In your work on the following problems you may use the theorems about limits in section 9 of the text.

1. (17 pts) Find a function $f(\epsilon)$ defined for $\epsilon > 0$ with the property that

$$\left|\frac{3n+5}{7n-11} - \frac{3}{7}\right| < \epsilon \quad \text{for all } n \in \mathbb{N} \text{ with } n > f(\epsilon) \tag{1}$$

Solution: Let $f(\epsilon) = \max\left\{\frac{68}{42\epsilon}, 11\right\}$ for $\epsilon > 0$, and let n be an element in $\mathbb{N}$ with $n > f(\epsilon)$, then

$$\frac{68}{7\epsilon} < 6n < 7n - 11$$

where the first inequality follows by multiplying each term in the inequality $\frac{68}{42\epsilon} < n$ by 6 and second inequality follows since $n > f(\epsilon)$ with $n \in \mathbb{N}$ implies that $n \geq 12$. Note that

$$\frac{68}{7\epsilon} < 7n - 11 \quad \text{implies} \quad \frac{68}{7(7n-11)} < \epsilon$$

and hence,

$$\begin{aligned} \left|\frac{3n+5}{7n-11} - \frac{3}{7}\right| &= \left|\frac{21n + 35 - 21n + 33}{7(7n-11)}\right| \\ &= \left|\frac{68}{7(7n-11)}\right| < \epsilon \end{aligned}$$

and the proof of inequality (1) is complete.

2. (17 pts) Find $\lim \sqrt{4n^2 + 3n + 2} - 2n$

Solution:

$$\begin{aligned} \lim_{n\to\infty} \sqrt{4n^2+3n+2} - 2n &= \lim_{n\to\infty}(\sqrt{4n^2+3n+2} - 2n)\left(\frac{\sqrt{4n^2+3n+2}+2n}{\sqrt{4n^2+3n+2}+2n}\right) \\ &= \lim_{n\to\infty} \frac{4n^2+3n+2-4n^2}{\sqrt{4n^2+3n+2}+2n} = \lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2+3n+2}+2n} \\ &= \lim_{n\to\infty} \frac{3+2/n}{\sqrt{4+3/n+2/n^2}+2} \\ &= \frac{3+0}{\sqrt{4+0+0}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4} \end{aligned}$$

where the last line follows from the line just before the last line using that $\lim_{n\to\infty} 1/n = \lim_{n\to\infty} 1/n^2 = 0$ (Theorem 9.7 (a) with $p = 1$ and then with $p = 2$), Theorem 9.3, Theorem 9.6, and Example 5 on page 42 of the text.

3. (17 pts) The squeeze theorem states that if $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim a_n = \lim b_n = L$, then the sequence (x_n) converges to L. Use the $N - \epsilon$ definition of limit to prove the squeeze theorem.

Solution: Note that given a number $\epsilon > 0$ and any number A, the set of numbers r with $|r - A| < \epsilon$ is the open interval of numbers r with $A - \epsilon < r < A + \epsilon$. To refer to this property in the steps below we write

$$\{r\colon |r - A| < \epsilon\} = \{r\colon A - \epsilon < r < A + \epsilon\} \quad \text{for } \epsilon > 0 \tag{2}$$

The proof of the squeeze theorem proceeds as follows. Let $\epsilon > 0$ be given. Then since $\lim_{n\to\infty} b_n = L$ there is an $N_1 \in \mathbb{N}$ such that $|b_n - L| < \epsilon$ for $n \geq N_1$, and hence, by equation (2) we have

$$b_n < L + \epsilon \quad \text{for } n \geq N_1$$

Similarly, since since $\lim_{n\to\infty} a_n = L$ there is an $N_2 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for $n \geq N_2$, and hence, by equation (2) we have

$$L - \epsilon < a_n \quad \text{for } n \geq N_2$$

Then for $n \geq N = \max\{N_1, N_2\}$ using the inequalities above along with $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$ we have

$$L - \epsilon < a_n \leq x_n \leq b_n < L + \epsilon \quad \text{for all } n \geq N$$

In particular

$$L - \epsilon < x_n < L + \epsilon \quad \text{for all } n \geq N$$

Hence, by equation (2) we have

$$|L - x_n| < \epsilon \quad \text{for all } n \geq N$$

and the proof is complete.

4. (17 pts) Use the $N - \epsilon$ definition of limit to show that the sequence with terms $x_n = \cos\left(\frac{n\pi}{3}\right)$ does not converge.

Solution: First, some preliminary observations. Note that the sequence starts as

$$\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \ldots$$

Due to the fact that the cosine function has period 2π, the terms of the sequence repeat with a period of $(2\pi)/(\pi/3) = 6$, that is, $x_{n+6} = x_n$ for all n, taking values ± 1 and $\pm\frac{1}{2}$.

Now suppose the sequence converges, that is, $\lim_{n\to\infty} x_n = x$, for some $x \in \mathbb{R}$. This means: For every $\epsilon > 0$, there is an $N(\epsilon) \in \mathbb{R}$ such that $|x_n - x| < \epsilon$ for all $n > N(\epsilon)$. In particular, taking $\epsilon = \frac{1}{2}$: there is an $N = N(\frac{1}{2})$ such that $|x_n - x| < \frac{1}{2}$ for all $n > N$. Consider two cases (we could consider more, but that's all we need):

- $n = 6k + 1$, with $k \geq 0$. Then $x_n = \frac{1}{2}$, and so $\left|\frac{1}{2} - x\right| < \frac{1}{2}$ for $n > N$.
- $n = 6k + 2$, with $k \geq 0$. Then $x_n = -\frac{1}{2}$, and so $\left|\frac{1}{2} + x\right| = \left|-\frac{1}{2} - x\right| < \frac{1}{2}$ for $n > N$.

From the triangle inequality and the above, we find that

$$1 = \frac{1}{2} + \frac{1}{2} = \left|\frac{1}{2} - x + \frac{1}{2} + x\right| \leq \left|\frac{1}{2} - x\right| + \left|\frac{1}{2} + x\right| < \frac{1}{2} + \frac{1}{2} = 1$$

or, $1 < 1$, a contradiction. Therefore, the sequence (x_n) does not converge.

5. Let $x_1 = 2$ and $x_{n+1} = \dfrac{6x_n^2 + 1}{5}$ for $n \geq 1$.

   (a) (5 pts) Show that, if $a = \lim x_n$, then $a = \frac{1}{2}$ or $a = \frac{1}{3}$.

   Solution: If the sequence converges, we can use the properties of limits to infer that
$$\lim x_{n+1} = \lim \frac{6x_n^2 + 1}{5} = \frac{6(\lim x_n^2) + 1}{5} = \frac{6(\lim x_n)^2 + 1}{5}.$$
   Therefore, since $a = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$, we get
$$a = \frac{6a^2 + 1}{5}, \text{ or } 6a^2 - 5a + 1 = 0, \text{ or } (3a - 1)(2a - 1) = 0,$$
   and so $a = \frac{1}{2}$ or $a = \frac{1}{3}$.

   (b) (5 pts) Does $\lim x_n$ exist?

   Solution: No. A reason is that every convergent sequence is bounded, but this sequence is not bounded above. To show this, it is enough to show that $x_n > n$ for all n (since the sequence of natural numbers itself is not bounded above). We prove this by induction on n. Clearly, $x_1 = 2 > 1$. Assume $x_n > n$; then
$$x_{n+1} = \frac{6x_n^2 + 1}{5} > \frac{6n^2 + 1}{5} > n + 1,$$
   where the last inequality is equivalent to $6n^2 - 5n - 4 = (3n - 4)(2n + 1) > 0$, which holds for all $n \geq 2$. So we showed that $x_n > n$ implies $x_{n+1} > n + 1$. Therefore, by induction, $x_n > n$ for all n, and this completes the proof that the sequence (x_n) is not bounded, and thus does not have a limit.

   (c) (5 pts) Discuss the apparent contradiction between parts (a) and (b).

   Solution: There is no contradiction between the two answers. Indeed, the computations in part (a)—which used properties of limits—were predicated upon the premise that the sequence (x_n) is convergent. Yet, as we saw in part (b), the sequence is not even bounded, let alone convergent, so the premise of part (a) was false.

6. (17 pts) Let (x_n) be a convergent sequence. Suppose that $x_n \geq a$ for all but finitely many n. Show that $\lim x_n \geq a$.

   Solution: Set $x = \lim x_n$. Then, for every $\epsilon > 0$, there is an $N(\epsilon) \in \mathbb{R}$ such that $|x_n - x| < \epsilon$ for all $n > N(\epsilon)$.

   The assumption that $x_n \geq a$ for all but finitely many n may be rephrased as follows: There is an $N_0 \in \mathbb{N}$ such that $x_n \geq a$ for all $n \geq N_0$.

   For $\epsilon > 0$, let $N_1 = \max\{N(\epsilon), N_0\}$. Then, for all $n > N_1$, we have:
$$x - a = (x - x_n) + (x_n - a) \geq x - x_n > -\epsilon$$
   Since the inequality $x - a > -\epsilon$ holds for all $\epsilon > 0$, we must have $x - a \geq 0$, and we are done.