1. (14 pts) Use the definitions of $\limsup, \liminf$, and Cauchy sequence to show that if $\limsup x_n = \liminf x_n$, then (x_n) is a Cauchy sequence.

   Solution: From the definition of Cauchy sequence, it suffices to show that given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

   $$|x_n - x_m| < \epsilon \quad \text{for all } n \geq N \text{ and } m \geq N.$$

   Let $\epsilon > 0$ be given. Let $L = \limsup x_n = \liminf x_n$. From the definition of limit, it follows that there are natural numbers $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

   $$\begin{aligned} |v_n - L| < \epsilon/2 &\quad \text{for all } n \geq N_1, \text{ and} \\ |u_n - L| < \epsilon/2 &\quad \text{for all } n \geq N_2, \end{aligned}$$

   where (v_n) and (u_n) are the sequences given in Definition 3 of the Notes for the Midterm. Let $N_{\max} = \max(N_1, N_2)$. From the definition of sup and inf it follows that for any nonempty set S

   $$\inf S \leq x \leq \sup S \quad \text{for all } x \in S$$

   In particular,

   $$x_n \in [u_N, v_N] \subset [L - \epsilon/2, L + \epsilon/2] \quad \text{for all } n \geq N_{\max}$$

   Hence, for all $n \geq N_{\max}$ and $m \geq N_{\max}$, both x_n and x_m are in the interval $[L-\epsilon/2, L+\epsilon/2]$ so by the triangle inequality we have

   $$|x_n - x_m| = |x_n - L + L - x_m| \leq |x_n - L| + |L - x_m| < \epsilon/2 + \epsilon/2 = \epsilon,$$

   and the proof is complete.

2. (13 pts) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and assume that there is a number $a > 0$ with the property that $f(x) \geq a$ for all $x \in \mathbb{R}$. Use the δ–ϵ definition of continuity to show that $1/f(x)$ is continuous for all $x \in \mathbb{R}$.

   Solution: Let $x_0 \in \mathbb{R}$, it suffices to show that given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{1}{f(x_0)} - \frac{1}{f(x)}\right| < \epsilon \quad \text{for all } x \in \mathbb{R} \text{ with } |x - x_0| < \delta.$$

   Since $f(x) \geq a > 0$ for all x, we have $0 < 1/f(x) < 1/a$ for all x, and hence,

$$\left|\frac{1}{f(x_0)} - \frac{1}{f(x)}\right| = \left|\frac{f(x) - f(x_0)}{f(x_0)f(x)}\right| \leq \left|\frac{f(x) - f(x_0)}{a \cdot a}\right| = \frac{1}{a^2}\left|f(x) - f(x_0)\right|. \tag{1}$$

   Since $f(x)$ is continuous at x_0, there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < a^2\epsilon \quad \text{for all } x \text{ with } |x - x_0| < \delta. \tag{2}$$

   Combining equations (1) and (2) it follows that that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ we have

$$\left|\frac{1}{f(x_0)} - \frac{1}{f(x)}\right| \leq \frac{1}{a^2}\left|f(x) - f(x_0)\right| < \frac{1}{a^2}|f(x) - f(x_0)| < \frac{a^2\epsilon}{a^2} = \epsilon,$$

   and the proof is complete.

3. For each series below, determine whether the series converges or diverges. Be sure to name any tests that you use.

   (a) (10 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

   Solution: Use the ratio test with $a_n = n^3/3^n$.

$$\begin{aligned}
\lim \left|\frac{a_{n+1}}{a_n}\right| &= \lim \left(\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right) \\
&= \lim \left(\frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}}\right) \\
&= \lim \left(\left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}\right) \\
&= \lim \left(\left(1+\frac{1}{n}\right)^3 \cdot \frac{1}{3}\right) \\
&= 1^3 \cdot \frac{1}{3} = \frac{1}{3}.
\end{aligned}$$

   Since $\lim |a_{n+1}/a_n| = 1/3 < 1$, the series converges by the ratio test.

   (b) (10 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

   Solution: $\sum 1/n^2$ is a p-series with $p = 2 > 1$, so $\sum 1/n^2$ converges by the p-series test. Alternatively, $\sum 1/n^2$ converges by the integral test, since $\int_1^{\infty} 1/x^2\,dx$ converges.

   Since $0 < 1/(n^2+n) < 1/n^2$, it then follows from the comparison test that $\sum 1/(n^2+n)$ converges.

4. Let (s_n) be the sequence with $s_1 = 11$ and $s_{n+1} = \frac{3}{4}(s_n + 1)$.

   (a) (10 pts) Show that the sequence converges

   Solution: The first step is to show by induction that the sequence is decreasing. For $n \geq 1$ let $P(n)$ be the statement that $s_{n+1} < s_n$. Since $s_1 = 11$, we have that

$$s_2 = \frac{3}{4}(11 + 1) = \frac{3}{4} \cdot 12 = 9.$$

   Since $s_2 = 9 < 11 = s_1$, it follows that $P(1)$ is true.

   Now assume by induction that $P(n)$ is true, then

$$\begin{aligned} s_{n+1} &< s_n \quad \text{so} \\ s_{n+1} + 1 &< s_n + 1 \quad \text{hence} \\ \frac{3}{4}(s_{n+1} + 1) &< \frac{3}{4}(s_n + 1) \quad \text{that is} \\ s_{n+2} &< s_{n+1}. \end{aligned}$$

   Thus, by mathematical induction the statement $P(n)$ is true for all $n \in \mathbb{N}$. This completes the proof that the sequence (s_n) is decreasing.

   Note that if $s_n > 0$, then $(3/4)(s_n + 1)$ is also greater than 0. Since $s_1 = 11 > 0$, it follows by induction that $s_n > 0$ for all $n \in \mathbb{N}$.

   Thus, the sequence (s_n) is decreasing and bounded below by 0. It follows from results in the text that every decreasing sequence bounded below converges, and hence, the sequence (s_n) converges.

   (b) (10 pts) Find the limit of the sequence

   Solution: Let $L = \lim s_n$. From the results in the text that the limit of a sum is the sum of the limits and the limit of a product is the product of the limits, we have

$$\begin{aligned} \lim s_{n+1} &= \frac{3}{4}(\lim s_n + 1) \\ L &= \frac{3}{4}(L + 1) \\ L &= \frac{3L}{4} + \frac{3}{4} \\ L - \frac{3L}{4} &= \frac{3}{4} \\ \frac{L}{4} &= \frac{3}{4} \\ L &= 3. \end{aligned}$$

5. (a) (10 pts) Let $f\colon [1,4] \to \mathbb{R}$ be a continuous function such that $f(1) > 1$ and $f(4) < 2$. Show that there is a number $c \in [1,4]$ such that $f(c) = \sqrt{c}$.

Solution: First consider the function $h\colon [1,4] \to \mathbb{R}$ given by $h(x) = \sqrt{x}$; we claim that this function is continuous.

One way to prove this claim is by using known facts from Calculus: the function h is differentiable (with derivative $h'(x) = \frac{1}{2\sqrt{x}}$), and thus, h is continuous.

Another way to establish the claim is to use the ϵ–δ definition of continuity. Let $\epsilon > 0$ and let $x_0 \in [1,4]$. Choose $\delta = \sqrt{x_0}\epsilon$. Then $|x - x_0| < \delta$ implies

$$\begin{aligned} |\sqrt{x} - \sqrt{x_0}| &= \left| \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}} \right| \\ &= \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{\delta}{\sqrt{x} + \sqrt{x_0}} \\ &= \frac{\sqrt{x_0}\epsilon}{\sqrt{x} + \sqrt{x_0}} < \frac{\sqrt{x_0}\epsilon}{\sqrt{x_0}} = \epsilon. \end{aligned}$$

Now consider the function $g\colon [1,4] \to \mathbb{R}$ given by $g(x) = f(x) - \sqrt{x}$. Then g is the difference of two continuous functions, and thus g is also continuous (by properties of continuous functions from the text). Moreover, note that

$$\begin{aligned} g(1) &= f(1) - \sqrt{1} = f(1) - 1 > 1 - 1 = 0, \\ g(4) &= f(4) - \sqrt{4} = f(4) - 2 < 2 - 2 = 0. \end{aligned}$$

Since g is continuous and $g(1) > 0 > g(4)$, the Intermediate Value Theorem guarantees there is a number $c \in [1,4]$ such that $g(c) = 0$, that is, $f(c) - \sqrt{c} = 0$, or $f(c) = \sqrt{c}$.

(b) (10 pts) Give an example of a (discontinuous) function $f\colon [1,4] \to \mathbb{R}$ such that $f(1) > 1$ and $f(4) < 2$ for which the equation $f(c) = \sqrt{c}$ has no solution $c \in [1,4]$.

Solution: Consider the function $f\colon [1,4] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } 1 \le x < 2 \\ 0 & \text{if } 2 \le x \le 4. \end{cases}$$

Clearly, the function f satisfies the hypothesis of the problem; indeed, $f(1) = 2 > 1$ and $f(4) = 0 < 2$. Note also that f is not continuous at $x = 2$, since $\lim_{x \to 2^-} f(x) = 2$, while $\lim_{x \to 2^+} f(x) = 0$. Finally, note that

$$\begin{aligned} \sqrt{x} < 2 = f(x) &\quad \text{if } 1 \le x < 2, \\ 0 = f(x) < \sqrt{x} &\quad \text{if } 2 \le x \le 4. \end{aligned}$$

Therefore, $f(x) \neq \sqrt{x}$ for $1 \le x \le 4$; that is, the equation $f(c) = \sqrt{c}$ has no solution $c \in [1,4]$.

6. (13 pts) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function which is *not* uniformly continuous on $\mathbb{R}$. Show that there is a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) such that

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Solution: By definition, a function $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$ if the following condition is satisfied: for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x)-f(y)| < \epsilon$ for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$.

Thus, a function $f\colon \mathbb{R} \to \mathbb{R}$ is *not* uniformly continuous on $\mathbb{R}$ if the following condition is satisfied: there exists $\epsilon_0 > 0$ such that

$$\text{for all } \delta > 0, \text{ there exist } x, y \in \mathbb{R} \text{ with } |x - y| < \delta \text{ but } |f(x) - f(y)| \geq \epsilon_0.$$

For each $n \in \mathbb{N}$, let us take $\delta = 1/n$ in the above display, and let $x_n, y_n \in \mathbb{R}$ the corresponding numbers, such that $|x_n - y_n| < 1/n$, yet $|f(x_n) - f(y_n)| \geq \epsilon_0$.

Since $0 < |x_n - y_n| < 1/n$ and the sequence $(1/n)$ converges to 0, the sequence $(|x_n - y_n|)$ must also converge to 0, by the Squeeze Theorem for sequences. In other words, $|x_n - y_n| \to 0$.

To recap, we have found sequences (x_n) and (y_n) such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$, and so we are done.