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MATH 4565 **TOPOLOGY** **Fall 2023**

MIDTERM EXAM

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1. Let $X = \{0, 1\}$ be the 2-element set equipped with the topology $\mathcal{T} = \{\varnothing, \{0\}, X\}$.

(a) Is every continuous map $X \to \mathbb{R}$ constant?

(b) Is every continuous map $\mathbb{R} \to X$ constant?

(c) Is X a Hausdorff space?

(d) Is X connected?

(e) Is X path-connected?

(f) Is X locally connected?

(g) Is X locally path-connected?

(h) Determine all self-homeomorphisms of X.

2. Let A be a subspace of a topological space X. A *retraction* of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. If such a map exists, we say that A is a *retract* of X.

(a) Prove the following: If X is Hausdorff and A is a retract of X, then A is closed.

(b) By the above, the open interval $(0, 1)$ is *not* a retract of the real line $\mathbb{R}$. Nevertheless, show that the closed interval $[0, 1]$ *is* a retract of $\mathbb{R}$.

3. Let $(X, \mathcal{T})$ be a topological space, let $\mathcal{B}$ be a basis for the topology $\mathcal{T}$, and let $\sim$ be an equivalence relation on X.

(a) Show that if the projection $q\colon X \to X/\!\sim$ is an open map, then $q(\mathcal{B}) := \{q(B) : B \in \mathcal{B}\}$ is a basis for the quotient topology of $X/\!\sim$.

(b) Show that in general (if the projection map q is not open), the conclusion of part (a) does not hold.

4. Let $q\colon X \to Y$ be a quotient map. Suppose Y is connected, and, for each $y \in Y$, the subspace $q^{-1}(\{y\})$ is connected. Show that X is also connected.

5. Let $q\colon X \to Y$ be a quotient map, let U be an open subset of X, and let $q|_U\colon U \to q(U)$ be the restriction of q to U (co-restricted to its image).

(a) Suppose U is a saturated open subset of X. Show that $q|_U\colon U \to q(U)$ is again a quotient map.

(b) Give an example showing that the saturation hypothesis is necessary.

6. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$, with the topology induced from the standard (Euclidean) topology on $\mathbb{R}^{n+1}$, for $n \geqslant 1$. Also let $\mathbb{RP}^n = S^n/\sim$ be the projective n-space, obtained as the quotient space of S^n by the equivalence relation $x \sim y$ if $y = x$ or $y = -x$.

(a) Show that S^n is path-connected, by constructing for any two points $x, y \in S^n$ an explicit path connecting them.

(b) Show that S^n is locally path-connected.

(c) Show that $\mathbb{RP}^n$ is path-connected and locally path-connected.

7. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ be the set of all points in the plane with at least one rational coordinate. Show that X, with the subspace topology, is a path-connected space.

8. Let X be a locally path-connected space. Let U be an open, connected subset of X. Show that U is path-connected.