**Definition 1.** A sequence $(s_n)$ converges to the number $S$ if given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_n - S| < \epsilon$ for all $n \geq N$.

**Definition 2.** A number $U$ is called the *least upper bound or* sup of a set $S$ if

(1) $s \leq U$ for all $s \in S$, and
(2) If $L$ is any number with $L < U$, then there is an element $s \in S$ with $L < s$.

**Completeness Axiom.** Every nonempty subset $S$ of the real numbers $\mathbb{R}$ that is bounded above has a least upper bound.

**Definition 3.** Given a sequence $(x_n)$ and $N \in \mathbb{N}$, let $v_N = \sup\{x_n : n \geq N\}$. Then $(v_N)$ is a decreasing sequence, and hence, has a limit. We define $\limsup x_n$ to be the limit of the sequence $(v_N)$.
Set $u_N = \inf\{x_n : n \geq N\}$. Then $(u_N)$ is an increasing sequence, and hence, has a limit. We define $\liminf x_n$ to be the limit of the sequence $(u_N)$. Moreover, $u_N \leq v_N$ for all $N \in \mathbb{N}$.

**Definition 4.** A sequence $(x_n)$ is called a *Cauchy sequence* if given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|s_m - s_n| < \epsilon$ for all $n \geq N$ and $m \geq N$.

**Theorem 1.** Every Cauchy sequence is bounded. A sequence of real numbers is a Cauchy sequence if and only if it is convergent.

**Theorem 2.** Every bounded monotone sequence of real numbers converges.

**Theorem 3.** Every subsequence $(x_{n_k})$ of a convergent sequence $(x_n)$ converges, and $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$.

**Theorem 4** (Bolzano–Weierstrass)**.** Every bounded sequence of real numbers has a convergent subsequence.

**Definition 5** ($\delta - \epsilon$ definition of continuity)**.** Let $f$ be a real-valued function whose domain, dom$(f)$, is a subset of $\mathbb{R}$. The function $f$ is *continuous* at a point $x_0 \in$ dom$(f)$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{for all } x \in \text{dom}(f) \text{ with } |x - x_0| < \delta.$$

**Theorem 5** (Extreme Value Theorem)**.** Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then $f$ is bounded and it assumes its maximum and minimum values on $[a, b]$.

**Theorem 6** (Intermediate Value Theorem)**.** Let $f\colon I \to \mathbb{R}$ be a continuous function defined on an interval $I$. Suppose $a, b$ are two numbers in $I$ such that $a < b$ and suppose $y$ is a real number that lies between $f(a)$ and $f(b)$. Then there is an $x \in (a, b)$ such $f(x) = y$.

**Definition 6** (Uniform continuity)**.** Let $f$ be a real-valued function whose domain, dom$(f)$, is a subset of $\mathbb{R}$. The function $f$ is *uniformly continuous* on dom$(f)$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in \text{dom}(f) \text{ with } |x - y| < \delta.$$

**Theorem 7.** If $f\colon S \to \mathbb{R}$ is uniformly continuous of a set $S \subset \mathbb{R}$, and if $(x_n)$ is a Cauchy sequence in $S$, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

**Theorem 8.** If $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then $f$ is uniformly continuous on $[a, b]$.