Resonance schemes, Koszul modules, and Hilbert series

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Resonance varieties

- Let $A^\bullet$ be a graded, graded-commutative, algebra (cga) over a field $\Bbbk$ of characteristic 0, with multiplication maps $A^i \otimes_{\Bbbk} A^j \to A^{i+j}$.
- We assume A is connected ($A^0 = \Bbbk$) and of finite-type ($\dim_{\Bbbk} A^i < \infty$).
- For each $a \in A^1$, graded commutativity gives $a^2 = -a^2$, and so $a^2 = 0$.
- We then have a cochain complex,

$$(A^\bullet, \delta_a)\colon \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u$, for all $u \in A^i$.
- The *resonance varieties* of A are the homogeneous sets

$$\mathcal{R}^i(A) = \{a \in A^1 \mid H^i(A^\bullet, \delta_a) \neq 0\}.$$

- $\mathcal{R}^0(A) = \{0\}$
- $\mathcal{R}^1(A) = \{a \in A^1 \mid \exists\, b \in A^1 \text{ s.t. } a \wedge b \in K \setminus \{0\}\} \cup \{0\}$, where $K = \ker(A^1 \wedge A^1 \to A^2)$.

The BGG correspondence

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^\vee$, and identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.

- The BGG correspondence yields a cochain complex of finitely generated, free S-modules,

$$(A^\bullet \otimes_\Bbbk S, \delta_A) \colon \ \cdots \longrightarrow A^i \otimes_\Bbbk S \xrightarrow{\delta_A^i} A^{i+1} \otimes_\Bbbk S \xrightarrow{\delta_A^{i+1}} A^{i+2} \otimes_\Bbbk S \longrightarrow \cdots,$$

where $\delta_A^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j$.

- The specialization of $(A \otimes_\Bbbk S, \delta_A)$ at $a \in A^1$ coincides with (A, δ_a).

- $a \in A^1$ belongs to $\mathcal{R}^i(A)$ iff $\operatorname{rank} \delta_a^{i-1} + \operatorname{rank} \delta_a^i < b_i(A)$. Hence,

$$\mathcal{R}^i(A) = V\Big(I_{b_i(A)}\big(\delta_A^{i-1} \oplus \delta_A^i\big)\Big),$$

where $I_r(\psi)$ is the ideal of $r \times r$ minors of a matrix ψ.

Examples

- If $E = \bigwedge \mathbb{k}^n$, then $\mathrm{L}(E)$ is the usual Koszul complex. E.g., for $n = 3$:

$$\mathrm{L}(E):\quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}} S\ .$$

  Hence, $\mathcal{R}^i(E) = \{0\}$ for $0 \leqslant i \leqslant n$ and empty otherwise.

- If $A = \bigwedge(e_1, e_2, e_3)/\langle e_1 e_2 \rangle$, then

$$\mathrm{L}(A):\quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_3 & 0 \\ 0 & x_3 \\ -x_1 & -x_2 \end{pmatrix}} S^2\ .$$

  Hence, $\mathcal{R}^1(A) = \{x_3 = 0\}$.

- If $A = \bigwedge(e_1, \dots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4 \rangle$, then

$$\mathrm{L}(A):\quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x_4 & 0 & -x_2 \\ 0 & x_3 & x_1 \\ 0 & -x_2 & x_4 \\ -x_1 & 0 & -x_3 \end{pmatrix}} S^3\ .$$

  Hence, $\mathcal{R}^1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$.

Koszul modules

- Set $A_i = (A^i)^\vee$ and $\partial_i = (\delta^{i-1})^\vee$ and consider the chain complex

$$(A_\bullet \otimes_{\Bbbk} S, \partial)\colon \cdots \longrightarrow A_{i+1} \otimes_{\Bbbk} S \xrightarrow{\partial^A_{i+1}} A_i \otimes_{\Bbbk} S \xrightarrow{\partial^A_i} A_{i-1} \otimes_{\Bbbk} S \longrightarrow \cdots .$$

- The *Koszul modules* of A are the graded S-modules

$$W_i(A) := H_i(A_\bullet \otimes_{\Bbbk} S).$$

- Setting $E_\bullet = \bigwedge A_1$, the first one has presentation

$$\big(E_3 \oplus K^\perp\big) \otimes_{\Bbbk} S \xrightarrow{\partial^E_3 + \iota \otimes \mathrm{id}} E_2 \otimes_{\Bbbk} S \twoheadrightarrow W_1(A), \qquad (*)$$

where $K^\perp = \{\varphi \in A_1 \wedge A_1 = (A^1 \wedge A^1)^\vee \mid \varphi|_K \equiv 0\} \overset{\iota}{\hookrightarrow} A_1 \wedge A_1 = E_2$.

- The *resonance schemes* of A are defined as

$$\boldsymbol{\mathcal{R}}_i(A) = \operatorname{Spec}(S/\operatorname{ann} W_i(A)).$$

- The underlying sets, $\mathrm{R}_i(A) = \operatorname{supp} W_i(A)$, are related to the resonance varieties by $\bigcup_{i \leqslant q} \mathrm{R}_i(A) = \bigcup_{i \leqslant q} \mathcal{R}^i(A)$. In particular, $\mathrm{R}_1(A) = \mathcal{R}^1(A)$.

- Recall $K = \ker(A^1 \wedge A^1 \to A^2)$.
- Let $L \subseteq A^1$ be a linear subspace. We say:
  - L is *isotropic* if $L \wedge L \subseteq K$.
  - L is *separable* if $K \cap \langle L \rangle_E \subseteq L \wedge L$, where $E = \bigwedge A^1$ and $\langle L \rangle_E$ is the ideal of E generated by L.

Example

- If $K = 0$, then every subspace $L \subseteq A^1$ is separable
- If $K = A^1 \wedge A^1$, then every subspace $L \subseteq A^1$ is isotropic, but the only separable subspace is the trivial one.

Example

Let $A = E/(K)$, where $E = \bigwedge(e_1, \dots, e_4)$ and

$$K = \langle e_1 \wedge e_2, e_1 \wedge e_3 + e_2 \wedge e_4 \rangle.$$

Then $\mathcal{R}^1(A) = \langle e_1, e_2 \rangle$ is isotropic but not separable.

Reduced resonance schemes

- Let $\mathcal{R}^1(A) = L_1 \cup \cdots \cup L_s$ be the decomposition of $\mathcal{R}^1(A) \subset A^1$ into irreducible components.
- Letting $K_j = K \cap (L_j \wedge L_j)$, we define S-modules $W_1^j(A)$ as in (??).
- Assume each component of $\mathcal{R}^1(A)$ is a linear subspace of A^1.

Theorem (AFRS)

(1) If each L_j is separable, then the projectivized resonance scheme is reduced and its components are disjoint.

(2) If the projectivized resonance scheme is reduced and each L_j are isotropic, then all its components are separable and disjoint.

(3) If each L_j is separable, then $\dim[W_1(A)]_q = \sum_{j=1}^{s} \dim[W_1^j(A)]_q$.

(4) If each L_j is separable and isotropic, then

$$\dim[W_1(A)]_q = \sum_{j=1}^{s} (q+1)\binom{q+\dim L_j}{q+2}.$$

Resonance varieties of spaces and groups

- The resonance varieties of a connected, finite-type CW-complex X are those of its cohomology algebra: $\mathcal{R}^i(X) := \mathcal{R}^i(H^\bullet(X, \Bbbk))$.
- $\mathcal{R}^1(X)$ depends only on $G = \pi_1(X)$.
- The geometry of these varieties provides obstructions to the formality of X (or 1-formality of G).
- They allow to distinguish between various classes of groups, such as
  - Kähler groups
  - Quasi-projective groups
  - Arrangement groups
  - 3-manifold groups
  - Right-angled Artin groups
- Through their connections with other types of cohomology jump loci (characteristic varieties, Bieri–Neumann–Strebel–Renz invariants), they also inform on the homological and geometric finiteness properties of spaces and groups.

Hyperplane arrangements

- Let $\mathcal{A}$ be a complex hyperplane arrangement, with complement $M(\mathcal{A})$. Then $A = H^\bullet(M, \Bbbk)$ is the Orlik–Solomon algebra of $\mathcal{A}$.
- The components of the varieties $\mathcal{R}^i(M(\mathcal{A}))$ are linear subspaces of $A^1 = \Bbbk^{|\mathcal{A}|}$, which depend solely on the intersection lattice $L(\mathcal{A})$.
- The components $L_1, \dots, L_s$ of $\mathcal{R}^1(M(\mathcal{A}))$ admit an explicit combinatorial description, in terms of "multinets" on $L(\mathcal{A})$.
- Moreover, each linear subspace $L_j \subset A^1$ is *isotropic* (i.e., $ab = 0 \in A^2$, for every $a, b \in L_j$), and $L_i \cap L_j = \{0\}$ for $i \neq j$.
- It is not known whether the above properties hold for the resonance varieties of the OS-algebra of a non-realizable matroid.

Lower central series

Let G be a finitely-generated group. Define:

- *LCS series*: $G = G_1 \rhd G_2 \rhd \cdots \rhd G_k \rhd \cdots$, where $G_{k+1} = [G_k, G]$.
- *LCS quotients*: $\operatorname{gr}_k(G) = G_k/G_{k+1}$ (f.g. abelian groups).
- *LCS ranks*: $\phi_k(G) = \operatorname{rank} \operatorname{gr}_k(G)$.
- *Associated graded Lie algebra*: $\operatorname{gr}(G) = \bigoplus_{k\geqslant 1} \operatorname{gr}_k(G)$, with Lie bracket $[\ ,\ ]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by group commutator.
- *Chen Lie algebra*: $\operatorname{gr}(G/G'')$, where $G' = [G, G]$, $G'' = [G', G']$.
- *Chen ranks*: $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$ ($\theta_k \leqslant \phi_k$, equality for $k \leqslant 3$).

Resonance and Chen ranks

Example (Witt, Magnus, Chen)

Let $G = F_n$ (free group of rank n). Then:

- $\mathrm{gr}(F_n) = \mathrm{Lie}_n$ (free Lie algebra of rank n) is torsion free, with LCS ranks $\phi_k(F_n) = \frac{1}{k}\sum_{d|k}\mu(d)n^{k/d}$, where μ is Möbius function.
- $\mathrm{gr}(F_n/F_n'')$ is torsion-free, $\theta_1 = n$ and $\theta_k = (k-1)\binom{n+k-2}{k}$ for $k \geqslant 2$.

Theorem (Papadima–S. 2004)

If G is 1-formal, then $\theta_k(G) = \dim_{\Bbbk}[W_1(G)]_{k-2}$.

Theorem (Cohen–Schenck 2015, AFRS)

Let G be a 1-formal group, and assume $\mathcal{R}^1(G)$ has linear components $L_1, \dots, L_s$ which are separable and isotropic. Then, for all $k \gg 0$,

$$\theta_k(G) = \sum_{j=1}^{s}(k-1)\binom{k+\dim L_j - 2}{k}.$$

Toric complexes

- Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex on vertex set $[n] = \{1, \dots, n\}$.
- Let T_Δ be the subcomplex of the n-torus T^n obtained by deleting the cells corresponding to the missing simplices of Δ.
- T_Δ is a connected, formal CW-complex of dimension $\dim(\Delta) + 1$.
- (Kim–Roush 1980, Charney–Davis 1995) The cohomology algebra $H^\bullet(T_\Delta; \Bbbk)$ is the exterior Stanley–Reisner ring

$$\Bbbk\langle\Delta\rangle = \bigwedge V^\vee / (e_\sigma \mid \sigma \notin \Delta),$$

where

- $V = \Bbbk^n$, with basis $v_1, \dots, v_n$.
- $V^\vee = \mathrm{Hom}_\Bbbk(V, \Bbbk)$, with dual basis $e_1, \dots, e_n$.
- $e_\sigma = e_{i_1} \wedge \cdots \wedge e_{i_s}$ for $\sigma = \{i_1, \dots, i_s\} \subseteq [n]$.

Resonance of simplicial complexes

- (Papadima–S. 2006/2009) The resonance varieties

$$\mathcal{R}^i(\Delta) := \mathcal{R}^i(T_\Delta) = \mathcal{R}^i(\Bbbk\langle\Delta\rangle)$$

are finite unions of coordinate subspaces of $V^\vee$:

$$\mathcal{R}^i(\Delta) = \bigcup_{\substack{\mathsf{W}\subseteq[n] \\ \exists\sigma\in\Delta_{[n]\setminus\mathsf{W}},\ \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{\Delta_\mathsf{W}}(\sigma),\Bbbk)\neq 0}} \Bbbk^\mathsf{W},$$

where
  - Δ_W is the induced simplicial subcomplex on vertex set $\mathsf{W}\subseteq[n]$.
  - $\mathrm{lk}_{\Delta_\mathsf{W}}(\sigma)$ is the link of a simplex $\sigma\subset\Delta$ in Δ_W.
  - $\Bbbk^\mathsf{W}$ is the coordinate subspace of $\Bbbk^n$ spanned by $\{\mathrm{e}_i \mid i\in\mathsf{W}\}$.

- (Denham–S.–Yuzvinsky 2017) Suppose Δ is Cohen–Macaulay over $\Bbbk$ ($\widetilde{H}^\bullet(\mathrm{lk}(\sigma),\Bbbk)$ is concentrated in degree $\dim\Delta-|\sigma|$, for all $\sigma\in\Delta$). Then resonance propagates: $\mathcal{R}^1(\Delta)\subseteq\mathcal{R}^2(\Delta)\subseteq\cdots\subseteq\mathcal{R}^{\dim\Delta+1}(\Delta)$.

Resonance of graphs

- If Γ is a (simple) graph on n vertices, then:

$$\mathcal{R}^1(\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq[n] \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \Bbbk^{\mathsf{W}}.$$

- The irreducible components of $\mathcal{R}^1(\Gamma)$ are the coordinate subspaces $\Bbbk^{\mathsf{W}}$, maximal among those for which Γ_{W} is disconnected.
- The codimension of $\mathcal{R}^1(\Gamma)$ equals the connectivity of Γ. In particular, if Γ is disconnected, then $\mathcal{R}^1(\Gamma) = \Bbbk^n$.

Proposition (AFRS)

Let Γ be a connected graph, let Γ' be a maximally disconnected full subgraph, and let L' be the corresponding component of $\mathcal{R}^1(\Gamma)$. Then:

- *L' is isotropic if and only if Γ' is discrete.*
- *L' is separable if and only if $\Gamma = \Gamma' * \Gamma''$.*

Hence, isotropic implies separable for the resonance varieties of graphs.

Square-free modules

- Consider the standard $\mathbb{N}^n$-multigrading on $S = \Bbbk[x_1, \dots, x_n]$, defined by $\deg(x_i) = \mathrm{e}_i \in \mathbb{N}^n$, where $\mathrm{e}_i = (0, \dots, 1, \dots, 0)$.
- For $\mathrm{a} = (a_1, \dots, a_n) \in \mathbb{N}$, set $\operatorname{Supp}(\mathrm{a}) := \{i \mid a_i > 0\}$.

Definition (Yanagawa 2000)

An $\mathbb{N}^n$-graded S-module M is called *square-free* if for any $\mathrm{a} \in \mathbb{N}^n$ and any $i \in \operatorname{Supp}(\mathrm{a})$, the multiplication map $x_i \colon M_{\mathrm{a}} \to M_{\mathrm{a}+\mathrm{e}_i}$ is an isomorphism.

- An ideal $I \subseteq S$ is a square-free module $\iff$ I is a square-free monomial ideal $\iff$ S/I is a square-free module.
- A free $\mathbb{N}^n$-graded S-module is square-free if and only it is generated in square-free multidegrees.

Proposition

If $f\colon M \to N$ is a morphism of $\mathbb{N}^n$-graded S-modules, and M and N are square-free modules, then $\ker(f)$ and $\operatorname{coker}(f)$ are also square-free. Moreover, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $\mathbb{N}^n$-graded S-modules, and M' and M'' are square-free, then so is M.

Corollary

Let M be an $\mathbb{N}^n$-graded square-free S-module. Then all the modules in the minimal free $\mathbb{N}^n$-graded resolution of M are square-free.

Corollary

If F is a bounded complex of free square-free S-modules, then the homology modules of F are also square-free.

Theorem (AFRSS)

If M is an $\mathbb{N}^n$-graded, square-free S-module, then its annihilator is a square-free monomial ideal. In particular, $\operatorname{ann} M$ is a radical ideal.

Koszul complexes and reduced resonance

- Fix a basis $v_1, \dots, v_n$ for V, and let $K_\bullet = (\bigwedge V) \otimes_{\Bbbk} S$ be the Koszul complex of $x_1, \dots, x_n$, whose i-th free S-module is $K_i = \bigwedge^i V \otimes_{\Bbbk} S$.
- Set $\deg(v_i) = e_i \in \mathbb{N}^n$. Then $K_\bullet$ is a complex of $\mathbb{N}^n$-graded square-free S-modules.
- For a simplicial complex Δ on vertex set $[n]$ we have $\Bbbk\langle\Delta\rangle \otimes_{\Bbbk} S = K_\bullet^\Delta$, where $K_\bullet^\Delta$ is the subcomplex of $K_\bullet$ whose i-th module K_i^Δ is the free S-module generated by $\{v_\sigma \mid \sigma \in \Delta\}$.

Proposition

For each $i > 0$, the Koszul module $W_i(\Delta) = H_i(K_\bullet^\Delta)$ is an $\mathbb{N}^n$-graded, square-free S-module.

- By definition, the i-th resonance scheme $\mathcal{R}^i(\Delta)$ is the affine subscheme of $V^\vee$ defined by the annihilator of $W_i(\Delta)$.

Corollary

The resonance schemes $\mathcal{R}^i(\Delta)$ are reduced.

- As an application, we obtain upper bounds on the Castelnuovo–Mumford regularity and the projective dimension of the Koszul modules of any simplicial complex Δ on n vertices.

Proposition

$W_i(\Delta)$ has regularity at most n and projective dimension at most $n-i-1$. Moreover, if Δ is a graph and $n \geqslant 4$, then $\operatorname{reg} W_1(\Delta) \leqslant n-4$.

- We also compute the Hilbert series of the Koszul modules $W_i(\Delta)$.

Theorem

$$\sum_{a\in\mathbb{N}} \dim_{\Bbbk}[W_i(\Delta)]_a\, t^a = \sum_{\substack{\mathrm{b}\in\mathbb{N}^n \\ \mathrm{b}\ \text{square-free}}} \dim_{\Bbbk} \widetilde{H}_{i-1}(\Delta_{\mathrm{b}};\Bbbk) \left(\frac{t}{1-t}\right)^{|\mathrm{b}|},$$

where $\Delta_{\mathrm{b}} = \Delta_{\mathrm{Supp(b)}}$ and $|\mathrm{b}| = b_1 + \cdots + b_n$.

Right angled Artin groups

- The fundamental group $G_\Gamma = \pi_1(T_\Delta)$ is the RAAG associated to the graph $\Gamma = \Delta^{(1)} = (\mathsf{V}, \mathsf{E})$,
$$G_\Gamma = \langle v \in V \mid [v,w] = 1 \text{ if } \{v,w\} \in E \rangle.$$
- Moreover, $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.
- (Kim–Makar-Limanov–Neggers–Roush 1980, Droms 1987)
$$\Gamma \cong \Gamma' \Longleftrightarrow G_\Gamma \cong G_{\Gamma'}.$$
- (Papadima–S. 2006) The associated graded Lie algebra $\operatorname{gr}(G_\Gamma)$ has (quadratic) presentation
$$\operatorname{gr}(G_\Gamma) = \operatorname{Lie}(\mathsf{V})/([v,w] = 0 \text{ if } \{v,w\} \in \mathsf{E}).$$
- (Duchamp–Krob 1992, PS06) The lower central series quotients of G_Γ are torsion-free, with ranks ϕ_k given by
$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t),$$
where $P_\Gamma(t) = \sum_{k \geqslant 0} f_k(\Delta_\Gamma) t^k$ is the clique polynomial of Γ.

Chen ranks

- (PS 06) $\mathrm{gr}(G_\Gamma/G''_\Gamma)$ is torsion-free, with ranks given by $\theta_1 = |\mathsf{V}|$ and

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right).$$

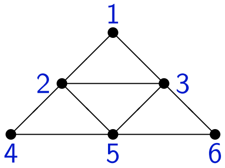
- Here $Q_\Gamma(t) = \sum_{j\geqslant 2} c_j(\Gamma)t^j$ is the "cut polynomial" of Γ, with

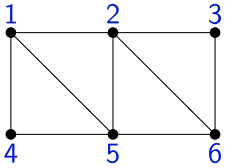
$$c_j(\Gamma) = \sum_{\mathsf{W}\subset\mathsf{V}\colon\, |\mathsf{W}|=j} \tilde{b}_0(\Gamma_\mathsf{W}).$$

Example

Let Γ be a pentagon and Γ' a square with edge attached to a vertex. Then:

- $P_\Gamma = P_{\Gamma'} = 1 + 5t + 5t^2$, and so $\phi_k(G_\Gamma) = \phi_k(G_{\Gamma'})$, for all $k \geqslant 1$.
- $Q_\Gamma = 5t^2 + 5t^3$ but $Q_{\Gamma'} = 5t^2 + 5t^3 + t^4$, and so $\theta_k(G_\Gamma) \neq \theta_k(G_{\Gamma'})$, for $k \geqslant 4$.





Example

Let Γ and Γ' be the two graphs above. Both have

$$P(t) = 1 + 6t + 9t^2 + 4t^3, \quad \text{and} \quad Q(t) = t^2(6 + 8t + 3t^2).$$

Thus, G_Γ and $G_{\Gamma'}$ have the same LCS and Chen ranks.
Each resonance variety has 3 components, of codimension 2:

$$\mathcal{R}^1(\Gamma) = \Bbbk^{\overline{23}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{35}}, \qquad \mathcal{R}^1(\Gamma') = \Bbbk^{\overline{15}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{26}}.$$

Yet the two varieties are not isomorphic, since

$$\dim(\Bbbk^{\overline{23}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{35}}) = 3, \quad \text{but} \quad \dim(\Bbbk^{\overline{15}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{26}}) = 2.$$

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