$ and hence $n_N \geq N_2$ and $n_N \geq N_3$. If n_N is even, then since $n_N \geq N_2$, it follows that $|x_{n_N} - 1| < \epsilon$ which contradicts $|x_{n_N} - a| < \epsilon$. If n_N is odd, then since $n_N \geq N_3$, it follows that $|x_{n_N} - 5| < \epsilon$ which contradicts $|x_{n_N} - a| < \epsilon$. Since n_N is either even or odd, it follows that the assumption that there is a convergent subsequence of (x_n) with limit not equal to either 1 or 5 leads to a contradiction.

4. (10 pts) Let (x_n) and (y_n) be two bounded sequences of non-negative numbers. Show that

$$\liminf(x_n y_n) \geq \liminf(x_n) \cdot \liminf(y_n).$$

Solution: *Step* 1. As a first step, let us prove the following statement: Let A, B be two subsets of $\mathbb{R}_+ = [0, \infty)$, and let $A \cdot B = \{ab : a \in A, b \in B\}$; then

$$\inf(A \cdot B) = \inf(A)\inf(B). \tag{3}$$

To establish this inequality, first note that $A \subseteq \mathbb{R}_+$ implies $\inf(A) \geq 0$, and similarly $\inf(B) \geq 0$. Next, by the definition of infimum, we have that $a \geq \inf(A)$ and $b \geq \inf(B)$, for all $a \in A$ and $b \in B$. Since all these quantities are non-negative, we multiply these inequalities (without changing the direction of those inequalities), and conclude that

$$ab \geq \inf(A)\inf(B), \quad \text{for all } a \in A \text{ and } b \in B.$$

This shows that the set $A \cdot B$ is bounded below by $\inf(A)\inf(B)$, and so

$$\inf(A \cdot B) \geq \inf(A)\inf(B).$$

To prove the reverse inequality, let $c = ab$ be an arbitrary element in $A \cdot B$. By definition of infimum, $c \geq \inf(A \cdot B)$. If $c = 0$, then $\inf(A \cdot B) = 0$, and so $\inf(A \cdot B) \leq \inf(A)\inf(B)$, in which case we are done. Thus, we may assume $c \neq 0$, which implies $a > 0$ and $b > 0$ for all $a \in A$ and $b \in B$.

Now fix $b \in B$; then the following holds. For all $a \in A$ (writing $c = ab$),

$$a = c/b \geq \inf(A \cdot B)/b,$$

and so $\inf(A) \geq \inf(A \cdot B)/b$. If $\inf(A) = 0$, then this implies $\inf(A \cdot B) \leq 0$, which forces $\inf(A \cdot B) = 0$, and so $\inf(A \cdot B) \leq \inf(A)\inf(B)$. Otherwise, we get $b \geq \inf(A \cdot B)/\inf(A)$. Since this inequality holds for all $b \in B$, we deduce that $\inf(B) \geq \inf(A \cdot B)/\inf(A)$, that is, $\inf(A \cdot B) \leq \inf(A)\inf(B)$. Therefore, the equality (3) has been established.

Step 2. For the second step, let us first recall the definition of $\liminf x_n$ and $\liminf y_n$. For each $N \geq 1$, put $A_N := \{x_n : n > N\}$ and $B_N := \{y_n : n > N\}$, and set

$$u_N = \inf(A_N) \text{ and } w_N = \inf(B_N).$$

Then both (u_N) and (w_N) are bounded, decreasing sequences; therefore, both are convergent sequences. By definition, their respective limits are:

$$\liminf x_n = \lim_{N\to\infty} u_N \text{ and } \liminf y_n = \lim_{N\to\infty} w_N.$$

Furthermore, since both (u_N) and (w_N) are convergent, the product of the two sequences, $(u_N w_N)$, also converges, and

$$\lim_{N\to\infty} u_N w_N = \left(\lim_{N\to\infty} u_N\right) \cdot \left(\lim_{N\to\infty} w_N\right) = \liminf(x_n) \cdot \liminf(y_n). \tag{4}$$

Since $A_N \subset \mathbb{R}_+$ and $B_N \subset \mathbb{R}_+$, we have by Step 1 that $\inf(A_N)\inf(B_N) = \inf(A_N \cdot B_N)$. Using the fact that $\{x_n y_n : n > N\} \subset A_N B_N$ and monotonicity of inf, we get

$$u_N w_N = \inf(A_N)\inf(B_N) = \inf(A_N \cdot B_N) \leq \inf\{x_n y_n : n > N\}$$

for all $N \geq 1$. Taking the limit as $N \to \infty$ on both sides, we obtain

$$\lim_{N\to\infty} u_N w_N \leq \lim_{N\to\infty} \inf\{x_n y_n : n > N\} = \liminf(x_n y_n). \tag{5}$$

Putting together equations (4) and (5) proves the desired inequality.

5. For each of the following series, determine whether the series converges or diverges. Justify your answers.

   (a) (10 pts) $\sum \frac{1}{n\ln(n)^3}$

Solution: Use the integral test. The substitution $u = \ln(x)$ gives $du = (1/x)dx$ so

$$\begin{aligned}\int \frac{1}{x\ln(x)^3}\,dx &= \int \frac{1}{u^3}\,du \\ &= \int u^{-3}\,du \\ &= \frac{-1}{2}u^{-2} + C \\ &= \frac{-1}{2\ln(x)^2} + C\end{aligned}$$

Since $\lim_{a\to\infty} \ln(a) = +\infty$, it follows that $\lim_{a\to\infty}(1/\ln(a)^2) = 0$. Thus, for the improper integral we have

$$\begin{aligned}
\int_2^\infty \frac{1}{x\ln(x)^3}\,dx &= \lim_{a\to\infty}\int_2^a \frac{1}{x\ln(x)^3}\,dx \\
&= \lim_{a\to\infty} \left.\frac{-1}{2\ln(x)^2}\right|_2^a \\
&= \lim_{a\to\infty}\left(\frac{1}{2\ln(2)^2} - \frac{1}{2\ln(a)^2}\right) \\
&= \frac{1}{2\ln(2)^2} - 0 = \frac{1}{2\ln(2)^2}
\end{aligned}$$

Since the improper integral converges, it follows from the integral test that the series $\sum \frac{1}{n\ln(n)^3}$ converges.

(b) (10 pts) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+2n+7}{2^n-1}$

Solution: Use the ratio test with $a_n = \frac{n^2+2n+7}{2^n-1}$.

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)^2+2(n+1)+7}{n^2+2n+7}\cdot\frac{2^n-1}{2^{n+1}-1} \\
&= \lim_{n\to\infty}\frac{([n+1]/n)^2+2([n+1]/n^2)+7/n^2}{1+2/n+7/n^2}\cdot\frac{1-1/2^n}{2-1/2^n} \\
&= \lim_{n\to\infty}\frac{(1+1/n)^2+2(1/n+1/n^2)+7/n^2}{1+2/n+7/n^2}\cdot\frac{1-1/2^n}{2-1/2^n} \\
&= \frac{1+0+0}{1+0+0}\cdot\frac{1-0}{2-0} \\
&= \frac{1}{2}
\end{aligned}$$

Since $\lim_{n\to\infty} a_{n+1}/a_n = 1/2 < 1$, it follows from the ratio test that the series $\displaystyle\sum_{n=2}^{\infty}\frac{n^2+2n+7}{2^n-1}$ converges.

(c) (10 pts) $\sum (1+2/n)^n$

Solution: Since $(1+2/n)^n > 1$ for all n, it follows that $\lim(1+2/n)^n$ can not be 0, and hence, the series diverges by the term test which is Corollary 14.5 in the text.