1. Given a number x_n with $x_n \geq -13$ set $x_{n+1} = \sqrt{x_n + 13}$.

(a) (10 pts) Show by mathematical induction that if $x_1 = -13$, then $x_{n+1} > x_n$ for all integers $n \geq 1$.

Solution: For $n \in \mathbb{N}$, let $P(n)$ be the statement that $x_{n+1} > x_n$. By the Principle of Mathematical Induction, it follows that to show $P(n)$ is true for all $n \in \mathbb{N}$ it suffices to show that

(1) $P(1)$ is true and

(2) If $P(n)$ is true, then $P(n + 1)$ is true.

To prove item (1), note that since $x_1 = -13$, we have that

$$x_2 = \sqrt{x_1 + 13} = \sqrt{-13 + 13} = \sqrt{0} = 0$$

Thus, $x_2 = 0 > -13 = x_1$, and hence, $P(1)$ is true. The next step is to prove item (2). Assume $P(k)$ is true for some $k \in \mathbb{N}$, then

$$\begin{aligned} x_{k+1} &< x_k \\ x_{k+1} + 13 &< x_k + 13 \\ \sqrt{x_{k+1} + 13} &< \sqrt{x_k + 13} \\ x_{k+2} &< x_{k+1} \end{aligned}$$

From the last line, it follows that $P(k + 1)$ is true and the proof of item (2) is complete. The result that $P(n)$ is true for all $n \in \mathbb{N}$ now follows by mathematical induction.

(b) (10 pts) Show by mathematical induction that if $x_1 = 12$, then $x_{n+1} < x_n$ for all integers $n \geq 1$.

Solution: For $n \in \mathbb{N}$, let $S(n)$ be the statement that $x_{n+1} < x_n$. By the Principle of Mathematical Induction, it follows that to show $S(n)$ is true for all $n \in \mathbb{N}$ it suffices to show that

(3) $S(1)$ is true and

(4) If $S(n)$ is true, then $S(n + 1)$ is true.

To prove item (3), note that since $x_1 = 12$, we have that

$$x_2 = \sqrt{12 + 13} = \sqrt{25} = 5$$

Thus, $x_2 = 5 < 12 = x_1$, and hence, $S(1)$ is true. The next step is to prove item (4). Assume $S(k)$ is true for some $k \in \mathbb{N}$, then

$$\begin{aligned} x_{k+1} &> x_k \\ x_{k+1} + 13 &> x_k + 13 \\ \sqrt{x_{k+1} + 13} &> \sqrt{x_k + 13} \\ x_{k+2} &> x_{k+1} \end{aligned}$$

From the last line, it follows that $S(k+1)$ is true and the proof of item (4) is complete. The result that $S(n)$ is true for all $n \in \mathbb{N}$ now follows by mathematical induction.

2. (20 pts) Use the Rational Zeros Theorem to find all rational solutions, if any, to the equation

$$p(x) = 3x^4 + x^3 + 4x^2 + 2x - 4 = 0.$$

Explain your reasoning.

Solution: From the Rational Zeros Theorem it follows that if x is a rational number $x = p/q$ with $p(x) = 0$, then p divides 4 and q divides 3. Thus $p = \pm 1, \pm 2, \pm 4$ and $q = \pm 1, \pm 3$. The table below lists the possible values for x and the corresponding value of $p(x)$ rounded to 2 decimal places.

x	$p(x)$	x	$p(x)$
1	6	-1	0
$1/3$	-2.82	$-1/3$	-4.22
2	72	-2	48
$2/3$	0	$-2/3$	-3.26
4	900	-4	756
$4/3$	17.63	$-4/3$	7.56

It follows that $x = 2/3$ and $x = -1$ are the only rational numbers x for which $p(x) = 0$.

3. Determine whether the following numbers are rational or irrational. In each case, explain your reasoning.

(a) (10 pts) $\sqrt{6 + \sqrt{5}}$.

Solution: Write $r = \sqrt{6 + \sqrt{5}}$. Squaring both sides, we obtain:

$$\begin{aligned} r^2 &= 6 + \sqrt{5} \\ r^2 - 6 &= \sqrt{5} \\ (r^2 - 6)^2 &= 5 \\ (r^2)^2 - 2 \cdot 6 \cdot r^2 + 6^2 &= 5 \\ r^4 - 12r^2 + 31 &= 0. \end{aligned}$$

This computation shows that r is a root of the polynomial $P(x) = x^4 - 12x + 31$.

Now suppose r is rational, and write $r = p/q$, with $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$. Then, by the Rational Zeroes Theorem, p divides 31 and q divides 1. It follows that $p \in \{\pm 1, \pm 31\}$ and $q = \pm 1$; therefore, $r = \pm 1$ or $r = \pm 31$. But

$$\begin{aligned} (\pm 1)^4 - 12 \cdot (\pm 1)^2 + 31 &= 1 - 12 + 31 = 20 \neq 0 \\ (\pm 31)^4 - 12 \cdot (\pm 31)^2 + 31 &= 923,521 - 11,532 + 31 = 912,020 \neq 0, \end{aligned}$$

showing that none of these numbers is a root of the polynomial $P(x)$. This contradiction invalidates our assumption that the number $r = \sqrt{6 + \sqrt{5}}$ is rational. Therefore, we have shown that r is irrational.

(b) (10 pts) $\sqrt{6 + 2\sqrt{5}} - \sqrt{5}$.

Solution: First note that

$$(1 + \sqrt{5})^2 = 1^2 + 2\sqrt{5} + (\sqrt{5})^2 = 1 + 2\sqrt{5} + 5 = 6 + 2\sqrt{5}.$$

Therefore,

$$\sqrt{6 + \sqrt{5}} - \sqrt{5} = \sqrt{(1 + \sqrt{5})^2} - \sqrt{5} = 1 + \sqrt{5} - \sqrt{5} = 1,$$

and this is a rational number (in fact, a natural number).

4. (20 pts) Let x be a real number. Show that if $|x - 1| < 1$, then $\left|x^2 - 1\right| < 3$.

Solution: We have

$$\left|x^2 - 1\right| = |(x-1)(x+1)| = |x-1| \cdot |x+1|$$

By assumption, $|x - 1| < 1$. This inequality is equivalent to $-1 < x - 1 < 1$, or $0 < x < 2$, or $1 < x + 1 < 3$. Since $-3 < 1$, we get $-3 < x + 1 < 3$, which is equivalent to

$$|x + 1| < 3.$$

Putting things together, we find:

$$\begin{aligned} \left|x^2 - 1\right| &= |x-1| \cdot |x+1| \\ &< 1 \cdot 3 \\ &= 3, \end{aligned}$$

which shows that $\left|x^2 - 1\right| < 3$.

5. (20 pts) Given nonempty subsets A and B of $\mathbb{R}$, with $A \cap B$ not equal to the empty set, prove directly from the definition of inf and sup that

$$\inf A \leq \inf(A \cap B) \leq \sup A$$

Solution: Recall the following: A number ℓ is the *greatest lower bound of a set* S if $\ell \leq s$ for all $s \in S$ and if $b > \ell$, then b is not a lower bound for S; that is, given $b > \ell$ there is an element $s \in S$ with $s < b$

The inf of a set is the greatest lower bound of the set if the set is bounded below and $-\infty$ if the set is not bounded below.

Similarly, a number u is the *least upper bound of a set* S if $u \geq s$ for all $s \in S$ and if $b < u$, then b is not an upper bound for S. The sup of a set is the least upper bound of the set if the set is bounded above and $+\infty$ if the set is not bounded above.

As a first step to proving

$$\inf A \leq \inf(A \cap B) \leq \sup A$$

consider the case where A is a bounded set. Let L be a lower bound for A, and let U be an upper bound for A. Then

(1) $$L \leq a \leq U \quad \text{for all } a \in A$$

Since every element in $A \cap B$ is an element in A, it follows from equation (1) that

$$L \leq c \leq U \quad \text{for all } c \in A \cap B \tag{2}$$

Note that by definition $\inf A$ is a lower bound for A and $\sup A$ is an upper bound for A. Hence the equation

$$\inf A \leq c \leq \sup A \quad \text{for all } c \in A \cap B \tag{3}$$

is the special case of equation (2) with $L = \inf A$ and $U = \sup A$.

The next step is to show that $\inf A \leq \inf(A \cap B)$. The proof is by contradiction. Suppose that $\inf A > \inf(A \cap B)$. Since $\inf(A \cap B) < \inf A$, it follows from the definition $\inf(A \cap B)$ that there is an element $c \in A \cap B$ with $c < \inf A$. Note c is an element in A so we have an element $c \in A$ with $c < \inf A$. This contradicts the property that the $\inf A$ is a lower bound for A. So the proof that $\inf A \leq \inf(A \cap B)$ is complete.

The proof that $\inf(A \cap B) \leq \sup A$ also follows by contradiction. Suppose that $\inf(A \cap B) > \sup A$. Let $e \in A \cap B$, then from the definition of inf, it follows that $e \geq \inf(A \cap B)$. Since e is an element in A,we have $e \geq \inf(A \cap B) > \sup A$. This contradicts the property that $\sup A$ is an upper bound for A.

This completes the proof that

$$\inf A \leq \inf(A \cap B) \leq \sup A$$

in the case where A is bounded. Note that the proof that $\inf A \leq \inf(A \cap B)$ used the assumption that A is bounded below but did not use the assumption that A is bounded above. Similarly, the proof that $\inf(A \cap B) \leq \sup A$ used the assumption that A is bounded above but did not use the assumption that is A is bounded below.

If A is not bounded below and is bounded above, then $\inf A \leq \inf(A \cap B)$ follows since $\inf A = -\infty$, and the inequality $\inf(A \cap B) \leq \sup A$ follows from the argument above.

If A is not bounded above and is bounded below, then $\inf(A \cap B) \leq \sup A$ follows since $\sup A = +\infty$, and the inequality $\inf A \leq \inf(A \cap B)$ follows from the argument in the case where A is bounded.

If A is not bounded below and not bounded above, then the inequality $\inf A \leq \inf(A \cap B) \leq \sup A$ follows since $\inf A = -\infty$ and $\sup A = +\infty$.

This completes the proof that $\inf A \leq \inf(A \cap B) \leq \sup A$.