Resonance schemes of simplicial complexes

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Resonance varieties

- Let $A^\bullet$ be a graded, graded-commutative, algebra (cga) over a field $\Bbbk$ of characteristic 0, with multiplication maps $A^i \otimes_{\Bbbk} A^j \to A^{i+j}$.
- We assume A is connected ($A^0 = \Bbbk$) and of finite-type ($\dim_{\Bbbk} A^i < \infty$).
- For each $a \in A^1$, graded commutativity gives $a^2 = -a^2$, and so $a^2 = 0$.
- We then have a cochain complex,

$$(A^\bullet, \delta_a)\colon\ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u$, for all $u \in A^i$.
- The *resonance varieties* of A are the homogeneous sets

$$\mathcal{R}^i(A) = \{a \in A^1 \mid H^i(A^\bullet, \delta_a) \neq 0\}.$$

- $\mathcal{R}^0(A) = \{0\}$; $\mathcal{R}^1(A) = \{a \in A^1 \mid \exists\, b \in A^1 \text{ s.t. } a \wedge b \in K \backslash \{0\}\} \cup \{0\}$, where $K = \ker(A^1 \wedge A^1 \to A^2)$.

The BGG correspondence

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^\vee$, and identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.

- The BGG correspondence yields a cochain complex of finitely generated, free S-modules, $\mathrm{L}(A) := (A^\bullet \otimes_\Bbbk S, \delta)$,

$$\cdots \longrightarrow A^i \otimes_\Bbbk S \xrightarrow{\delta_A^i} A^{i+1} \otimes_\Bbbk S \xrightarrow{\delta_A^{i+1}} A^{i+2} \otimes_\Bbbk S \longrightarrow \cdots,$$

  where $\delta_A^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j$.

- The specialization of $(A \otimes_\Bbbk S, \delta)$ at $a \in A^1$ coincides with (A, δ_a).

- $a \in A^1$ belongs to $\mathcal{R}^i(A)$ iff $\operatorname{rank} \delta_a^{i-1} + \operatorname{rank} \delta_a^i < b_i(A)$. Hence,

$$\mathcal{R}^i(A) = V\Big(I_{b_i(A)}\big(\delta_A^{i-1} \oplus \delta_A^i\big)\Big),$$

  where $I_r(\psi)$ is the ideal of $r \times r$ minors of a matrix ψ.

Koszul modules

- The *Koszul modules* of A: the graded S-modules $W_i(A) = H_i(\mathsf{L}(A))$.
- Setting $E^\bullet = \bigwedge A^1$, the first one has presentation

$$\left(E^{3\vee} \oplus K^\perp\right) \otimes_{\Bbbk} S \xrightarrow{(\delta_E^3)^\vee + \iota\otimes \mathrm{id}} E^{2\vee} \otimes_{\Bbbk} S \twoheadrightarrow W_1(A), \qquad (*)$$

where
$K^\perp = \{\varphi \in A_1 \wedge A_1 = (A^1 \wedge A^1)^\vee \mid \varphi_K \equiv 0\} \xhookrightarrow{\iota} A_1 \wedge A_1 = E^{2\vee}$.

- The *resonance schemes* of A are defined by the annihilator ideals of these S-modules:

$$\boldsymbol{\mathcal{R}}_i(A) = \mathrm{Spec}(S/\operatorname{Ann} W_i(A)).$$

- The underlying sets, $\mathcal{R}_i(A) = \operatorname{supp} W_i(A) \subset A^1$, are related to the resonance varieties by:

$$\bigcup_{i\leqslant q} \mathcal{R}_i(A) = \bigcup_{i\leqslant q} \mathcal{R}^i(A).$$

- Recall $K = \ker(A^1 \wedge A^1 \to A^2)$.
- Let $L \subseteq A^1$ be a linear subspace. We say:
  - L is *isotropic* if $L \wedge L \subseteq K$.
  - L is *separable* if $K \cap \langle L \rangle_E \subseteq L \wedge L$, where $E = \bigwedge A^1$ and $\langle L \rangle_E$ is the ideal of E generated by L.

Example

- If $K = 0$, then every subspace $L \subseteq A^1$ is separable
- If $K = A^1 \wedge A^1$, then every subspace $L \subseteq A^1$ is isotropic, but the only separable subspace is the trivial one.

Example

Let $A = E/(K)$, where $E = \bigwedge(e_1, \dots, e_4)$ and

$$K = \langle e_1 \wedge e_2, e_1 \wedge e_3 + e_2 \wedge e_4 \rangle.$$

Then $\mathcal{R}^1(A) = \langle e_1, e_2 \rangle$ is isotropic but not separable.

Reduced resonance schemes

- Let $\mathcal{R}^1(A) = L_1 \cup \cdots \cup L_s$ be the decomposition of $\mathcal{R}^1(A) \subset A^1$ into irreducible components.
- Letting $K_j = K \cap (L_j \wedge L_j)$, we define S-modules $W_1^j(A)$ as in (*).
- Assume each component of $\mathcal{R}^1(A)$ is a linear subspace of A^1.

Theorem (AFRS)

(1) If each L_j is separable, then the projectivized resonance scheme is reduced and its components are disjoint.

(2) If the projectivized resonance scheme is reduced and each L_j are isotropic, then all its components are separable and disjoint.

(3) If each L_j is separable, then $\dim[W_1(A)]_q = \sum_{j=1}^{s} \dim[W_1^j(A)]_q$.

(4) If each L_j is separable and isotropic, then

$$\dim[W_1(A)]_q = \sum_{j=1}^{s} (q+1)\binom{q+\dim L_j}{q+2}.$$

Resonance varieties of spaces and groups

- The resonance varieties of a connected, finite-type CW-complex X are those of its cohomology algebra:

$$\mathcal{R}^i(X) := \mathcal{R}^i(H^\bullet(X,\mathbb{k})) \text{ and } \mathcal{R}_i(X) := \mathcal{R}_i(H^\bullet(X,\mathbb{k})).$$

- $\mathcal{R}^1(X)$ depends only on $G = \pi_1(X)$.
- The geometry of these varieties provides obstructions to the formality of X (or the 1-formality of G). E.g., if G is 1-formal, then all components of $R^1(G)$ are linear (Dimca–Papadima–S. 2009).
- They allow to distinguish between various classes of groups, such as Kähler groups, quasi-projective groups, hyperplane arrangement groups, 3-manifold groups, and right-angled Artin groups.
- Through their connections with other types of cohomology jump loci (characteristic varieties, BNSR invariants), they inform on the homological and geometric finiteness properties of spaces and groups.

Resonance and Chen ranks

Let G be a finitely-generated group. Define:

- *LCS series*: $G = G_1 \rhd G_2 \rhd \cdots \rhd G_k \rhd \cdots$, where $G_{k+1} = [G_k, G]$.
- *LCS quotients*: $\operatorname{gr}_k(G) = G_k/G_{k+1}$ (f.g. abelian groups).
- *Associated graded Lie algebra*: $\operatorname{gr}(G) = \bigoplus_{k\geqslant 1} \operatorname{gr}_k(G)$, with Lie bracket $[\, , \,]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by group commutator.
- *Chen Lie algebra*: $\operatorname{gr}(G/G'')$, where $G' = [G, G]$, $G'' = [G', G']$.
- *Chen ranks*: $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$.

Example (K.-T. Chen 1951)

Let F_n be the free group of rank $n \geqslant 2$. Then $\theta_1 = n$ and $\theta_k = (k-1)\binom{n+k-2}{k}$ for $k \geqslant 2$.

Example (Cohen–S. 1995)

Let P_n be the pure braid group on $n \geqslant 2$ strings. Then $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, and $\theta_k = (k-1)\binom{n+1}{4}$ for $k \geqslant 3$.

- Let $W_1(G) := W_1(H^{\leqslant 2}(G, \Bbbk))$ be the (first) Koszul module of G, viewed as a graded module over $S = \Bbbk[x_1, \dots, x_n]$, where $n = b_1(G)$.

Theorem (Papadima–S. 2004)

If G is 1-formal, then $\theta_k(G) = \dim_{\Bbbk}[W_1(G)]_{k-2}$ for all $k \geqslant 2$.

Theorem (Cohen–Schenck 2015, AFRS 2023)

Let G be a 1-formal group, and assume $\mathcal{R}^1(G)$ has linear components $L_1, \dots, L_s$ which are separable and isotropic. Then, for all $k \gg 0$,

$$\theta_k(G) = \sum_{j=1}^{s} (k-1)\binom{k + \dim L_j - 2}{k}.$$

Square-free modules

- Consider the standard $\mathbb{N}^n$-multigrading on $S = \Bbbk[x_1, \dots, x_n]$, defined by $\deg(x_i) = \mathrm{e}_i \in \mathbb{N}^n$, where $\mathrm{e}_i = (0, \dots, 1, \dots, 0)$.
- For $\mathrm{a} = (a_1, \dots, a_n) \in \mathbb{N}$, set $\mathrm{Supp}(\mathrm{a}) := \{i \mid a_i > 0\}$.

Definition (Yanagawa 2000)

An $\mathbb{N}^n$-graded S-module M is called *square-free* if for any $\mathrm{a} \in \mathbb{N}^n$ and any $i \in \mathrm{Supp}(\mathrm{a})$, the multiplication map $x_i \colon M_{\mathrm{a}} \to M_{\mathrm{a}+\mathrm{e}_i}$ is an isomorphism.

- An ideal $I \subseteq S$ is a square-free module $\iff$ I is a square-free monomial ideal $\iff$ S/I is a square-free module.
- A free $\mathbb{N}^n$-graded S-module is square-free if and only it is generated in square-free multidegrees.

Proposition

If $f\colon M \to N$ is a morphism of $\mathbb{N}^n$-graded S-modules, and M and N are square-free modules, then $\ker(f)$ and $\operatorname{coker}(f)$ are also square-free. Moreover, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $\mathbb{N}^n$-graded S-modules, and M' and M'' are square-free, then so is M.

Corollary

Let M be an $\mathbb{N}^n$-graded square-free S-module. Then all the modules in the minimal free $\mathbb{N}^n$-graded resolution of M are square-free.

Corollary

If F is a bounded complex of free, square-free S-modules, then the homology modules of F are also square-free.

Proposition (AFRSS 2023)

If M is an $\mathbb{N}^n$-graded, square-free S-module, then its annihilator is a square-free monomial ideal. In particular, $\operatorname{Ann} M$ is a radical ideal.

Toric complexes and Stanley–Reisner rings

- Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex on vertex set $[n] = \{1, \dots, n\}$.
- Let T_Δ be the subcomplex of the n-torus T^n obtained by deleting the cells corresponding to the missing simplices of Δ.
- T_Δ is a connected, formal CW-complex of dimension $\dim(\Delta) + 1$. It is a $K(G, 1)$ if and only if Δ is a flag complex.
- The fundamental group $G_\Gamma = \pi_1(T_\Delta)$ is the RAAG associated to the graph $\Gamma = \Delta^{(1)} = (\mathsf{V}, \mathsf{E})$,

$$G_\Gamma = \langle v \in V \mid [v, w] = 1 \text{ if } \{v, w\} \in E \rangle.$$

- (Papadima–S. 2006) The associated graded Lie algebra $\operatorname{gr}(G_\Gamma)$ has (quadratic) presentation

$$\operatorname{gr}(G_\Gamma) = \operatorname{Lie}(\mathsf{V})/([v, w] = 0 \text{ if } \{v, w\} \in \mathsf{E}).$$

- Moreover, $\mathrm{gr}(G_\Gamma/G''_\Gamma)$ is torsion-free, with ranks given by $\theta_1 = |\mathsf{V}|$ and

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right),$$

where $Q_\Gamma(t) = \sum_{j\geqslant 2} c_j(\Gamma) t^j$ is the "cut polynomial" of Γ, with

$$c_j(\Gamma) = \sum_{\mathsf{W}\subset \mathsf{V}\colon |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}}).$$

- (Kim–Roush 1980, Charney–Davis 1995) The cohomology algebra $H^\bullet(T_\Delta;\Bbbk)$ is the exterior Stanley–Reisner ring

$$\Bbbk\langle\Delta\rangle = \bigwedge V^\vee/(e_\sigma \mid \sigma \notin \Delta),$$

where

  - $V = \Bbbk^n$, with basis $v_1, \dots, v_n$.
  - $V^\vee = \mathrm{Hom}_\Bbbk(V, \Bbbk)$, with dual basis $e_1, \dots, e_n$.
  - $e_\sigma = e_{i_1} \wedge \cdots \wedge e_{i_s}$ for $\sigma = \{i_1, \dots, i_s\} \subseteq [n]$.

Koszul modules of simplicial complexes

- Let $\mathrm{K}_\bullet = \mathrm{L}(\bigwedge V)$ be the Koszul complex of $x_1, \dots, x_n$, whose i-th free S-module is $\mathrm{K}_i = \bigwedge^i V \otimes_{\Bbbk} S$.
- Set $\deg(v_i) = \mathrm{e}_i \in \mathbb{N}^n$. Then $\mathrm{K}_\bullet$ is a complex of $\mathbb{N}^n$-graded, square-free S-modules.
- For a simplicial complex Δ on vertex set $[n]$ we have $\mathrm{L}(\Bbbk\langle\Delta\rangle) = \mathrm{K}^\Delta_\bullet$, where $\mathrm{K}^\Delta_\bullet$ is the subcomplex of $\mathrm{K}_\bullet$ whose i-th module K^Δ_i is the free S-module generated by $\{v_\sigma \mid \sigma \in \Delta\}$.
- We let $W_i(\Delta) := H_i(\mathrm{K}^\Delta_\bullet)$ be the Koszul modules of Δ.

Proposition (AFRSS 2023)

Each Koszul module $W_i(\Delta)$ is an $\mathbb{N}^n$-graded, square-free S-module.

- Proof: $\mathrm{K}^\Delta_\bullet$ is a bounded complex of free, square-free S-modules; thus, its homology modules are also square-free.

Resonance of simplicial complexes

- We define the resonance varieties of a simplicial complex Δ as

$$\mathcal{R}^i(\Delta) := \mathcal{R}^i(T_\Delta) = \mathcal{R}^i(\Bbbk\langle\Delta\rangle)$$

- Likewise, we set $\mathcal{R}_i(\Delta) := \mathcal{R}_i(\Bbbk\langle\Delta\rangle)$.

Theorem (Papadima–S. 2006/2009)

Let Δ be a simplicial complex on vertex set $\mathsf{V} = [n]$. The resonance varieties of Δ are finite unions of coordinate subspaces of $V^\vee = \Bbbk^{\mathsf{V}}$,

$$\mathcal{R}^i(\Delta) = \bigcup_{\substack{\mathsf{W}\subseteq \mathsf{V} \\ \exists\sigma\in\Delta_{\mathsf{V}\setminus\mathsf{W}},\ \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{\Delta_\mathsf{W}}(\sigma),\Bbbk)\neq 0}} \Bbbk^{\mathsf{W}},$$

where

- Δ_W *is the induced simplicial subcomplex on vertex set* $\mathsf{W} \subseteq \mathsf{V}$.
- $\mathrm{lk}_{\Delta_\mathsf{W}}(\sigma)$ *is the link in* Δ_W *of a simplex* $\sigma \in \Delta$.
- $\Bbbk^\mathsf{W}$ *is the coordinate subspace of* $\Bbbk^\mathsf{V}$ *spanned by* $\{e_i \mid i \in \mathsf{W}\}$.

Theorem (AFRSS 2023)

For each $i \geqslant 1$, the scheme structure on the support resonance locus $\mathcal{R}_i(\Delta)$ is reduced. Moreover, the decomposition into irreducible components is given by

$$\mathcal{R}_i(\Delta) = \bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V} \text{ maximal with} \\ \widetilde{H}^{i-1}(\Delta_{\mathsf{W}};\Bbbk)\neq 0}} \Bbbk^{\mathsf{W}}.$$

- Whereas the schemes $\mathcal{R}_i(\Delta)$ are always reduced, the corresponding jump resonance loci $\mathcal{R}^i(\Delta)$ are not necessarily reduced (with the Fitting scheme structure), even when $i = 1$.

Example

Let Γ be a path on 4 vertices. Then

$$\operatorname{Fitt}_0(W_1(\Gamma)) = (x_2) \cap (x_3) \cap (x_1, x_2^2, x_3^2, x_4)$$

is not reduced, although $\operatorname{Ann}(W_1(\Gamma)) = (x_2) \cap (x_3)$ is reduced. Therefore, the Fitting scheme structure on $\mathcal{R}^1(\Gamma)$ has an embedded component at 0.

Cohen–Macaulay complexes and propagation

- A simplicial complex Δ of dimension d is *Cohen–Macaulay (over $\Bbbk$)* if $\widetilde{H}^\bullet(\mathrm{lk}(\sigma);\Bbbk)$ is concentrated in degree $d-|\sigma|$, for all $\sigma\in\Delta$.

Theorem (Denham–S.–Yuzvinsky 2017)

If Δ is Cohen–Macaulay over $\Bbbk$, then the resonance of Δ propagates:

$$\mathcal{R}^1(\Delta)\subseteq\mathcal{R}^2(\Delta)\subseteq\cdots\subseteq\mathcal{R}^{d+1}(\Delta).$$

- In general, though, the resonance varieties do not always propagate.

Example (Papadima–S. 2009)

Let Δ be the disjoint union of two edges. Then $\mathcal{R}^1(\Delta)=\Bbbk^4$, whereas $\mathcal{R}^2(\Delta)=\Bbbk^2\cup\Bbbk^2$, the union of two transversal coordinate planes. Thus, $\mathcal{R}^1(\Delta)\not\subseteq\mathcal{R}^2(\Delta)$.

- If Δ is Cohen–Macaulay, it follows that $\mathcal{R}^i(\Delta) = \bigcup_{j\leqslant i} \mathcal{R}_j(\Delta)$.

Question

Suppose Δ is Cohen–Macaulay. Do the support resonance varieties $\mathcal{R}_i(\Delta)$ propagate? Or, equivalently in this case, is $\mathcal{R}^i(\Delta) = \mathcal{R}_i(\Delta)$?

- For an arbitrary Δ, the support resonance varieties may fail to propagate, and we may well have $\mathcal{R}^i(\Delta) \neq \mathcal{R}_i(\Delta)$ for some $i > 1$.

Example

Let Δ be the disjoint union of two edges. Then $\mathcal{R}_1(\Delta) = \mathcal{R}^1(\Delta) = \Bbbk^4$ but $\mathcal{R}_2(\Delta) = \varnothing$ whereas, as we saw earlier, $\mathcal{R}^2(\Delta) = \Bbbk^2 \cup \Bbbk^2$. Thus,

$$\mathcal{R}_1(\Delta) \nsubseteq \mathcal{R}_2(\Delta) \;\text{ and }\; \mathcal{R}_2(\Delta) \neq \mathcal{R}^2(\Delta).$$

Resonance of graphs

- If Γ is a (simple) graph on n vertices, then:

$$\mathcal{R}^1(\Gamma) = \bigcup_{\substack{W\subseteq[n]\\ \Gamma_W \text{ disconnected}}} \Bbbk^W.$$

- The irreducible components of $\mathcal{R}^1(\Gamma)$ are the coordinate subspaces $\Bbbk^W$, maximal among those for which Γ_W is disconnected.
- The codimension of $\mathcal{R}^1(\Gamma)$ equals the connectivity of Γ. In particular, if Γ is disconnected, then $\mathcal{R}^1(\Gamma) = \Bbbk^n$.

Proposition (AFRS 2023)

Let Γ be a connected graph, let Γ' be a maximally disconnected full subgraph, and let L' be the corresponding component of $\mathcal{R}^1(\Gamma)$. Then:

- *L' is isotropic if and only if Γ' is discrete.*
- *L' is separable if and only if $\Gamma = \Gamma' * \Gamma''$.*

Hence, isotropic implies separable for the resonance varieties of graphs.

Regularity and Hilbert series

- The next result gives upper bounds on the Castelnuovo–Mumford regularity and the projective dimension of the Koszul modules of a simplicial complex Δ.

Proposition (AFRSS 2023)

If Δ has n vertices, then $W_i(\Delta)$ has regularity at most n and projective dimension at most $n-i-1$. Moreover, if Γ is a graph and $n \geqslant 4$, then $\operatorname{reg} W_1(\Gamma) \leqslant n-4$.

- These bounds are sharp. E.g., if $\Gamma = \mathcal{C}_n$ is a cycle on $n \geqslant 4$ vertices, then $\operatorname{pdim} W_\Gamma = n-2$ and $\operatorname{reg} W_\Gamma = n-4$.

- We also compute in (AFRSS 2023) the (multigraded) Hilbert series of the Koszul modules of a simplicial complex Δ.

Theorem

- *For any $i \geqslant 1$ and any square-free multi-index b, there are natural isomorphisms of vector spaces*

$$[W_i(\Delta)]_{\mathsf{b}} \cong \Big[\operatorname{Tor}^S_{|\mathsf{b}|-i}(\Bbbk, \Bbbk[\Delta])\Big]^\vee_{\mathsf{b}} \cong \widetilde{H}^{i-1}(\Delta_{\mathsf{b}};\Bbbk)^\vee \cong \widetilde{H}_{i-1}(\Delta_{\mathsf{b}};\Bbbk),$$

where $\Delta_{\mathsf{b}} = \Delta_{\mathsf{Supp(b)}}$ and $|\mathsf{b}| = b_1 + \cdots + b_n$.

- *Moreover,*

$$\sum_{\mathsf{a}\in\mathbb{N}^n} \dim_{\Bbbk}[W_i(\Delta)]_{\mathsf{a}}\, \mathsf{t}^{\mathsf{a}} = \sum_{\substack{\mathsf{b}\in\mathbb{N}^n\\ \mathsf{b}\text{ square-free}}} \dim_{\Bbbk}(\widetilde{H}_{i-1}(\Delta_{\mathsf{b}};\Bbbk)) \frac{\mathsf{t}^{\mathsf{b}}}{\prod_{j\in\mathsf{Supp(b)}}(1-t_j)}.$$