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Cohomology, Bocksteins, and resonance varieties in characteristic 2

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ABSTRACT. We use the action of the Bockstein homomorphism on the cohomology ring $H^{\bullet}(X,\mathbb{Z}_2)$ of a finite-type CW-complex X in order to define the resonance varieties of X in characteristic 2. Much of the theory is done in the more general framework of the Maurer–Cartan sets and the resonance varieties attached to a finite-type commutative differential graded algebra. We illustrate these concepts with examples mainly drawn from closed manifolds, where Poincaré duality over $\mathbb{Z}_2$ has strong implications on the nature of the resonance varieties.

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1. Introduction

1.1. Resonance varieties. The cohomology ring of a space X captures deep, albeit incomplete information about the homotopy type of X. A fruitful idea, which arose in the late 1990s from the theory of hyperplane arrangements [8, 4, 16] turns the cohomology algebra $H^{\bullet}(X,\Bbbk)$ over a coefficient field $\Bbbk$ into a family of cochain complexes with differentials given by multiplication by an element in the vector space $H^1(X,\Bbbk)$. One extracts from these data the *resonance varieties* of X over $\Bbbk$, as the loci where the cohomology of the aforementioned cochain complexes jumps in a certain degree, by a specified amount.

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Originally, the ground field was $\mathbb{C}$ and the space X was the complement of an arrangement of n hyperplanes in $\mathbb{C}^\ell$, in which case each of these resonance varieties was shown to be a finite union of linear subspaces in $\mathbb{C}^n$. Starting with [20], the scope of investigation of the resonance varieties broadened, with fields $\Bbbk$ of arbitrary characteristic and more general spaces X being allowed, as long as X had the homotopy type of a connected, finite-type CW-complex, its homology groups $H_\bullet(X,\mathbb{Z})$ were free abelian, and the cohomology ring $H^\bullet(X,\mathbb{Z})$ was generated in degree 1. New phenomena were discovered in this wider generality; for instance, the resonance varieties over $\mathbb{C}$ no longer need to be linear, while the resonance varieties over $\mathbb{Z}_p$ (for p a prime) carry useful information regarding the index-p subgroups of the second nilpotent quotient of $\pi_1(X)$ (see also [31]). Further investigations of the resonance varieties over finite fields were done in [9].

The theory was developed in even greater generality in [23, 24], where the only condition required was that either $\operatorname{char}(\Bbbk) \neq 2$, or $\operatorname{char}(\Bbbk) = 2$ and $H_1(X,\mathbb{Z})$ to be torsion-free, so as to insure that $a^2 = 0$ for every $a \in H^1(X,\Bbbk)$. Unfortunately, this torsion-freeness condition rules out many interesting spaces. In this note, we remedy this situation, by re-defining the resonance varieties in characteristic 2, so as to take into account the action of the Bockstein operator, $\beta_2 = \operatorname{Sq}^1$, on the cohomology algebra $H^\bullet(X,\mathbb{Z}_2)$.

1.2. Commutative differential graded algebras. The key idea is to appeal to a construction done in an even wider framework. This construction, which was first introduced and studied in [18, 25, 34] and is being further developed in [5, 36, 37], starts with a commutative differential graded algebra (A, d). For every element a in the *Maurer–Cartan set* of this CDGA,

$$\mathrm{MC}(A) = \{a \in A^1 \mid a^2 + \mathrm{d}(a) = 0 \in A^2\}, \tag{1}$$

one constructs a cochain complex $(A^\bullet, \delta_a)$ with differentials $\delta^i_a \colon A^i \to A^{i+1}$ the $\Bbbk$-linear maps given by $\delta^i_a(u) = a \cdot u + \mathrm{d}(u)$ for $u \in A^i$. The resonance varieties of A (in degree $q \geq 0$ and depth $s \geq 0$) are the loci

$$\mathcal{R}^q_s(A) = \{a \in \mathrm{MC}(A) \mid \dim_\Bbbk H^q(A, \delta_a) \geq s\}. \tag{2}$$

Under the assumption that all the graded pieces A^i are finite-dimensional, these sets are Zariski closed subsets of the algebraic variety $\mathrm{MC}(A)$; when A has zero differential, the resonance varieties are homogeneous, but in general that is not the case.

Especially interesting is the case when (A, d) is a Poincaré duality CDGA of formal dimension m, that is, the underlying graded algebra $A^\bullet$ satisfies Poincaré duality and the differential d vanishes on A^{m-1}. Building on work from [27, 35], we show in Theorem 4.6 that, for such CDGAS, the involution $a \mapsto -a$ on A^1 restricts to isomorphisms

$$\mathcal{R}^q_s(A) \xrightarrow{\simeq} \mathcal{R}^{m-q}_s(A) \tag{3}$$

for all $q, s \geq 0$.

1.3. The Bockstein operator and resonance in characteristic 2. We apply the general theory outlined above to the case when $A = H^\bullet(X,\mathbb{Z}_2)$ is the cohomology algebra of a connected, finite-type CW-complex X, equipped with the differential $\beta_2 \colon A^\bullet \to A^{\bullet+1}$ given by the Bockstein homomorphism associated to the coefficient exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$. We denote the resonance varieties of the CDGA (A, β_2) by $\widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2)$. Slightly more generally, for a field $\Bbbk$ of characteristic 2, we define the *Bockstein resonance varieties* of X as

$$\widetilde{\mathcal{R}}^q_s(X,\Bbbk) = \widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2) \times_{\mathbb{Z}_2} \Bbbk. \tag{4}$$

If the group $H_1(X,\mathbb{Z})$ has no 2-torsion, one may also define the "usual" resonance varieties, $\mathcal{R}^q_s(X,\Bbbk)$, as the resonance varieties of the CDGA $(A,0)$. It turns out that $\mathcal{R}^1_s(X,\Bbbk) = \widetilde{\mathcal{R}}^1_s(X,\Bbbk)$ for all $s \geq 0$, due to the vanishing of the Bockstein $\beta_2\colon H^1(X,\mathbb{Z}_2) \to H^2(X,\mathbb{Z}_2)$ in this situation. Nevertheless, the resonance varieties $\mathcal{R}^q_s(X,\Bbbk)$ and $\widetilde{\mathcal{R}}^q_s(X,\Bbbk)$ may differ in degrees $q > 1$; in fact, we give examples showing that either variety may be properly included in the other when $s = 1$. For instance, if M is a smooth, closed, non-orientable manifold of dimension m and $H_1(M,\mathbb{Z})$ has no 2-torsion, we then show in Corollary 7.11 that $\mathcal{R}^m_1(M,\mathbb{Z}_2) = \{0\}$, whereas $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) = \mathbb{Z}_2$.

1.4. Poincaré duality, orientability, and resonance. In fact, much more can be said in the case when our space is a topological manifold M. We always assume M to be a closed manifold (that is, compact, connected, with no boundary). Our first main result in this context (proved in Corollary 7.3 and Proposition 7.10) characterizes the orientability of M in terms of its Bockstein CDGA and its top-degree Bockstein resonance variety.

THEOREM 1.1. *Let M be a closed, smooth manifold of dimension m. The following conditions are equivalent.*

(1) M is orientable.
(2) $(H^\bullet(M,\mathbb{Z}_2),\beta_2)$ is a Poincaré duality differential graded algebra (of formal dimension m).
(3) $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) = \{0\}$.

Our second main result (proved in Propositions 7.5 and 7.7) is a Poincaré duality analog for the resonance varieties (of both types) of closed, orientable manifolds.

THEOREM 1.2. *Let M be a closed, orientable manifold of dimension m.*

(1) If $\operatorname{char}(\Bbbk) \neq 2$, then $\mathcal{R}^q_s(M;\Bbbk) = \mathcal{R}^{m-q}_s(M;\Bbbk)$ for all $q, s \geq 0$.
(2) If $\operatorname{char}(\Bbbk) = 2$, then $\widetilde{\mathcal{R}}^q_s(M;\Bbbk) = \widetilde{\mathcal{R}}^{m-q}_s(M;\Bbbk)$ for all $q, s \geq 0$.

We illustrate these results with several classes of manifolds, including the orientable and non-orientable surfaces, some lens spaces and other 3-manifolds, the real projective spaces $\mathbb{RP}^n$, and the Dold manifolds $P(m,n)$.

1.5. Betti numbers in finite covers. In the final section we discuss some of the ways in which the resonance varieties $\mathcal{R}^q_s(X,\Bbbk)$ and $\widetilde{\mathcal{R}}^q_s(X,\Bbbk)$, as well as the Betti numbers $b_q(X,\Bbbk) = \dim_{\Bbbk} H^q(X,\Bbbk)$ behave when passing to finite covers.

For instance, suppose $Y \to X$ is a connected 2-fold cover classified by a non-zero class $\alpha \in H^1(X,\mathbb{Z}_2)$ with $\alpha^2 = 0$. In Proposition 8.5 we strengthen results from [39] and show that

$$b_q(Y,\mathbb{Z}_2) = b_q(X,\mathbb{Z}_2) + \dim_{\mathbb{Z}_2} H^q(H^\bullet(X,\mathbb{Z}_2),\delta_\alpha), \tag{5}$$

for all $q \geq 1$. In particular, the mod-2 Betti numbers of Y are at least as large as those of X, which is not necessarily the case when $\alpha^2 \neq 0$.

2. Maurer–Cartan elements and Koszul complexes

We start with the Maurer–Cartan sets and the Koszul complexes associated to a commutative differential graded algebra.

2.1. Commutative graded algebras. Throughout this work, $\Bbbk$ will denote a ground field. A *graded* $\Bbbk$*-vector space* is a vector space A over $\Bbbk$, together with a direct sum decomposition, into vector subspaces, $A = \bigoplus_{i\geq 0} A^i$, called its graded pieces. An element $a \in A^i$ is said to be homogeneous; we write $|a| = i$ for its degree.

A *graded algebra* over $\Bbbk$ is a graded $\Bbbk$-vector space, $A^\bullet = \bigoplus_{i\geq 0} A^i$, equipped with an associative multiplication map, $\cdot\colon A \times A \to A$, making A into a $\Bbbk$-algebra with unit $1 \in A^0$ such that $|a \cdot b| = |a| + |b|$ for all homogenous elements $a, b \in A$. A graded algebra A is said to be *graded-commutative* (for short, a CGA), if $a \cdot b = (-1)^{|a||b|} b \cdot a$ for all homogeneous $a, b \in A$, and *strictly commutative* if $ab = ba$ for all $a, b \in A$. Note that the two versions of commutativity agree when $\operatorname{char}(\Bbbk) = 2$.

A morphism between two graded algebras is a $\Bbbk$-linear map $\varphi\colon A \to B$ such that φ sends A^i to B^i for all $i \geq 0$ and satisfies $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ for all $a, b \in A$.

We say that a graded algebra A is *connected* if A^0 is the $\Bbbk$-span of the unit 1 (and thus $A^0 = \Bbbk$). We also say that A is of *finite-type* (or, locally finite) if all the graded pieces A^i are finite-dimensional.

Given a vector space V, one may construct several basic examples of graded algebras from it: the tensor algebra $T = T(V)$, the symmetric algebra $S = \operatorname{Sym}(V)$, and the exterior algebra $E = \bigwedge(V)$. All three algebras are connected; moreover, E is graded-commutative, S is strictly commutative, and T has neither of these properties. If V is finite-dimensional, then T and S are of finite-type and E is finite-dimensional.

2.2. Commutative differential graded algebras. We now enrich the notion of a CGA by introducing a differential, which we require to be compatible with both the grading and the multiplication. By definition, a *commutative differential graded algebra* over $\Bbbk$ (for short, a $\Bbbk$-CDGA) is a pair $A = (A^\bullet, \mathrm{d})$, where $A^\bullet$ is a $\Bbbk$-CGA and $\mathrm{d}\colon A \to A$ is a *graded derivation*; that is, d is a $\Bbbk$-linear map such that $\mathrm{d} \circ \mathrm{d} = 0$, $\mathrm{d}(A^i) \subseteq A^{i+1}$ for all $i \geq 0$, and $\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^{|a|} a \cdot \mathrm{d}(b)$, for all homogeneous elements $a, b \in A$.

Using only the underlying cochain complex structure of the CDGA, we let $Z^i(A) = \ker(\mathrm{d}\colon A^i \to A^{i+1})$ be the subspace of cocycles and $B^i(A) = \operatorname{im}(\mathrm{d}\colon A^{i-1} \to A^i)$ the subspace of coboundaries, and define the i-th cohomology group of $(A^\bullet, \mathrm{d})$ as the quotient $\Bbbk$-vector space $H^i(A) = Z^i(A)/B^i(A)$. The direct sum of these vector spaces, $H^\bullet(A) = \bigoplus_{i\geq 0} H^i(A)$, inherits from A the structure of a graded, graded-commutative $\Bbbk$-algebra. If A is of finite-type, then clearly $H^\bullet(A)$ is also of finite-type; in this case, we define the Betti numbers of A to be the ranks of the graded pieces of $H^\bullet(A)$, and write them as $b_i(A) \coloneqq \dim_{\Bbbk} H^i(A)$.

For a cocycle $z \in Z^i(A)$, we will denote by $[z] \in H^i(A)$ the cohomology class it represents. Observe that $\mathrm{d}(1) = \mathrm{d}(1 \cdot 1) = \mathrm{d}(1) + \mathrm{d}(1)$, and so $\mathrm{d}(1) = 0$. Therefore, if A is connected, the differential $\mathrm{d}\colon A^0 \to A^1$ vanishes, and so we may identify $H^0(A) = \Bbbk$ and $H^1(A) = Z^1(A)$.

A morphism between two CDGAS, $\varphi\colon (A, \mathrm{d}_A) \to (B, \mathrm{d}_B)$, is both a map of graded algebras and a cochain map; that is, φ is a $\Bbbk$-linear map that preserves gradings, multiplicative structures, and differentials. Denoting by $\varphi^i\colon A^i \to B^i$ the restriction of φ to i-th graded pieces, we have that $\varphi^i(ab) = \varphi^i(a)\varphi^i(b)$ and $\mathrm{d}_B(\varphi^i(a)) = \varphi^{i+1}(\mathrm{d}_A(a))$, for all $a, b \in A^i$. The morphism φ induces a morphism $\varphi^*\colon H^\bullet(A) \to H^\bullet(B)$ between the respective cohomology algebras. We say that φ is a *quasi-isomorphism* if φ^* is an isomorphism.

A *weak equivalence* between two CDGAS, A and B, is a finite sequence of quasi-isomorphisms going either way and connecting A to B; one such zig-zag of quasi-isomorphisms is given in the diagram below,

$$A \xleftarrow{\ \varphi_1\ } A_1 \xrightarrow{\ \varphi_2\ } \cdots \longleftarrow A_{\ell-1} \xrightarrow{\ \varphi_\ell\ } B. \tag{6}$$

Note that a weak equivalence induces a well-defined isomorphism $H^\bullet(A) \cong H^\bullet(B)$. If a weak equivalence between A and B exists, we say that the two CDGAs are *weakly equivalent*, and write $A \simeq B$. Clearly, $\simeq$ is an equivalence relation among CDGAs.

2.3. Maurer–Cartan sets. Let (A, d) be a CDGA over a field $\Bbbk$. We define the *Maurer–Cartan set* of A by

$$\mathrm{MC}(A) = \left\{a \in A^1 \mid a^2 + \mathrm{d}(a) = 0 \in A^2\right\}. \tag{7}$$

If A^1 is finite-dimensional, then the set of Maurer–Cartan elements is a quadratic algebraic subvariety of the affine space A^1. Clearly, this variety contains the point $0 \in A^1$. Two general classes of examples are worth singling out.

Example 2.1. Suppose $a^2 + \mathrm{d}(a) = 0$ for all $a \in A^1$. In this case (which we will concentrate on in the latter sections), we clearly have $\mathrm{MC}(A) = A^1$.

Example 2.2. Suppose $a^2 = 0$, for every $a \in A^1$—a condition that is always satisfied if $\mathrm{char}(\Bbbk) \neq 2$. In this case, $\mathrm{MC}(A) = Z^1(A)$. If, additionally, A is connected, then a previous observation gives $\mathrm{MC}(A) = H^1(A)$.

In all the above examples, the Maurer–Cartan set is a linear subspace of A^1. In general, though, $\mathrm{MC}(A)$ is cut out by quadratic and linear equations; here is a simple example illustrating this fact.

Example 2.3. Let $A = (\mathbb{Z}_2[a_1, a_2], \mathrm{d})$, with differential given by $\mathrm{d}(a_1) = a_1a_2$ and $\mathrm{d}(a_2) = a_2^2$. Then $\mathrm{MC}(A) = \{(x_1, x_2) \in \mathbb{Z}_2^2 \mid x_1(x_1 + x_2) = 0\}$.

The next lemma (which shall be of use in §4.2) shows that the Maurer–Cartan set in invariant under negation in the ambient vector space.

Lemma 2.4. *Let (A, d) be a $\Bbbk$-CDGA. Then the linear involution $A^1 \to A^1$, $a \mapsto -a$ restricts to an involution* $\mathrm{MC}(A) \to \mathrm{MC}(A)$.

Proof. If $\mathrm{char}(\Bbbk) = 2$, then the claim is obviously true. On the other hand, if $\mathrm{char}(\Bbbk) \neq 2$, then $\mathrm{MC}(A) = Z^1(A)$, which is a linear subspace of A^1, and we are done. ◻

The construction of the Maurer–Cartan sets is functorial. More precisely, we have the following lemma, which follows directly from the definitions.

Lemma 2.5. *Let $\varphi\colon A \to B$ be a morphism of* CDGAs*. Then the linear map $\varphi^1\colon A^1 \to B^1$ restricts to a map, $\bar{\varphi}\colon$ $\mathrm{MC}(A) \to \mathrm{MC}(B)$, between the respective subsets; moreover, $\overline{\psi \circ \varphi} = \bar{\psi} \circ \bar{\varphi}$.*

If both A^1 and B^1 are finite-dimensional, then the map $\bar{\varphi}\colon \mathrm{MC}(A) \to \mathrm{MC}(B)$ from above is a morphism of algebraic varieties. In particular, if $\varphi\colon A \to B$ is an isomorphism, then $\bar{\varphi}$ is also an isomorphism.

It also follows from the definitions that the construction is compatible with restriction and extension of the base field.

Lemma 2.6. *Let A be a $\Bbbk$-*CDGA *with $\dim_\Bbbk A^1 < \infty$, and let $\Bbbk \subset \mathbb{K}$ be a field extension. Then,*

(1) $\mathrm{MC}(A) = \mathrm{MC}(A \otimes_\Bbbk \mathbb{K}) \cap A^1$.
(2) $\mathrm{MC}(A \otimes_\Bbbk \mathbb{K}) = \mathrm{MC}(A) \times_\Bbbk \mathbb{K}$.

2.4. The cochain complex associated to a Maurer–Cartan element. Let (A, d) be a finite-type $\Bbbk$-CDGA. For each element $a \in \mathrm{MC}(A)$, we have a cochain complex of finite-dimensional $\Bbbk$-vector spaces,

$$(A^\bullet, \delta^A_a)\colon \ A^0 \xrightarrow{\delta^0_a} A^1 \xrightarrow{\delta^1_a} A^2 \xrightarrow{\delta^2_a} \cdots, \tag{8}$$

with differentials $\delta^i_a\colon A^i \to A^{i+1}$ the $\Bbbk$-linear maps given by

$$\delta^i_a(u) = a \cdot u + \mathrm{d}(u), \tag{9}$$

for $u \in A^i$. The fact that these maps are differentials is readily verified:

$$\begin{aligned} \delta^{i+1}_a \delta^i_a(u) &= a^2 u + a \cdot \mathrm{d}(u) + \mathrm{d}(a) \cdot u - a \cdot \mathrm{d}(u) + \mathrm{d}(\mathrm{d}(u)) \\ &= (a^2 + \mathrm{d}(a)) \cdot u \\ &= 0, \end{aligned} \tag{10}$$

where at the last step we used the assumption that a belongs to $\mathrm{MC}(A)$. We let

$$b_i(A, a) := \dim_{\Bbbk} H^i(A, \delta^A_a) \tag{11}$$

be the Betti numbers of this cochain complex. Observe that $\delta_0 = \mathrm{d}$, and thus $b_i(A, 0) = b_i(A)$.

The cochain complex associated to a Maurer–Cartan element enjoys the following naturality property. Related statements (in different levels of generality) can be found in [**5**, **35**].

Lemma 2.7. *Let $\varphi\colon (A, \mathrm{d}_A) \to (B, \mathrm{d}_B)$ be a morphism of finite-type $\Bbbk$-CDGAs. For each $a \in \mathrm{MC}(A)$, the map φ induces a chain map,*

$$\begin{tikzcd} \left(A^\bullet, \delta^A_a\right)\colon \arrow[d, "\varphi_a"] & A^0 \arrow[r, "\delta^0_a"] \arrow[d, "\varphi^0"] & A^1 \arrow[r] \arrow[d, "\varphi^1"] & \cdots \arrow[r] & A^i \arrow[r, "\delta^i_a"] \arrow[d, "\varphi^i"] & A^{i+1} \arrow[r, "\delta^{i+1}_a"] \arrow[d, "\varphi^{i+1}"] & \cdots \\ \left(B^\bullet, \delta^B_{\varphi(a)}\right)\colon & B^0 \arrow[r, "\delta^0_{\varphi(a)}"] & B^1 \arrow[r] & \cdots \arrow[r] & B^i \arrow[r, "\delta^i_{\varphi(a)}"] & B^{i+1} \arrow[r, "\delta^{i+1}_{\varphi(a)}"] & \cdots . \end{tikzcd} \tag{12}$$

Proof. Let $u \in A^i$. By definition, $\varphi_a(u) = \varphi^i(u)$. Hence,

$$\begin{aligned} \varphi_a(\delta^A_a(u)) &= \varphi^{i+1}(au + \mathrm{d}_A(u)) \\ &= \varphi^1(a)\varphi^i(u) + \mathrm{d}_B(\varphi^i(u)) \\ &= \delta^B_{\varphi(a)}(\varphi_a(u)), \end{aligned} \tag{13}$$

and the claim is proved. ◻

Consequently, for each $a \in \mathrm{MC}(A)$, the chain map $\varphi_a\colon (A^\bullet, \delta^A_a) \to (B^\bullet, \delta^B_{\varphi(a)})$ induces homomorphisms in cohomology,

$$\varphi^i_a\colon H^i\left(A, \delta^A_a\right) \longrightarrow H^i\left(B, \delta^B_{\bar\varphi(a)}\right). \tag{14}$$

Lemma 2.8. *Let $\varphi\colon (A, \mathrm{d}_A) \to (B, \mathrm{d}_B)$ be a morphism of finite-type $\Bbbk$-CDGAs, let $a \in \mathrm{MC}(A)$, and let i be a positive integer.*

(1) Suppose φ^i is injective and φ^{i-1} is surjective. Then φ^i_a is injective.
(2) Suppose φ^i is surjective and φ^{i+1} is injective. Then φ^i_a is surjective.

Proof. (1) Suppose $\varphi^i_a([u]) = 0$, for some $u \in A^i$ with $au + d_A(u) = 0$. Then $\varphi^i(u) = \varphi^1(a)v + d_B(v)$, for some $v \in B^{i-1}$. By our surjectivity assumption on φ^{i-1}, there is an element $w \in A^{i-1}$ such that $\varphi^{i-1}(w) = v$, and so $\varphi^i(u) = \varphi^i(aw) + \varphi^i(d_A(w))$. Our injectivity assumption on φ^i now implies that $u = aw + d_A(w)$, and so $[u] = 0$.

(2) Let $[v] \in H^i(B, \delta^B_{\bar\varphi(a)})$, for some $v \in B^i$ with $\varphi^1(a)v + d_B(v) = 0$. Since φ^i is surjective, there is an element $u \in A^i$ such that $v = \varphi^i(u)$. We then have

$$\begin{aligned}\varphi^{i+1}(au + d_A(u)) &= \varphi^1(a)\varphi^i(u) + d_B(\varphi^i(u))\\ &= \varphi^1(a)v + d_B(v)\\ &= 0,\end{aligned}\tag{15}$$

and, since φ^{i+1} is injective, we have that $au + d_A(u) = 0$. Therefore, $[v] = \varphi^i_a([u])$, and the proof is complete. ∎

2.5. A generalized Koszul complex. We now fix a basis $\{e_1, \dots, e_n\}$ for the finite-dimensional $\Bbbk$-vector space A^1, and we let $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for the dual vector space $A_1 = (A^1)^\vee$. In what follows, we shall identify the symmetric algebra $\mathrm{Sym}(A_1)$ with the polynomial ring $R = \Bbbk[x_1, \dots, x_n]$.

The coordinate ring of the affine variety $\mathrm{MC}(A) \subset A^1$ is the quotient, $S = R/I$, of the ring R by the defining ideal of $\mathrm{MC}(A)$. Consider the cochain complex of free S-modules,

$$(A^\bullet \otimes_\Bbbk S, \delta_A)\colon \cdots \longrightarrow A^i \otimes_\Bbbk S \xrightarrow{\delta^i} A^{i+1} \otimes_\Bbbk S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes_\Bbbk S \longrightarrow \cdots,\tag{16}$$

where the differentials $\delta^i = \delta^i_A$ are the S-linear maps defined by

$$\delta^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j + d(u) \otimes s\tag{17}$$

for all $u \in A^i$ and $s \in S$. The fact that this is a cochain complex is easily verified; indeed, $\delta^{i+1}\delta^i(u \otimes s)$ is equal to

$$\begin{aligned}&\sum_k e_k\Big(\sum_j e_j u \otimes s x_j + d\,u \otimes s\Big) \otimes x_k + d\Big(\sum_j e_j u \otimes s x_j + d\,u \otimes s\Big)\\ &\qquad = \sum_{j,k} e_k e_j u \otimes s x_j x_k + \sum_k e_k\, d\,u \otimes s x_k - \sum_j e_j\, d\,u \otimes s x_j\\ &\qquad = 0,\end{aligned}\tag{18}$$

where at the last step we used the fact that $e_k e_j = -e_j e_k$.

Remark 2.9. The cochain complex (16) is independent of the choice of basis for the vector space A^1. Indeed, under the canonical identification $A^1 \otimes_\Bbbk A_1 \cong \mathrm{Hom}(A^1, A^1)$, the element $\iota = \sum_{j=1}^n e_j \otimes x_j$ used in defining the differentials δ^i corresponds to the identity map of A^1.

Example 2.10. Let $E = \bigwedge(e_1, \dots, e_n)$ be an exterior algebra over a field $\Bbbk$ of characteristic not equal to 2 (with generators e_i in degree 1 and with zero differential), and let $S = \Bbbk[x_1, \dots, x_n]$ be its Koszul dual. Then the cochain complex $(E^\bullet \otimes_\Bbbk S, \delta)$ is the classical Koszul complex $K_\bullet(x_1, \dots, x_n)$.

More generally, if the cdga A has zero differential, each boundary map $\delta^i\colon A^i \otimes_\Bbbk S \to A^{i+1} \otimes_\Bbbk S$ is given by a matrix whose entries are linear forms in the variables $x_1, \dots, x_n$. If the differential of A is non-zero, though, the entries of δ^i may also have non-zero constant terms. We give a simple example, extracted from [18, 34].

Example 2.11. Let $\bigwedge(a,b)$ be the exterior algebra on generators a,b in degree 1 over a field $\Bbbk$ with $\operatorname{char}(\Bbbk)\neq 2$, and let $A=(\bigwedge(a,b),\mathrm{d})$ be the cdga with differential given by $\mathrm{d}(a)=0$ and $\mathrm{d}(b)=b\cdot a$. Then $\mathrm{MC}(A)=H^1(A)$ is 1-dimensional, generated by a. Writing $S=\Bbbk[x]$, the cochain complex (16) takes the form

$$S \xrightarrow{\delta^0=(\,x\ 0\,)} S^2 \xrightarrow{\delta^1=\left(\begin{smallmatrix}0\\ x-1\end{smallmatrix}\right)} S\,. \tag{19}$$

The relationship between the cochain complexes (8) and (16) is given by the following lemma, which is a slight generalization of a known result (see e.g. [34]).

Lemma 2.12. *The specialization of the cochain complex $(A\otimes_{\Bbbk} S,\delta)$ at an element $a\in \mathrm{MC}(A)$ coincides with the cochain complex (A,δ_a).*

Proof. Write $a=\sum_{j=1}^n a_je_j\in A^1$, and let $\mathfrak{m}_a=(x_1-a_1,\dots,x_n-a_n)$ be the maximal ideal at a. The evaluation map $\mathrm{ev}_a\colon S\to S/\mathfrak{m}_a=\Bbbk$ is the ring morphism given by $g\mapsto g(a_1,\dots,a_n)$. The resulting cochain complex, $A(a)=A\otimes_S S/\mathfrak{m}_a$, has differentials $\delta^i(a)\colon A^i\to A^{i+1}$ given by

$$\begin{aligned}\delta^i(a)(u)&=\sum_{j=1}^n e_ju\otimes \mathrm{ev}_a(x_j)+\mathrm{d}(u)\\ &=\sum_{j=1}^n e_ju\cdot a_j+\mathrm{d}(u)\\ &=a\cdot u+\mathrm{d}(u),\end{aligned}\tag{20}$$

for all $u\in A^i$. Thus, $A(a)=(A,\delta_a)$, as claimed. □

3. Resonance varieties of commutative differential graded algebras

In this section we study the resonance varieties associated to a finite-type commutative differential graded algebra over an arbitrary field.

3.1. Resonance varieties. Let (A,d) be a finite-type cdga over a field $\Bbbk$. Computing the homology of the cochain complexes (A,δ_a) for various values of the parameter $a\in \mathrm{MC}(A)$ and keeping track of the resulting Betti numbers carves out noteworthy subsets of the Maurer–Cartan set of A.

Definition 3.1. The *resonance varieties* (in degree $q\geq 0$ and depth $s\geq 0$) of a finite-type $\Bbbk$-cdga (A,d) are the sets

$$\mathcal{R}^q_s(A)=\{a\in \mathrm{MC}(A)\mid \dim_{\Bbbk} H^q(A,\delta_a)\geq s\}. \tag{21}$$

As we shall see below, these sets are, in fact, Zariski closed subsets of $\mathrm{MC}(A)$. Note that, for a fixed $q\geq 0$, the degree-q resonance varieties form a descending filtration of the Maurer–Cartan set,

$$\mathrm{MC}(A)=\mathcal{R}^q_0(A)\supseteq \mathcal{R}^q_1(A)\supseteq \mathcal{R}^q_2(A)\supseteq\cdots\,. \tag{22}$$

Here is a more concrete description of these varieties, which follows at once from the definitions.

Lemma 3.2. *Fix integers $q\geq 1$ and $s\geq 0$. An element $a\in \mathrm{MC}(A)$ belongs to $\mathcal{R}^q_s(A)$ if and only if there exist elements $u_1,\dots,u_s\in A^q$ such that $au_1+\mathrm{d}(u_1)=\cdots=au_s+\mathrm{d}(u_s)=0$ in A^{q+1}, and the set $\{av+\mathrm{d}(v),u_1,\dots,u_s\}$ is linearly independent in A^q, for all $v\in A^{q-1}$.*

When the algebra A is connected, the differential $\mathrm{d}\colon A^0 \to A^1$ vanishes, and the next lemma readily follows.

Lemma 3.3. *Let (A, d) be a connected, finite-type $\Bbbk$-cdga. Then,*

(1) The point $0 \in \mathrm{MC}(A)$ belongs to $\mathcal{R}^q_s(A)$ if and only if $s \le b_q(A)$.
(2) $\mathcal{R}^0_1(A) = \{0\}$ and $\mathcal{R}^0_s(A) = \emptyset$ for $s > 1$.
(3) A non-zero element $a \in \mathrm{MC}(A)$ belongs to $\mathcal{R}^1_1(A)$ if and only if there is an element $u \in A^1$ not proportional to a and satisfying $au + \mathrm{d}(u) = 0$.
(4) If $\dim_{\Bbbk} A^1 = 1$, then $\mathcal{R}^1_1(A) = \{0\}$ if $b_1(A) = 1$ and $\mathcal{R}^1_1(A) = \emptyset$ if $b_1(A) = 0$.

3.2. Equations for the resonance varieties. Once again, let (A, d) be a finite-type $\Bbbk$-cdga. By definition, an element $a \in \mathrm{MC}(A)$ belongs to $\mathcal{R}^q_s(A)$ if and only if

$$\operatorname{rank} \delta^{q-1}_a + \operatorname{rank} \delta^q_a \le c_q - s, \tag{23}$$

where $c_q = \dim_{\Bbbk} A^q$. Let $r = \Bbbk[x_1, \dots, x_n]$, let $S = R/I$ be the coordinate ring of $\mathrm{MC}(A)$, and let $(A^\bullet \otimes_{\Bbbk} S, \delta_A)$ be the cochain complex of free, finitely generated S-modules from (16). It follows from Lemma 2.12 that

$$\mathcal{R}^q_s(A) = V\Big(I_{c_q - s+1}(\delta^{q-1}_A \oplus \delta^q_A)\Big). \tag{24}$$

Here, $\oplus$ denotes block-sum of matrices; for an $m \times n$ matrix ψ with entries in S, we let $I_r(\psi)$ denote the ideal of $r \times r$ minors of ψ, with the convention that $I_0(\psi) = S$ and $I_r(\psi) = 0$ if $r > \min(m, n)$; finally, $V(\mathfrak{a}) \subset \mathrm{MC}(A)$ denotes the zero set of an ideal $\mathfrak{a} \subset S$. This shows that the resonance sets $\mathcal{R}^q_s(A)$ are indeed subvarieties of the Maurer–Cartan variety $\mathrm{MC}(A) \subset A^1$.

Example 3.4. Let (A, d) be the cdga with $A = \mathbb{Z}_2[a_1, a_2]/(a_1^2, a_2^3, a_1 a_2)$, where $|a_i| = 1$ and with differential given by $\mathrm{d}(a_1) = 0$, $\mathrm{d}(a_2) = a_2^2$. Then $\mathrm{MC}(A)$ is equal to $A^1 = \mathbb{Z}_2^2$, and it follows that $S = \mathbb{Z}_2[x_1, x_2]/(x_1^2 + x_1, x_2^2 + x_2)$, see the discussion from §5.2 below. The chain complex $(A \otimes_{\mathbb{Z}_2} S, \delta)$ now takes the form

$$S \xrightarrow{\delta^0 = (\,x_1 \;\; x_2\,)} S^2 \xrightarrow{\delta^1 = \left(\begin{smallmatrix} 0 \\ x_2+1 \end{smallmatrix}\right)} S,$$

By formula (24), the resonance varieties $\mathcal{R}^1_s(A)$ are the zero loci of the ideals of minors of size $3 - s$ of the matrix

$$\delta^0 \oplus \delta^1 = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_2 + 1 \end{pmatrix}.$$

Thus, $\mathcal{R}^1_0(A) = \mathbb{Z}_2^2$, $\mathcal{R}^1_1(A) = \{x_1 x_2 + x_1 = 0\}$, and $\mathcal{R}^1_2(A) = \emptyset$.

3.3. Naturality properties. The next result describes a (partial) functoriality property enjoyed by the resonance varieties. Other results of this type (in various levels of generality, though not in the generality we work in here) can be found in [18, 35].

Proposition 3.5. *Let $\varphi\colon (A, \mathrm{d}_A) \to (B, \mathrm{d}_B)$ be a morphism of finite-type cdgas. Suppose φ^i is an isomorphism for all $i \le q$ and a monomorphism for $i = q + 1$, for some $q \ge 0$.*

(1) If $q = 0$, the map $\bar{\varphi}\colon \mathrm{MC}(A) \to \mathrm{MC}(B)$ is an embedding which sends $\mathcal{R}^1_s(A)$ into $\mathcal{R}^1_s(B)$ for all $s \ge 0$.
(2) If $q \ge 1$, the map $\bar{\varphi}\colon \mathrm{MC}(A) \to \mathrm{MC}(B)$ is an isomorphism which identifies $\mathcal{R}^i_s(A)$ with $\mathcal{R}^i_s(B)$ for all $i \le q$ and sends $\mathcal{R}^{q+1}_s(A)$ into $\mathcal{R}^{q+1}_s(B)$, for all $s \ge 0$.

Proof. (1) First assume that $q = 0$. By our hypothesis, the map φ^1 is injective, and thus $\bar{\varphi}$ is also injective. Let $a \in \mathrm{MC}(A)$. By Lemma 2.7, part (1), the chain map $\varphi_a\colon (A^\bullet, \delta^A_a) \to (B^\bullet, \delta^B_{\bar{\varphi}(a)})$ induces a monomorphism, $\varphi^1_a\colon H^1(A, \delta^A_a) \to H^1(B, \delta^B_{\bar{\varphi}(a)})$. Now suppose $a \in \mathcal{R}^1_s(A)$, that is, $\dim_{\Bbbk} H^1(A, \delta^A_a) \geq s$; then $H^1(B, \delta^B_{\bar{\varphi}(a)}) \geq s$, and so $\bar{\varphi}(a) \in \mathcal{R}^1_s(B)$. This shows that $\bar{\varphi}(\mathcal{R}^1_s(A)) \subseteq \mathcal{R}^1_s(B)$, and the first claim is proved.

(2) Next, assume that $q \geq 1$. By hypothesis, the map φ^1 is an isomorphism; thus, $\bar{\varphi}$ is also an isomorphism. For each $a \in \mathrm{MC}(A)$, Lemma 2.7 implies that the homomorphism $\varphi^i_a\colon H^i(A, \delta^A_a) \to H^i(B, \delta^B_{\bar{\varphi}(a)})$ is an isomorphism for $i \leq q$ and a monomorphism for $i = q + 1$. The second claim follows. ◻

A similar argument yields the following proposition, which shows that the resonance varieties of a $\Bbbk$-cdga (A, d) only depend on the isomorphism type of A.

Proposition 3.6. *Let $\varphi\colon A \to B$ be an isomorphism of* cdgas*. Then φ restricts to an isomorphism $\bar{\varphi}\colon \mathrm{MC}(A) \to \mathrm{MC}(B)$ which identifies $\mathcal{R}^q_s(A)$ with $\mathcal{R}^q_s(B)$ for all $q, s \geq 0$.*

The conclusions of Proposition 3.6 do not always hold if we only assume that the map $\varphi\colon A \to B$ is a quasi-isomorphism. We give a simple example of this phenomenon, adapted from [18, 34].

Example 3.7. Let $A = (\bigwedge(a, b), \mathrm{d})$ be the cdga from Example 2.11, with $\mathrm{d}(a) = 0$ and $\mathrm{d}(b) = ba$, and let $A' = (\bigwedge(a), 0)$ be the sub-cdga generated by a. It is readily seen that the inclusion map, $\varphi\colon A' \hookrightarrow A$, induces an isomorphism $\varphi^*\colon H^\bullet(A') \xrightarrow{\simeq} H^\bullet(A)$; moreover, φ^1 identifies $\mathrm{MC}(A') = \Bbbk$ with $\mathrm{MC}(A)$. On the other hand, $\mathcal{R}^1_1(A') = \{0\}$ is strictly included in $\mathcal{R}^1_1(A) = \{0, 1\}$.

Noteworthy is the fact that the resonance varieties are compatible with restriction and extension of the base field.

Lemma 3.8. *Let A be a finite-type $\Bbbk$-*cdga*, and let $\Bbbk \subset \mathbb{K}$ be a field extension. Then, for all $q, s \geq 0$,*

(1) $\mathcal{R}^q_s(A) = \mathcal{R}^q_s(A \otimes_{\Bbbk} \mathbb{K}) \cap \mathrm{MC}(A)$.
(2) $\mathcal{R}^q_s(A \otimes_{\Bbbk} \mathbb{K}) = \mathcal{R}^q_s(A) \times_{\Bbbk} \mathbb{K}$.

Proof. By (24), the resonance varieties $\mathcal{R}^q_s(A)$ are determinantal varieties of matrices defined over $\Bbbk$. The two claims follow. ◻

3.4. Resonance varieties of tensor products. Let (A, d_A) and (B, d_B) be two $\Bbbk$-cdgas. The tensor product of these two $\Bbbk$-vector spaces, $A \otimes_{\Bbbk} B$, has a natural cdga structure, with grading $(A \otimes_{\Bbbk} B)^q = \bigoplus_{i+j=q} A^i \otimes_{\Bbbk} B^j$, multiplication $(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|}(ab \otimes a'b')$, and differential D given on homogeneous elements by $D(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b)$.

If both A and B are connected, finite-type cdgas then clearly $A \otimes_{\Bbbk} B$ is also a connected, finite-type cdga. Moreover, upon identifying $(A \otimes_{\Bbbk} B)^1$ with $A^1 \oplus B^1 = A^1 \times B^1$, we have

$$\mathrm{MC}(A \otimes_{\Bbbk} B) = \mathrm{MC}(A) \times \mathrm{MC}(B). \tag{25}$$

The resonance varieties of a tensor product of cdgas obey a type of Künneth formula. Such formulas were obtained in [24, 38, 35] under the assumption that the differentials vanish; a proof of one inclusion was given in full generality in [25]. We give here a proof which is both complete and in full generality.

Proposition 3.9. *Let (A, d_A) and (B, d_B) be two connected, finite-type* cdgas*. Then,*

(1) $\mathcal{R}^q_1(A \otimes_{\Bbbk} B) = \bigcup_{i+j=q} \mathcal{R}^i_1(A) \times \mathcal{R}^j_1(B)$, for all $q \geq 1$.

(2) $\mathcal{R}^1_s(A \otimes_{\Bbbk} B) = \mathcal{R}^1_s(A) \times \{0\} \cup \{0\} \times \mathcal{R}^1_s(B)$*, for all* $s \geq 1$.

Proof. Set $C = A \otimes_{\Bbbk} B$, and let $c = (a, b)$ be an element in $\mathrm{MC}(C) = \mathrm{MC}(A) \times \mathrm{MC}(B)$. The cochain complex (C, δ^C_c) splits as a tensor product of cochain complexes, $(A, \delta^A_a) \otimes_{\Bbbk} (B, \delta^B_b)$. Therefore,

$$b_q(C, c) = \sum_{i+j=q} b_i(A, a) \cdot b_j(B, b), \tag{26}$$

and part (1) follows. When $q = 1$, we obtain from (26) the following equalities:

$$\begin{aligned} &b_1(C, (0, 0)) = b_1(A, 0) + b_1(B, 0), \\ &b_1(C, (a, 0)) = b_1(A, a) && \text{if } a \neq 0, \\ &b_1(C, (0, b)) = b_1(B, b) && \text{if } b \neq 0, \\ &b_1(C, (a, b)) = 0 && \text{if } a \neq 0 \text{ and } b \neq 0. \end{aligned} \tag{27}$$

Part (2) readily follows. ◻

3.5. Resonance varieties of coproducts. Let (A, d_A) and (B, d_B) be two connected cdgas. Their wedge sum, $A \vee B$, is a new connected cdga, whose underlying graded vector space in positive degrees is the direct sum $A^+ \oplus B^+$, whose multiplication is given by $(a, b) \cdot (a', b') = (aa', bb')$, and whose differential is $D = d_A + d_B$. Note that $(A \vee B)^1 = A^1 \oplus B^1$; therefore, we may identify the Maurer–Cartan set $\mathrm{MC}(A \vee B)$ with the product $\mathrm{MC}(A) \times \mathrm{MC}(B)$.

The following proposition recovers a result proved in various levels of generality in [24, 25, 38, 35].

Proposition 3.10. *Let* $C = A \vee B$ *be the coproduct of two connected, finite-type* cdgas *with* $b_1(A) > 0$ *and* $b_1(B) > 0$. *Then, we have for all* $s \geq 1$

$$\mathcal{R}^q_s(A \vee B) = \begin{cases} \bigcup_{j+k=s-1} \mathcal{R}^1_j(A) \times \mathcal{R}^1_k(B) & \text{if } q = 1, \\ \bigcup_{j+k=s} \mathcal{R}^q_j(A) \times \mathcal{R}^q_k(B) & \text{if } q \geq 2. \end{cases} \tag{28}$$

Proof. Let $c = (a, b)$ be an element in $\mathrm{MC}(C) = \mathrm{MC}(A) \times \mathrm{MC}(B)$. The chain complex (C, c) splits (in positive degrees) as a direct sum of chain complexes, $(C^+, \delta^C_c) \cong (A^+, \delta^A_a) \oplus (B^+, \delta^B_b)$. The Betti numbers of the respective chain complexes are related, as follows:

$$b_q(C, c) = \begin{cases} b_q(A, a) + b_q(B, b) + 1 & \text{if } q = 1 \text{ and } a \neq 0,\, b \neq 0, \\ b_q(A, a) + b_q(B, b) & \text{otherwise.} \end{cases} \tag{29}$$

The claim now follows by a case-by-case analysis of formula (29). ◻

4. Poincaré duality algebras and resonance varieties

In this section we study the interplay between Poincaré duality, the Koszul complex, and the resonance varieties of a differential graded algebra.

4.1. Poincaré duality. We start with a basic algebraic concept that abstracts the classical notion of Poincaré duality for manifolds. For more information on this subject, we refer to [21, 30, 35].

Definition 4.1. Let A be a connected, finite-type CGA over a field $\Bbbk$. We say that A is a *Poincaré duality $\Bbbk$-algebra* of formal dimension m (for short, an m-PDA) if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that the bilinear forms

$$A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk, \quad a\otimes b \mapsto \varepsilon(ab) \tag{30}$$

are non-singular, for all $i \geq 0$.

It follows straight from the definition that the map ε is an isomorphism, and that $A^i = 0$ for $i > m$. Furthermore, for each $0 \leq i \leq m$, there is an isomorphism

$$\mathrm{PD}^i\colon A^i \to (A^{m-i})^*, \quad \mathrm{PD}^i(a)(b) = \varepsilon(ab). \tag{31}$$

Consequently, each element $a \in A^i$ has a Poincaré dual, $a^\vee \in A^{m-i}$, which is uniquely determined by the formula $\varepsilon(aa^\vee) = 1$. We define the orientation class $\omega_A \in A^m$ as the Poincaré dual of $1 \in A^0$, that is, $\omega_A = 1^\vee$. Conversely, a choice of orientation class $\omega_A \in A^m$ defines an orientation $\varepsilon\colon A^m \to \Bbbk$ by setting $\varepsilon(\omega_A) = 1$.

Definition 4.2. A $\Bbbk$-CDGA $(A^\bullet, \mathrm{d})$ is a *Poincaré duality differential graded algebra* of formal dimension m (for short, an m-PD-CDGA) if

(1) The underlying graded algebra $A^\bullet$ is an m-PDA.
(2) $\mathrm{d}(A^{m-1}) = 0$.

Condition (2) can also be stated as $\varepsilon(\mathrm{d}(u)) = 0$ for all $u \in A^{m-1}$. By condition (1), the algebra A is connected and $A^m \cong A^0$; thus, condition (2) is equivalent to $H^m(A) = \Bbbk$. As noted in [14], if $(A^\bullet, \mathrm{d})$ is an m-PD-CDGA, then $H^\bullet(A)$ is an m-PDA, with orientation $[\varepsilon]\colon H^m(A) \to \Bbbk$ given by $[\varepsilon]([u]) = \varepsilon(u)$ for every $u \in Z^m(A)$. Lambrechts and Stanley also showed in [15] that if A is a CDGA such that $H^1(A) = 0$ and $H^\bullet(A)$ is an m-PDA, then A is weakly equivalent to an m-PD-CDGA.

Clearly, if A is an m-PDA, then A endowed with the zero differential is an m-PD-CDGA. Also, if (A, d) is a CDGA such that $A^1 = 0$ and A is an m-PDA, then $A^{m-1} = 0$, and so (A, d) is m-PD-CDGA. We give a simple example of a PD-CDGA (A, d) with $\mathrm{d} \neq 0$ and $A^1 \neq 0$.

Example 4.3. Let $A = (\mathbb{Z}_2[x]/(x^{2k}), \mathrm{d})$, where $|x| = 1$, $k \geq 2$, and $\mathrm{d}(x) = x^2$. Clearly, A is a Poincaré duality algebra of dimension $m = 2k - 1$; since $\mathrm{d}(x^{2k-2}) = 0$, we have that A is an m-PD-CDGA.

Next, we provide an example of a CDGA (A, d) such that A is an m-PDA, yet condition (2) from Definition 4.2 is not satisfied.

Example 4.4. Let $A = (\mathbb{Z}_2[x, y]/(x^2, y^2), \mathrm{d})$ with $|x| = 1$, $|y| = k$, and differential given by $\mathrm{d}(x) = 0$, $\mathrm{d}(y) = xy$. Clearly, the underlying graded algebra $A^\bullet$ is a $\mathbb{Z}_2$-Poincaré duality algebra of dimension $k + 1$, with orientation class $\omega_A = xy$. Nevertheless, $\mathrm{d}(y) \neq 0$, and so (A, d) is not a PD-CDGA.

4.2. Resonance varieties of PD-CDGAs. Poincaré duality imposes some rather stringent constraints on the resonance varieties of a PD-CDGA A. To ascertain those constraints, we start by analyzing the boundary maps in the Aomoto complexes (A, δ_a). Versions of the next lemma were given in [27, Lemma 7.4] when $\mathrm{char}(\Bbbk) = 0$ and in [35, Lemma 5.1] when $\mathrm{char}(\Bbbk) \neq 2$ and $\mathrm{d} = 0$. For completeness and to fix a sign issue in the latter reference, we give a self-contained proof here, valid in full generality.

LEMMA 4.5. *Let $(A^\bullet, \mathrm{d})$ be an m-PD-CDGA over an arbitrary field $\Bbbk$. Then, for all $a \in A^1$ and all $0 \leq i \leq m$, we have a commuting square,*

$$\begin{tikzcd} (A^{m-i})^* \arrow[r, "(\delta_{-a}^{m-i-1})^*"] & (A^{m-i-1})^* \\ A^i \arrow[u, "\cong"', "\mathrm{PD}^i"] \arrow[r, "\delta_a^i"] & A^{i+1} . \arrow[u, "\cong"', "(-1)^{i+1}\,\mathrm{PD}_{i+1}"] \end{tikzcd} \tag{32}$$

PROOF. Let $b \in A^i$ and $c \in A^{m-i-1}$. Then

$$\begin{aligned} (-1)^{i+1}\,\mathrm{PD}^{i+1} \circ \delta_a^i(b)(c) &= (-1)^{i+1}\,\mathrm{PD}^{i+1}(ab + \mathrm{d}(b))(c) \\ &= (-1)^{i+1}\varepsilon(abc + \mathrm{d}(b)c), \end{aligned} \tag{33}$$

while

$$\begin{aligned} (\delta_{-a}^{m-i-1})^* \circ \mathrm{PD}^i(b)(c) &= \mathrm{PD}^i(b)(\delta_{-a}^{m-i-1}(c)) \\ &= \mathrm{PD}^i(b)(-ac + \mathrm{d}(c)) \\ &= \varepsilon(-bac + b\,\mathrm{d}(c)). \end{aligned} \tag{34}$$

Now, since A is a CDGA, we have that $ab = (-1)^i ba$ and $\mathrm{d}(b)c + (-1)^i b\,\mathrm{d}(c) = \mathrm{d}(bc)$. Moreover, since A is an m-PD-CDGA and $bc \in A^{m-1}$, we have that $\mathrm{d}(bc) = 0$, and the claim follows. □

We are now in a position to state and prove the main result of this section, which recovers results from [27, 35] and extends them to our present context. First recall from Lemma 2.4 that the Maurer–Cartan set $\mathrm{MC}(A)$ is invariant under the involution of A^1 sending a to $-a$.

THEOREM 4.6. *Let $(A^\bullet, \mathrm{d})$ be a Poincaré duality* CDGA *of formal dimension m over a field $\Bbbk$. Then,*

(1) $H^i(A, \delta_a)^* \cong H^{m-i}(A, \delta_{-a})$ for all $a \in \mathrm{MC}(A)$ and $i \geq 0$.
(2) The linear isomorphism $A^1 \xrightarrow{\simeq} A^1$, $a \mapsto -a$ restricts to isomorphisms $\mathcal{R}_s^i(A) \xrightarrow{\simeq} \mathcal{R}_s^{m-i}(A)$ for all $i, s \geq 0$.
(3) $\mathcal{R}_1^m(A) = \{0\}$.

PROOF. Part (1) is a direct consequence of Lemma 4.5. Part (2) follows from (1). Finally, Part (3) follows from (2) and the fact that A is connected, and so $\mathcal{R}_1^0(A) = \{0\}$. □

5. The Aomoto–Bockstein complex

We now use the mod-2 cohomology algebra of a topological space X, equipped with the differential given by the Bockstein operator to define the Aomoto–Bockstein complex of X.

5.1. The Bockstein operator. We start by reviewing some standard material on Bocksteins, following the treatment in Hatcher [11, §3.E]. For each prime p, let $A = H^\bullet(X, \mathbb{Z}_p)$ be the cohomology algebra of X with $\mathbb{Z}_p$ coefficients, with multiplication given by the cup-product. For each $q \geq 0$, we have a *Bockstein operator*, $\beta_p\colon H^q(X, \mathbb{Z}_p) \to H^{q+1}(X, \mathbb{Z}_p)$, which is defined as the coboundary homomorphism associated to the coefficient exact sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p \longrightarrow 0. \tag{35}$$

Alternatively, if $\beta_0\colon H^q(X,\mathbb{Z}_p) \to H^{q+1}(X,\mathbb{Z})$ is the coboundary map associated to the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$ and $\rho_p\colon H^\bullet(X,\mathbb{Z}) \to H^\bullet(X,\mathbb{Z}_p)$ is the morphism induced by the projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_p$, then $\beta_p = \rho_p \circ \beta_0$.

Note that a cohomology class $u \in H^q(X,\mathbb{Z}_p)$ is the reduction mod-p of a cohomology class in $H^q(X,\mathbb{Z}_{p^2})$ if and only if $\beta_p(u) = 0$. Consequently, if $u = \rho_p(v)$ for some $v \in H^q(X,\mathbb{Z})$, then $\beta_p(u) = 0$. Hence, if all the integral cohomology groups of X are torsion-free, the Bockstein β_p vanishes identically.

The Bockstein operators in all degrees $q \geq 0$ assemble into a $\mathbb{Z}_p$-linear map, $\beta_p\colon A \to A$, with the properties that $\beta_p \circ \beta_p = 0$ and $\beta_p(a \cup b) = \beta_p(a) \cup b + (-1)^{|a|} a \cup \beta_p(b)$ for all homogeneous elements $a, b \in A$. Therefore, β_p is a graded derivation of the algebra A and the pair (A,β_p) is a cdga. Furthermore, the Bockstein is functorial: if $f\colon X \to Y$ is a continuous map, and $f^*\colon H^\bullet(Y,\mathbb{Z}_p) \to H^\bullet(X,\mathbb{Z}_p)$ is the induced homomorphism in cohomology, then $\beta^X_p(f^*(u)) = f^*(\beta^Y_p(u))$, for all $u \in H^\bullet(Y,\mathbb{Z}_p)$.

The homology groups of the Bockstein cochain complex (A,β_p), denoted $BH^q(X,\mathbb{Z}_p)$, are called the *Bockstein cohomology groups* of X (with coefficients in $\mathbb{Z}_p$). If $H^\bullet(X,\mathbb{Z})$ is of finite-type, then each $\mathbb{Z}$-summand of $H^q(X,\mathbb{Z})$ contributes a $\mathbb{Z}_p$-summand to $BH^q(X,\mathbb{Z}_p)$; the $\mathbb{Z}_p$-summands do not contribute to Bockstein cohomology; and each $\mathbb{Z}_{p^k}$-summand of $H^q(X,\mathbb{Z})$ with $k > 1$ contributes a $\mathbb{Z}_p$-summand to both $BH^{q-1}(X,\mathbb{Z}_p)$ and $BH^q(X,\mathbb{Z}_p)$. In particular, $b_q(X) \leq \dim_{\mathbb{Z}_p} BH^q(X,\mathbb{Z}_p)$.

When $p = 2$, the Bockstein operator β_2 may be identified with the Steenrod square operation $\operatorname{Sq}^1\colon H^\bullet(X,\mathbb{Z}_2) \to H^\bullet(X,\mathbb{Z}_2)$. The fact that $\beta_2 = \operatorname{Sq}^1$ is a derivation may be seen as a consequence of the Ádem relation $\operatorname{Sq}^1 \circ \operatorname{Sq}^1 = 0$ and the Cartan relation $\operatorname{Sq}^1(a \cup b) = \operatorname{Sq}^1(a) \cup \operatorname{Sq}^0(b) + \operatorname{Sq}^0(a) \cup \operatorname{Sq}^1(b)$, where $\operatorname{Sq}^0 = \mathrm{id}$.

5.2. The Aomoto–Bockstein cochain complex. Assume now that X is a connected, finite-type CW-complex, and consider the cohomology algebra $A = H^\bullet(X,\mathbb{Z}_2)$, endowed with the differential given by the Bockstein operator $\beta_2 = \operatorname{Sq}^1$. Since $\operatorname{Sq}^1(a) = a^2$ for all $a \in A^1$, the Maurer–Cartan set for the cdga (A,β_2) is then $\operatorname{MC}(A) = A^1$.

Definition 5.1. The *Aomoto–Bockstein complex* of $A = H^\bullet(X,\mathbb{Z}_2)$ with respect to an element $a \in A^1$ is the cochain complex of finite-dimensional $\mathbb{Z}_2$-vector spaces,

$$(A,\delta_a)\colon\ A^0 \xrightarrow{\delta_a} A^1 \xrightarrow{\delta_a} \cdots \xrightarrow{\delta_a} A^q \xrightarrow{\delta_a} A^{q+1} \xrightarrow{\delta_a} \cdots, \tag{36}$$

where $\delta_a(u) = au + \beta_2(u)$.

Note that for $a = 0$ this is the usual Bockstein cochain complex, (A,β_2); therefore, $H^q(A,\delta_0) = BH^q(X,\mathbb{Z}_2)$. On the other hand, if the Bockstein operator vanishes identically, then (36) is the usual Aomoto complex, with differentials $\delta_a(u) = au$.

Here is an alternative (and, in some ways, more convenient) interpretation. We pick a basis $\{e_1,\dots,e_n\}$ for $A^1 = H^1(X,\mathbb{Z}_2)$, we let $\{x_1,\dots,x_n\}$ be the Kronecker dual basis for $A_1 = H_1(X,\mathbb{Z}_2)$, and we identify the symmetric algebra $\operatorname{Sym}(A_1)$ with the polynomial ring $\mathbb{Z}_2[x_1,\dots,x_n]$. The coordinate ring of A^1 is then the quotient ring

$$S = \mathbb{Z}_2[x_1,\dots,x_n]/(x_1^2 + x_1,\dots,x_n^2 + x_n), \tag{37}$$

which may be viewed as the ring of (Boolean) functions on $\mathbb{Z}_2^n$. For detailed information on how to compute with polynomials, ideals, Gröbner bases, and varieties over this ring, we refer to [1, 2, 17].

Definition 5.2. The *universal Aomoto–Bockstein complex* of the cohomology algebra $A = H^{\bullet}(X, \mathbb{Z}_2)$ is the cochain complex of free S-modules,

(38)
$$(A \otimes_{\mathbb{Z}_2} S, \delta)\colon\ A^0 \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^0} A^1 \otimes_{\mathbb{Z}_2} S \longrightarrow \cdots \longrightarrow A^i \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^i} A^{i+1} \otimes_{\mathbb{Z}_2} S \longrightarrow \cdots,$$

where the differentials are defined by

$$\delta^i(u \otimes 1) = \sum_{j=1}^{n} e_j u \otimes x_j + \beta_2(u) \otimes 1 \tag{39}$$

for $u \in A^i$, and then extended by S-linearity.

If the Bockstein operator acts trivially on $H^{\bullet}(X, \mathbb{Z}_2)$, then each boundary map δ^i in the cochain complex (38) is given by a matrix whose entries are linear forms in the variables $x_1, \dots, x_n$. In general, though, the entries of δ^i may also have non-zero constant terms.

Example 5.3. Let $X = \mathbb{RP}^{\infty}$ be the infinite-dimensional real projective space. We identify $H^{\bullet}(X, \mathbb{Z}_2) = \mathbb{Z}_2[a]$ where $|a| = 1$ and note that the Bockstein β_2 sends a^i to a^{i+1} if i is odd and to 0 otherwise. Writing $S = \mathbb{Z}_2[x]/(x^2 + x)$, the resulting Aomoto–Bockstein complex has boundary maps $\delta^i\colon S \to S$ given by $\delta^i(a^i) = (x + 1)a^{i+1}$ if i is odd and $\delta^i(a^i) = xa^{i+1}$ if i is even, and thus has the form

$$S \xrightarrow{x} S \xrightarrow{x+1} S \xrightarrow{x} S \longrightarrow \cdots. \tag{40}$$

Note that this cochain complex is exact.

5.3. The twisted Bockstein operator. To conclude this section, we present an alternative interpretation of the differentials in the Aomoto–Bockstein complex (36), due to Samelson [29] and Greenblatt [10].

Fix a basepoint $x_0 \in X$ and let $\pi = \pi_1(X, x_0)$ be the fundamental group of X at x_0. Every mod 2 cohomology class $w \in H^1(X, \mathbb{Z}_2)$ defines a local system $\mathbb{Z}_w$ on X, as follows. Viewing w as a homomorphism from $H_1(X, \mathbb{Z})$ to $\mathbb{Z}_2$, and identifying the target group with $\{\pm 1\}$, regarded as the units of the ring $\mathbb{Z}$ puts a $\mathbb{Z}[H_1(X, \mathbb{Z})]$-module structure on the additive group $\mathbb{Z}$. Restricting scalars to the group ring $\mathbb{Z}[\pi]$ via the abelianization map $\pi \twoheadrightarrow \pi_{\mathrm{ab}} = H_1(X, \mathbb{Z})$ yields a $\mathbb{Z}[\pi]$-module structure on $\mathbb{Z}$, which is the module that we denote by $\mathbb{Z}_w$.

The twisted Bockstein operator $\beta_w\colon H^q(X, \mathbb{Z}_2) \to H^{q+1}(X, \mathbb{Z}_w)$ is defined as the coboundary map induced by the exact sequence

$$0 \longrightarrow \mathbb{Z}_w \xrightarrow{\times 2} \mathbb{Z}_w \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{41}$$

Let $\bar{\rho}_2\colon H^{\bullet}(X, \mathbb{Z}_w) \to H^{\bullet}(X, \mathbb{Z}_2)$ be the coefficient morphism induced by the projection $\mathbb{Z}_w \twoheadrightarrow \mathbb{Z}_2$, and set $\bar{\beta}_w \coloneqq \bar{\rho}_2 \circ \beta_w\colon H^q(X, \mathbb{Z}_2) \to H^{q+1}(X, \mathbb{Z}_2)$. By standard properties of the Bockstein operator, we have that $\bar{\beta}_w^2 = 0$. Furthermore, as shown in [10], the map

$$\bar{\beta}\colon H^1(X, \mathbb{Z}_2) \times H^q(X, \mathbb{Z}_2) \longrightarrow H^{q+1}(X, \mathbb{Z}_2) \tag{42}$$

given by $\bar{\beta}(w, u) = \bar{\beta}_w(u)$ is a cohomology operation. That is, if $f\colon X \to Y$ is a continuous map, and $f^*\colon H^{\bullet}(Y, \mathbb{Z}_2) \to H^{\bullet}(X, \mathbb{Z}_2)$ is the induced homomorphism in cohomology, then, for every $w \in H^1(X, \mathbb{Z}_2)$ and $u \in H^q(Y, \mathbb{Z}_2)$, we have that

$$f^*(\bar{\beta}(w, u)) = \bar{\beta}(f^*(w), f^*(u)). \tag{43}$$

Theorem 5.4 ([29, 10]). *For every $w \in H^1(X, \mathbb{Z}_2)$ and $u \in H^q(X, \mathbb{Z}_2)$, we have that $\bar{\beta}(w, u) = wu + \mathrm{Sq}^1(u)$.*

6. The resonance varieties of a space

In this section we discuss the resonance varieties of a space, with coefficients in a field $\Bbbk$. Throughout, X will be a connected, finite-type CW-complex and $A = H^{\bullet}(X, \Bbbk)$ will be its cohomology algebra over $\Bbbk$, with multiplication given by the cup-product.

6.1. The usual resonance varieties. We start by singling out certain situations in which all cohomology classes in degree 1 square to zero. The next (standard) lemma formalizes arguments from [**23**, **24**].

Lemma 6.1. *Suppose either* $\operatorname{char}(\Bbbk) \neq 2$*, or* $\operatorname{char}(\Bbbk) = 2$ *and* $H_1(X, \mathbb{Z})$ *has no* 2*-torsion. Then* $a^2 = 0$ *for all* $a \in H^1(X, \Bbbk)$*.*

Proof. By graded-commutativity of the cup-product, $a^2 = -a^2$. Thus, if $\operatorname{char}(\Bbbk) \neq 2$, then $a^2 = 0$.

Now suppose $\operatorname{char}(\Bbbk) = 2$ and $H_1(X, \mathbb{Z})$ has no 2-torsion. In this case, by the Universal Coefficients Theorem, every class $a \in H^1(X, \Bbbk)$ is the image of a class $\tilde{a} \in H^1(X, \mathbb{Z})$ under the coefficient homomorphism $\mathbb{Z} \to \Bbbk$. By obstruction theory, there is a map $f \colon X \to S^1$ and a class $\omega \in H^1(S^1, \mathbb{Z})$ such that $\tilde{a} = f^*(\omega)$. Hence, $\tilde{a} \cup \tilde{a} = f^*(\omega \cup \omega) = 0$. The claim then follows by naturality of cup products with respect to coefficient homomorphisms. $\square$

Under the assumptions of Lemma 6.1, we shall view the cohomology algebra $H^{\bullet}(X, \Bbbk)$ as a $\Bbbk$-cdga with differential $d = 0$ and we shall identify the Maurer–Cartan set $\operatorname{MC}(A)$ with the affine space $A^1 = H^1(X, \Bbbk)$.

Definition 6.2. Let X be a connected, finite-type CW-complex, let $\Bbbk$ be a field, and suppose either $\operatorname{char}(\Bbbk) \neq 2$, or $\operatorname{char}(\Bbbk) = 2$ and $H_1(X, \mathbb{Z})$ has no 2-torsion. The *resonance varieties* of X (over $\Bbbk$) are the resonance varieties of the cdga $A = (H^*(X, \Bbbk), 0)$; that is,

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in H^1(X, \Bbbk) \mid \dim_{\Bbbk} H^q(A, \delta_a) \geq s\}, \tag{44}$$

where $\delta_a \colon A^q \to A^{q+1}$ is given by $\delta_a(u) = au$.

These sets are homogeneous algebraic subvarieties of the affine space $H^1(X, \Bbbk)$. For each $q \geq 0$, they form a descending filtration,

$$H^1(X, \Bbbk) = \mathcal{R}^q_0(X, \Bbbk) \supseteq \mathcal{R}^q_1(X, \Bbbk) \supseteq \mathcal{R}^q_2(X, \Bbbk) \supseteq \cdots \supseteq \mathcal{R}^q_{b_q}(X, \Bbbk) \supseteq \{0\}, \tag{45}$$

where $b_q = b_q(X, \Bbbk) := \dim_{\Bbbk} H^q(X, \Bbbk)$; moreover, $\mathcal{R}^q_{b_q+1}(X, \Bbbk) = \emptyset$. By construction, the resonance varieties of X depend only on the cohomology algebra $H^{\bullet}(X, \Bbbk)$. We illustrate the concept with some well-known examples, see e.g. [**32**, **34**].

Example 6.3. Let T^n be the n-dimensional torus. Using the exactness of the Koszul complex from Example 2.10, we see that $\mathcal{R}^i_s(T^n, \Bbbk)$ is equal to $\{0\}$ if $0 < s \leq \binom{n}{i}$ and is empty for $s > \binom{n}{i}$.

Example 6.4. Let M_g be the orientable surface of genus $g > 1$. In depth $s > 0$ the resonance varieties are then given by

$$\mathcal{R}^q_s(M_g, \Bbbk) = \begin{cases} \Bbbk^{2g} & \text{if } q = 1,\ s < 2g-1, \\ \{0\} & \text{if } q = 1,\ s \in \{2g-1, 2g\} \text{ or } q \in \{0, 2\},\ s = 1, \\ \emptyset & \text{otherwise.} \end{cases} \tag{46}$$

Best understood are the degree 1, depth 1 resonance varieties $\mathcal{R}^1_1(X,\Bbbk)$. For low values of $n = b_1(X,\Bbbk)$, these varieties are easy to describe. If $n = 0$, then $\mathcal{R}^1_1(X,\Bbbk) = \emptyset$, while if $n = 1$, then $\mathcal{R}^1_1(X,\Bbbk) = \{0\}$. If $n = 2$, then $\mathcal{R}^1_1(X,\Bbbk) = \Bbbk^2$ or $\{0\}$, according to whether the cup product vanishes on $H^1(X,\Bbbk)$ or not. For $n \geq 3$, the variety $\mathcal{R}^1_1(X,\Bbbk)$ can be much more complicated; in particular, it may have irreducible components which are not linear subspaces.

Example 6.5. Let $X = \mathrm{Conf}(E,3)$ be the configuration space of 3 labeled points on a torus. Then $H^\bullet(X,\Bbbk)$ is the exterior algebra on generators a_1,a_2,a_3,b_1,b_2,b_3 in degree 1, modulo the ideal generated by$(a_1-a_2)(b_1-b_2)$, $(a_1-a_3)(b_1-b_3)$, and $(a_2-a_3)(b_2-b_3)$. A calculation reveals that

$$\mathcal{R}^1_1(X,\Bbbk) = \{(a,b) \in \Bbbk^6 \mid a_1+a_2+a_3 = b_1+b_2+b_3 = a_1b_2 - a_2b_1 = 0\}.$$

Hence, $\mathcal{R}^1_1(X,\Bbbk)$ is isomorphic to $\{a_1b_2 - a_2b_1 = 0\}$, a smooth quadric hypersurface in $\Bbbk^4$.

6.2. Resonance varieties in characteristic 2. We now treat the case when $\mathrm{char}(\Bbbk) = 2$, without imposing any restriction on the 2-torsion of $H_1(X,\mathbb{Z})$.

Definition 6.6. The *Bockstein resonance varieties* of a connected, finite-type CW-complex X are the resonance varieties of the cdga $A = (H^\bullet(X,\mathbb{Z}_2),\beta_2)$; that is,

$$\widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2) = \{a \in H^1(X,\mathbb{Z}_2) \mid \dim_{\mathbb{Z}_2} H^q(A,\delta_a) \geq s\}, \tag{47}$$

where $\beta_2 = \mathrm{Sq}^1\colon A^q \to A^{q+1}$ is the Bockstein operator and $\delta_a\colon A^q \to A^{q+1}$ is given by $\delta_a(u) = au + \beta_2(u)$.

More generally, we define the Bockstein resonance varieties of X over a field $\Bbbk$ of characteristic 2 as

$$\widetilde{\mathcal{R}}^q_s(X,\Bbbk) = \widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2) \times_{\mathbb{Z}_2} \Bbbk\,. \tag{48}$$

These sets are algebraic subvarieties of the affine space $H^1(X,\Bbbk)$. For each $q \geq 0$, they form a descending filtration,

$$H^1(X,\Bbbk) = \widetilde{\mathcal{R}}^q_0(X,\Bbbk) \supseteq \widetilde{\mathcal{R}}^q_1(X,\Bbbk) \supseteq \widetilde{\mathcal{R}}^q_2(X,\Bbbk) \supseteq \cdots\,. \tag{49}$$

Since all the essential features of the resonance varieties in characteristic 2 already appear over the field with 2 elements, we will concentrate mainly on the sets $\widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2)$.

6.3. Properties of the Bockstein resonance varieties. As a direct consequence of Lemma 3.3, we have the following result.

Lemma 6.7. *Let X be a connected, finite-type CW-complex. Then,*

(1) The element $0 \in H^1(X,\mathbb{Z}_2)$ belongs to $\widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2)$ if and only if $\dim_{\mathbb{Z}_2} BH^q(X,\mathbb{Z}_2) \geq s$.

(2) $\widetilde{\mathcal{R}}^0_1(X,\mathbb{Z}_2) = \{0\}$ and $\widetilde{\mathcal{R}}^0_s(X,\mathbb{Z}_2) = \emptyset$ for $s > 1$.

Since $\mathrm{Sq}^1(u) = u^2$ for all $u \in H^1(X,\mathbb{Z}_2)$, the varieties $\widetilde{\mathcal{R}}^1_s(X,\mathbb{Z}_2)$ depend only on the truncated cohomology ring $H^{\leq 2}(X,\mathbb{Z}_2)$. In depth $s = 1$, we can be even more concrete.

Lemma 6.8. *Let X be a connected, finite-type CW-complex. Then,*

(1) $0 \in \widetilde{\mathcal{R}}^1_1(X,\mathbb{Z}_2)$ if and only if there is an element $u \in H^1(X,\mathbb{Z}_2)$ such that $u^2 = 0$.

(2) A non-zero element $a \in H^1(X,\mathbb{Z}_2)$ belongs to $\widetilde{\mathcal{R}}^1_1(X,\mathbb{Z}_2)$ if and only if there is an element $u \in H^1(X,\mathbb{Z}_2)$ such that $au + u^2 = 0$ and u is not proportional to a.

For instance, if $H^1(X,\mathbb{Z}_2)=\{0,a\}$, then $\widetilde{\mathcal{R}}^1_1(X,\mathbb{Z}_2)$ is either equal to $\{0\}$ or is empty, according to whether $a^2=0$ or not.

It follows from Proposition 3.5, part (1) that both types of degree 1 resonance varieties, $\mathcal{R}^1_s(X,\Bbbk)$ and $\widetilde{\mathcal{R}}^1_s(X,\Bbbk)$, enjoy a (partial) naturality property.

Proposition 6.9. *Let $f\colon X\to Y$ be a continuous map, let $\Bbbk$ be a field, and suppose that the induced homomorphism $f^*\colon H^1(Y,\Bbbk)\to H^1(X,\Bbbk)$ is injective.*

(1) If either $\operatorname{char}(\Bbbk)\neq 2$, or $\operatorname{char}(\Bbbk)=2$ and both $H_1(X,\mathbb{Z})$ and $H_1(Y,\mathbb{Z})$ have no 2-torsion, then f^* restricts to embeddings $\mathcal{R}^1_s(Y,\Bbbk)\hookrightarrow\mathcal{R}^1_s(X,\Bbbk)$ for all $s\geq 1$.

(2) If $\operatorname{char}(\Bbbk)=2$, then f^* restricts to embeddings $\widetilde{\mathcal{R}}^1_s(Y,\Bbbk)\hookrightarrow\widetilde{\mathcal{R}}^1_s(X,\Bbbk)$ for all $s\geq 1$.

6.4. Sample computations of resonance. We now give several examples illustrating the computation of the Bockstein resonance varieties $\widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2)$ for some familiar spaces X. For simplicity, we will assume throughout that $s>0$.

Example 6.10. Let N_g be the non-orientable surface of genus $g\geq 1$. Then $H^{\bullet}(N_g,\mathbb{Z}_2)=\mathbb{Z}_2[a_1,\dots,a_g]/I$, where $|a_i|=1$ and I is the ideal generated by $a_i^2+a_j^2$ and a_ia_j for all $1\leq i<j\leq g$. Direct computation shows that

$$\widetilde{\mathcal{R}}^q_s(N_g,\mathbb{Z}_2)=\begin{cases}\{0,a_1+\cdots+a_g\} & \text{if } q=1 \text{ and } s=1,\\ \{0\} & \text{if } q=0 \text{ or } 2 \text{ and } s=1,\\ \emptyset & \text{otherwise.}\end{cases}\tag{50}$$

Example 6.11. Let $X=\mathbb{RP}^n$ be the real projective space of dimension n. Then $H^{\bullet}(\mathbb{RP}^n,\mathbb{Z}_2)=\mathbb{Z}_2[a]/(a^{n+1})$, with a in degree 1, and $\beta_2(a^q)=a^{q+1}$ if q is odd, and 0 otherwise. Thus, $BH^q(\mathbb{RP}^n,\mathbb{Z}_2)=0$ for $0<q<n$; moreover, $\delta_a(a^q)=a^{q+1}$ if a is even, and 0 otherwise. It follows that

$$\widetilde{\mathcal{R}}^q_s(\mathbb{RP}^n,\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{if } q=n,\ n \text{ is even, and } s=1,\\ \{0\} & \text{if } q=0 \text{ and } s=1,\\ \emptyset & \text{otherwise.}\end{cases}\tag{51}$$

Example 6.12. Let $\mathrm{Gr}_n=\mathrm{Gr}_n(\mathbb{R}^\infty)$ be the Grassmannian of (unoriented) n-planes in $\mathbb{R}^\infty$. Then $H^{\bullet}(\mathrm{Gr}_n,\mathbb{Z}_2)=\mathbb{Z}_2[w_1,\dots,w_n]$, where w_k is the (universal) Stiefel–Whitney class of degree k, and the Bockstein operator $\beta_2=\mathrm{Sq}^1$ is given by Wu's formula,

$$\beta_2(w_k)=w_1w_k+(k+1)w_{k+1},\tag{52}$$

see [22]. A straightforward computation (or the observation at the end of Example 5.3) shows that $\widetilde{\mathcal{R}}^q_s(\mathrm{Gr}_1,\mathbb{Z}_2)=\emptyset$ for all $q,s>0$. A slightly more involved computation yields

$$\widetilde{\mathcal{R}}^{2q}_s(\mathrm{Gr}_2,\mathbb{Z}_2)=\widetilde{\mathcal{R}}^{2q+1}_s(\mathrm{Gr}_2,\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{if } 0<s\leq q,\\ \{0\} & \text{if } q<s\leq 2q,\\ \emptyset & \text{if } s>2q.\end{cases}\tag{53}$$

The resonance varieties $\widetilde{\mathcal{R}}^q_s(\mathrm{Gr}_n,\mathbb{Z}_2)$ for $n>2$ can be computed in like fashion.

6.5. Comparing the two types of resonance. We end this section with a comparison between the two types of mod-2 resonance varieties, $\mathcal{R}^q_s(X,\mathbb{Z}_2)$ and $\widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2)$. We start by observing that these varieties do agree in degree $q=1$, provided the former are defined.

Lemma 6.13. *Suppose $H_1(X,\mathbb{Z})$ has no 2-torsion. Then $\mathcal{R}^1_s(X,\mathbb{Z}_2)=\widetilde{\mathcal{R}}^1_s(X,\mathbb{Z}_2)$ for all $s\geq 0$.*

Proof. By Lemma 6.1, we have that $u^2 = 0$ for all $u \in H^1(X, \mathbb{Z}_2)$. Thus, the Bockstein $\beta_2 \colon H^1(X, \mathbb{Z}_2) \to H^2(X, \mathbb{Z}_2)$ vanishes, and the claim follows straight from the definitions of the respective resonance varieties. □

In degrees $q > 1$, though, it may happen that $\mathcal{R}^q_s(X, \mathbb{Z}_2) \neq \widetilde{\mathcal{R}}^q_s(X, \mathbb{Z}_2)$. We shall see examples of spaces X for which $\mathcal{R}^q_1(X, \mathbb{Z}_2) \subsetneq \widetilde{\mathcal{R}}^q_1(X, \mathbb{Z}_2)$ in some degrees $q > 1$ in Example 6.15 below, and also in Corollary 7.11 and in Example 7.12 in the next section. We give now an example of a space X for which $\widetilde{\mathcal{R}}^q_1(X, \mathbb{Z}_2) \subsetneq \mathcal{R}^q_1(X, \mathbb{Z}_2)$ for some $q > 1$.

Example 6.14. Let $X = S^1 \vee \Sigma(\mathbb{RP}^2)$ be the wedge of a circle with a suspended projective plane. Then $H^i(X, \mathbb{Z}_2) = \mathbb{Z}_2$, generated by classes a_i for $1 \le i \le 3$, with all cup products among these generators vanishing; moreover, since $\beta_2 = \mathrm{Sq}^1$ commutes with suspensions, $\beta_2(a_2) = a_3$. It follows that $\mathcal{R}^2_1(X, \mathbb{Z}_2) = \mathbb{Z}_2$, whereas $\widetilde{\mathcal{R}}^2_1(X, \mathbb{Z}_2) = \emptyset$.

If the mod-2 cohomology ring of X is generated in degree 1, then the Bockstein $\beta_2 = \mathrm{Sq}^1$ is completely determined by the cup-product structure of $H^\bullet(X, \mathbb{Z}_2)$, and so the mod-2 resonance varieties $\widetilde{\mathcal{R}}^q_s(X, \mathbb{Z}_2)$ depend only on the cohomology ring $H^\bullet(X, \mathbb{Z}_2)$. In general, though, the varieties $\widetilde{\mathcal{R}}^q_s(X, \mathbb{Z}_2)$ also depend on the action of the Steenrod algebra on the cohomology ring, as the next example shows.

Example 6.15. Let $X = S^2 \times S^1$ and let $Y = S^2 \widetilde{\times} S^1$ be the non-trivial S^2-bundle over S^1, with monodromy given by the antipodal map. Clearly, $H_1(X, \mathbb{Z}) = H_2(Y, \mathbb{Z}) = \mathbb{Z}$. The mod-2 cohomology rings of both spaces are isomorphic to $\mathbb{Z}_2[a, b]/(a^2, b^2)$, where $|a| = 1$ and $|b| = 2$, while the two Bocksteins are given by $\beta^X_2(a) = \beta^X_2(b) = 0$ and $\beta^Y_2(a) = 0$, $\beta^Y_2(b) = ab$. The usual resonance varieties $\mathcal{R}^q_s$ of the two spaces are the same (over any field $\Bbbk$); in particular, $\mathcal{R}^2_1 = \mathcal{R}^3_1 = \{0\}$. Nevertheless, the mod-2 resonance varieties $\widetilde{\mathcal{R}}^q_s$ differ:

$$\begin{array}{lll} \widetilde{\mathcal{R}}^1_1(X, \mathbb{Z}_2) = \{0\}, & \widetilde{\mathcal{R}}^2_1(X, \mathbb{Z}_2) = \{0\}, & \widetilde{\mathcal{R}}^3_1(X, \mathbb{Z}_2) = \{0\}, \\ \widetilde{\mathcal{R}}^1_1(Y, \mathbb{Z}_2) = \{0\}, & \widetilde{\mathcal{R}}^2_1(Y, \mathbb{Z}_2) = \mathbb{Z}_2, & \widetilde{\mathcal{R}}^3_1(Y, \mathbb{Z}_2) = \mathbb{Z}_2. \end{array}$$

7. Manifolds, Poincaré duality, and resonance

In this section we study the resonance varieties of topological manifolds. All manifolds will be assumed to be without boundary, compact, and connected—for short, *closed* manifolds. Most of the theory works as well for Poincaré complexes, though at some point we will require manifolds to be smooth. We start by discussing the interplay between Poincaré duality, orientability, and the Bockstein operator.

7.1. Poincaré duality and orientability. Let M be a closed manifold of dimension m. If M is orientable, then its cohomology algebra with coefficients in an arbitrary field $\Bbbk$ is a Poincaré duality algebra, also of dimension m. We consider here the case when M is not necessarily orientable, and restrict our attention to the coefficient field $\Bbbk = \mathbb{Z}_2$, in which case $H^\bullet(M, \mathbb{Z}_2)$ is again a Poincaré duality algebra of dimension m.

We start with a well-known lemma, relating the orientability of M to the Bockstein operator $\beta_2 = \mathrm{Sq}^1$ on $H^\bullet(M, \mathbb{Z}_2)$, see Massey [19, Lemma 1]. For completeness, we give a full proof, since only a brief sketch is given in that reference.

Lemma 7.1 ([19]). *Let M be a closed manifold of dimension m. Then M is orientable if and only if the Bockstein $\beta_2 \colon H^{m-1}(M, \mathbb{Z}_2) \to H^m(M, \mathbb{Z}_2)$ is zero.*

Proof. If M is orientable, then $H_{m-1}(M,\mathbb{Z}) \cong H^1(M,\mathbb{Z})$ is free abelian. Thus, the Bockstein operator $\beta_0\colon H_m(M,\mathbb{Z}_2) \to H_{m-1}(M,\mathbb{Z})$ associated to the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ is trivial. Since β_2 is $\mathbb{Z}_2$-dual to the homomorphism $\rho_2 \circ \beta_0\colon H_m(M,\mathbb{Z}_2) \to H_{m-1}(M,\mathbb{Z}_2)$, we infer that $\beta_2 = 0$.

If M is non-orientable, then the torsion subgroup of $H_{m-1}(M,\mathbb{Z})$ is equal to $\mathbb{Z}_2$ and may be identified with the image of the Bockstein $\beta_0\colon H_m(M,\mathbb{Z}_2) \to H_{m-1}(M,\mathbb{Z})$, see [11, p. 238]. Since $\beta_2 = (\rho_2 \circ \beta_0)^*$, we conclude that $\beta_2 \neq 0$. □

Remark 7.2. For a smooth m-manifold M, an alternative proof can be given, using the Wu formulas (see also [12, Corollary 9.8.5]). These formulas relate the Wu classes $v_i \in H^i(M,\mathbb{Z}_2)$, the Stiefel–Whitney classes $w_i \in H^i(M,\mathbb{Z}_2)$, and the Steenrod squares $\operatorname{Sq}^i\colon H^q(M,\mathbb{Z}_2) \to H^{q+i}(M,\mathbb{Z}_2)$ via

$$w = \operatorname{Sq}(v) \text{ and } v \cup u = \operatorname{Sq}(u) \tag{54}$$

for all $u \in H^\bullet(M,\mathbb{Z}_2)$, where $w = \sum_{i\geq 0} w_i$ is the total Stiefel–Whitney class and likewise for v and Sq, see [22]. Interpreting these formulas in degree 1, we see that $w_1 = v_1$ and v_1 is Poincaré dual to $\varepsilon \circ \operatorname{Sq}^1 \in H^{m-1}(M,\mathbb{Z}_2)^*$. Since $\beta_2 = \operatorname{Sq}^1$ and since M is orientable if and only if $w_1 = 0$, the desired conclusion follows.

We now consider the graded $\mathbb{Z}_2$-algebra $A = H^\bullet(M,\mathbb{Z}_2)$, viewed as a cdga with differential given by the Bockstein $\beta_2\colon A \to A$.

Corollary 7.3. *Let M be a closed manifold of dimension m. Then $(H^\bullet(M,\mathbb{Z}_2),\beta_2)$ is a Poincaré duality differential graded algebra (of dimension m) if and only if M is orientable.*

Proof. By Poincaré duality, the cohomology algebra $H^\bullet(M,\mathbb{Z}_2)$ is an m-pda. The claim follows from Definition 4.2 and Lemma 7.1. □

Remark 7.4. As shown by Postnikov in [28], every 3-dimensional Poincaré duality algebra over $\mathbb{Z}_2$ can be realized as the cohomology ring, $H^\bullet(M,\mathbb{Z}_2)$, of some closed (not necessarily orientable) 3-manifold M.

7.2. Resonance varieties of closed manifolds. Poincaré duality has strong implications on the nature of the resonance varieties of closed manifolds (and Poincaré complexes). First, as an immediate consequence of Theorem 4.6, we have the following result.

Proposition 7.5. *Let M be a closed, orientable manifold of dimension m, and let $\Bbbk$ be a field of characteristic different from 2. Then $\mathcal{R}^q_s(M;\Bbbk) = \mathcal{R}^{m-q}_s(M;\Bbbk)$ for all $q, s \geq 0$. In particular, $\mathcal{R}^m_1(M,\Bbbk) = \{0\}$.*

Turning to the resonance varieties over a field $\Bbbk$ of characteristic 2, we have several results. For the first one, we impose no orientability condition (over $\mathbb{Z}$), but make instead an assumption guaranteeing that the usual resonance varieties $\mathcal{R}^m_1(M,\Bbbk)$ are defined.

Proposition 7.6. *Let M be a closed manifold of dimension m. Suppose $H_1(M,\mathbb{Z})$ has no 2-torsion and* $\operatorname{char}(\Bbbk) = 2$. *Then $\mathcal{R}^m_1(M,\Bbbk) = \{0\}$.*

Proof. By Poincaré duality, the cohomology algebra $H^\bullet(M,\mathbb{Z}_2)$ is an m-pda. Viewing it as an m-pd-cdga with $d = 0$, Theorem 4.6, Part (3) implies that $\mathcal{R}^m_1(M,\mathbb{Z}_2) = \{0\}$, and so $\mathcal{R}^m_1(M,\Bbbk) = \{0\}$, too.

The claim can also be proven directly, as follows. Let $\omega = 1^\vee$ be the generator of $H^m(M,\mathbb{Z}_2) = \mathbb{Z}_2$. Let $a \in H^1(M,\mathbb{Z}_2)$ and let $a^\vee \in H^{m-1}(M,\mathbb{Z}_2)$ be its Poincaré dual. Then $\delta_a(a^\vee) = aa^\vee = \omega$, and so $H^m(A,\delta_a) = 0$, once again showing that $\mathcal{R}^m_1(M,\mathbb{Z}_2) = \{0\}$. □

For the next result, we impose orientability (over $\mathbb{Z}$) but no other additional conditions on M, and draw some conclusions regarding the resonance varieties $\widetilde{\mathcal{R}}^m_1(M,\Bbbk)$.

Proposition 7.7. *Let M be a closed, orientable manifold of dimension m, and assume* $\mathrm{char}(\Bbbk) = 2$. *Then $\widetilde{\mathcal{R}}^q_s(M;\Bbbk) = \widetilde{\mathcal{R}}^{m-q}_s(M;\Bbbk)$ for all $q, s \geq 0$. In particular, $\widetilde{\mathcal{R}}^m_1(M,\Bbbk) = \{0\}$.*

Proof. By Corollary 7.3, the cdga $(H^\bullet(M,\mathbb{Z}_2),\beta_2)$ is an m-pd-cdga. Theorem 4.6 then gives $\widetilde{\mathcal{R}}^q_s(M;\mathbb{Z}_2) = \widetilde{\mathcal{R}}^{m-q}_s(M;\mathbb{Z}_2)$ for all $q, s \geq 0$, and the conclusions follow. ◻

Example 7.8. The lens space $L(4,1) = S^3/\mathbb{Z}_4$ and the projective space $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ are both closed, orientable manifolds of dimension 3. They share the same $\mathbb{Z}_2$-cohomology groups, but the multiplicative structure and the action of the Bockstein are different; indeed, $H^\bullet(L(4,1),\mathbb{Z}_2) = \mathbb{Z}_2[a,b]/(a^2,b^2)$ with $|a| = 1$, $|b| = 2$, and $\beta_2 = 0$, whereas $H^\bullet(\mathbb{RP}^3,\mathbb{Z}_2) = \mathbb{Z}_2[a]/(a^4)$ with $|a| = 1$ and $\beta_2(a) = a^2$. It follows that $\widetilde{\mathcal{R}}^1_1(L(4,1),\mathbb{Z}_2) = \widetilde{\mathcal{R}}^2_1(L(4,1),\mathbb{Z}_2) = \{0\}$, yet, as noted previously, $\widetilde{\mathcal{R}}^1_1(\mathbb{RP}^3,\mathbb{Z}_2) = \widetilde{\mathcal{R}}^2_1(\mathbb{RP}^3,\mathbb{Z}_2) = \emptyset$.

Example 7.9. For each $m, n \geq 0$, Dold constructed in [7] a smooth, closed manifold $P(m,n)$, defined as the quotient of $S^m \times \mathbb{CP}^n$ by the involution that acts as the antipodal map on S^m and complex conjugation of $\mathbb{CP}^n$. The Dold manifolds, which generate the unoriented cobordism ring, are orientable if and only if $m+n$ is odd. The cohomology algebra $H^\bullet(P(m,n),\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[a,b]/(a^{m+1},b^{n+1})$, where $|a| = 1$ and $|b| = 2$, and the Bockstein is given by $\beta_2(a) = a^2$ and $\beta_2(b) = ab$. It follows that $\widetilde{\mathcal{R}}^q_1(P(m,n),\mathbb{Z}_2) = \mathbb{Z}_2$ if $q < m+2n$ and q is even or $q = m+2n$ and $m+n$ is even; $\widetilde{\mathcal{R}}^q_1(P(m,n),\mathbb{Z}_2) = \{0\}$ if $0 \leq q \leq m+2n$; and $\widetilde{\mathcal{R}}^q_s(P(m,n),\mathbb{Z}_2) = \emptyset$, otherwise.

7.3. A resonant characterization of orientability. We conclude this section with a result that expresses the orientability of a manifold M in terms of its top-degree resonance variety $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2)$.

Proposition 7.10. *A smooth, closed manifold M of dimension m is orientable if and only if $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) = \{0\}$.*

Proof. If M is orientable, then Proposition 7.7 shows that $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) = \{0\}$. On the other hand, if M is not orientable, then the Stiefel–Whitney class $w_1 = w_1(M) \in H^1(M,\mathbb{Z}_2)$ is non-zero, and

$$\begin{aligned} \delta_{w_1}(u) &= w_1 u + \mathrm{Sq}^1(u) \\ &= v_1 u + \mathrm{Sq}^1(u) \\ &= \mathrm{Sq}^1(u) + \mathrm{Sq}^1(u) \\ &= 0 \end{aligned} \tag{55}$$

for every $u \in H^{m-1}(M,\mathbb{Z}_2)$, by the Wu formulas (54). Let $\omega = 1^\vee \in H^m(M,\mathbb{Z}_2)$; since $\delta_{w_1}(\omega) = 0$, this shows that $w_1 \in \widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2)$, and we are done. ◻

Corollary 7.11. *Let M be a smooth, closed, non-orientable manifold of dimension m. Suppose $H_1(M,\mathbb{Z})$ has no 2-torsion. Then $\mathcal{R}^m_1(M,\mathbb{Z}_2) = \{0\}$ whereas $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) = \mathbb{Z}_2$.*

Proof. Proposition 7.6 shows that $\mathcal{R}^m_1(M,\mathbb{Z}_2) = \{0\}$. The proof of Proposition 7.10 shows that $\{0, w_1\} \subseteq \widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2)$. It remain to show that this inclusion is an equality.

It follows from the proof of Lemma 7.1 and the discussion from Remark 7.2 that the map $\mathrm{Sq}^1\colon H^{m-1}(M,\mathbb{Z}_2) \to H^m(M,\mathbb{Z}_2)$ sends $w_1^\vee$ to ω and any other element to 0. Thus, if

$a \in H^1(M,\mathbb{Z}_2)$ is such that $a \neq 0$ and $a \neq w_1$, then $a^\vee \neq w_1^\vee$, and so

$$\delta_a(a^\vee) = aa^\vee + \mathrm{Sq}^1(a^\vee) = \omega + 0 = \omega. \tag{56}$$

Since ω generates $H^m(M,\mathbb{Z}_2) = \mathbb{Z}_2$, it follows that $a \notin \widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2)$, and the proof is complete. ◻

Example 7.12. Let G be a finitely presented group which admits an epimorphism $\nu\colon G \twoheadrightarrow \mathbb{Z}_2$. As explained in [3], standard surgery techniques, suitably adapted to this situation, produce a smooth, closed, non-orientable, 4-dimensional manifold M with $\pi_1(M) = G$ and with $w_1(M)$ corresponding to ν under the identification $H^1(M,\mathbb{Z}_2) = \mathrm{Hom}(G,\mathbb{Z}_2)$. If, furthermore, G_{ab} has no 2-torsion (for instance, if $G = \mathbb{Z}^n$, for some $n \geq 1$), then M satisfies the hypothesis of the above corollary, and so $\mathcal{R}^4_1(M,\mathbb{Z}_2) = \{0\}$, yet $\widetilde{\mathcal{R}}^4_1(M,\mathbb{Z}_2) = \mathbb{Z}_2$.

8. Resonance varieties and Betti numbers of finite covers

In this last section we compare the Betti numbers and the resonance varieties of certain finite covers to those of the base space.

8.1. Finite, regular covers and resonance. Let X be a connected, finite-type CW-complex and let $p\colon Y \to X$ be a connected, regular cover, with group of deck transformations Γ. We assume Γ is finite, and we let $\Bbbk$ be a field of characteristic 0 or a prime not dividing the order of Γ. Given these data, a transfer argument shows that the induced homomorphism $p^*\colon H^\bullet(X,\Bbbk) \to H^\bullet(Y,\Bbbk)$ is injective, with image the subgroup $H^\bullet(Y,\Bbbk)^\Gamma$ of cohomology classes invariant under the action of Γ, see e.g. [11, Proposition 3G.1].

The next proposition and its corollary were proved in [6] in the case when $\Bbbk$ has characteristic 0. For completeness, we include a proof, following the argument given in [33].

Proposition 8.1 ([6, 33]). *Let Γ be a finite group and let $p\colon Y \to X$ be a regular Γ-cover. Suppose that $\mathrm{char}(\Bbbk) \nmid |\Gamma|$, and also $\mathrm{char}(\Bbbk) \neq 2$ if $H_1(X,\mathbb{Z})$ has 2-torsion. Then $p^*(\mathcal{R}^q_s(X,\Bbbk)) \subseteq \mathcal{R}^q_s(Y,\Bbbk)$, for all $i, s \geq 0$.*

Proof. For a class $a \in H^1(Y,\Bbbk)^\Gamma = H^1(X,\Bbbk)$, the monodromy action of Γ on $H^\bullet(Y,\Bbbk)$ gives rise to an action on the chain complex $(H^\bullet(Y,\Bbbk),\delta_a)$, with the fixed subcomplex equal to $(H^\bullet(X,\Bbbk),\delta_a)$. Since Γ is finite and since $\mathrm{char}(\Bbbk) \nmid |\Gamma|$, a transfer argument shows once again that $H^\bullet(H^\bullet(Y,\Bbbk),\delta_a)^\Gamma = H^\bullet(H^\bullet(X,\Bbbk),\delta_a)$. We thus obtain an inclusion, $H^\bullet(H^\bullet(X,\Bbbk),\delta_a) \hookrightarrow H^\bullet(H^\bullet(Y,\Bbbk),\delta_a)$, and the claim follows. ◻

Corollary 8.2. *With notation as above, suppose the group Γ acts trivially on $H^1(Y,\Bbbk)$. Then the map $p^*\colon \mathcal{R}^1_s(X,\Bbbk) \to \mathcal{R}^1_s(Y,\Bbbk)$ is an isomorphism, for all $s \geq 0$.*

By way of contrast, the resonance varieties $\widetilde{\mathcal{R}}^q_s(Y,\mathbb{Z}_2)$ may vanish, even when the corresponding varieties $\widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2)$ are non-zero. This phenomenon is illustrated by the next result, which is an immediate corollary to Proposition 7.10.

Corollary 8.3. *Let M be a smooth, closed, non-orientable manifold of dimension m, and let $\widetilde{M}$ be its orientation double cover. Then $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) \neq \{0\}$ yet $\widetilde{\mathcal{R}}^m_1(\widetilde{M},\mathbb{Z}_2) = \{0\}$.*

8.2. Mod-2 Betti numbers of 2-fold covers. We now consider the case when the order of the deck group Γ is not coprime to the characteristic of the coefficient field $\Bbbk$. We will focus on the simplest possible situation: the one where Γ is a finite cyclic group of even order (mainly, $\Gamma = \mathbb{Z}_2$) and $\Bbbk = \mathbb{Z}_2$.

Every regular $\mathbb{Z}_n$-cover $p\colon Y \to X$ is classified by a homomorphism $\alpha\colon \pi_1(X) \to \mathbb{Z}_n$, or, equivalently, by a cohomology class (called its characteristic class), $\alpha \in H^1(X,\mathbb{Z}_n)$.

Furthermore, the covering space Y is connected if and only if the homomorphism α is surjective, in which case $\pi_1(Y) = \ker(\alpha)$. Note also that all 2-fold covers are regular.

We start with a simple lemma, which strengthens Lemma 3.2, part (i) from [39].

Lemma 8.4. *Let $p\colon Y \to X$ be a connected $\mathbb{Z}_2$-cover, with characteristic class $\alpha \in H^1(X,\mathbb{Z}_2)$. Then p lifts to a connected, regular $\mathbb{Z}_4$-cover $\bar{p}\colon \overline{Y} \to X$ if and only if $\alpha^2 = 0$.*

Proof. Since Y is connected, the homomorphism $\alpha\colon \pi_1(X) \to \mathbb{Z}_2$ is surjective, or, equivalently, the characteristic class α is non-zero. As noted in §5.1, the Bockstein of α is given by $\beta_2(\alpha) = \alpha^2$. Thus, $\alpha^2 = 0$ if and only if there is a class $\bar{\alpha} \in H^1(X,\mathbb{Z}_4)$ such that α is the reduction mod-2 of $\bar{\alpha}$. Let $\bar{p}\colon \overline{Y} \to X$ be the corresponding regular $\mathbb{Z}_4$-cover. Since $\alpha \neq 0$, the homomorphism $\bar{\alpha}\colon \pi_1(X) \to \mathbb{Z}_4$ must be surjective, and so $\overline{Y}$ must be connected. □

The next result generalizes Lemma 3.2, part (ii) and Theorem 3.7 from [39].

Proposition 8.5. *Let $p\colon Y \to X$ be a 2-fold cover, classified by a non-zero class $\alpha \in H^1(X,\mathbb{Z}_2)$. Suppose that $\alpha^2 = 0$. Then, for all $q \geq 1$,*

$$b_q(Y,\mathbb{Z}_2) = b_q(X,\mathbb{Z}_2) + \dim_{\mathbb{Z}_2} H^q(H^{\bullet}(X,\mathbb{Z}_2),\delta_\alpha). \tag{57}$$

In particular, $b_q(Y,\mathbb{Z}_2) \geq b_q(X,\mathbb{Z}_2)$.

Proof. The transfer exact sequence for the 2-fold cover with characteristic class α (see [11, 12]) takes the form

$$\cdots \longrightarrow H^q(X,\mathbb{Z}_2) \xrightarrow{\cdot\alpha} H^q(X,\mathbb{Z}_2) \xrightarrow{p^*} H^q(Y,\mathbb{Z}_2) \xrightarrow{\tau} H^q(X,\mathbb{Z}_2) \xrightarrow{\cdot\alpha} H^{q+1}(X,\mathbb{Z}_2) \longrightarrow \cdots. \tag{58}$$

Since by assumption $\beta_2(\alpha) = \alpha^2$ vanishes, we have that $\delta_\alpha(u) = \alpha u$ for all $u \in H^i(X,\mathbb{Z}_2)$. Hence, the above long exact sequence splits into short exact sequences,

$$0 \longrightarrow \operatorname{im}(\delta_\alpha) \longrightarrow H^q(X,\mathbb{Z}_2) \xrightarrow{p^*} H^q(Y,\mathbb{Z}_2) \longrightarrow \ker(\delta_\alpha) \longrightarrow 0, \tag{59}$$

and the claim follows. □

Remark 8.6. Yoshinaga [39] proved this result in the case when either $q = 1$ or X is the complement of a complex hyperplane arrangement. He used it to detect a $\mathbb{Z}_2$-summand in the first homology of the Milnor fiber of the icosidodecahedral arrangement.

If the 2-fold cover $Y \to X$ is classified by an element $\alpha \in H^1(X,\mathbb{Z}_2)$ with $\alpha^2 \neq 0$, the mod-2 Betti numbers of Y may actually be smaller than those of X.

Example 8.7. For $n > 1$, let $S^n \to \mathbb{RP}^n$ be the orientation double cover, classified by the degree-1 generator $\alpha = w_1(\mathbb{RP}^n)$ of the cohomology algebra $H^{\bullet}(\mathbb{RP}^n,\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$.

Note added in proof

After acceptance of this paper, Ishibashi, Sugawara, and Yoshinaga posted a preprint, [13], in which they expand on the work described in Section 8.2. Namely, suppose $Y \to X$ is a connected $\mathbb{Z}_2$-cover with characteristic class $\alpha \in H^1(X,\mathbb{Z}_2)$. Assuming $H_{\bullet}(X,\mathbb{Z})$ is torsion-free, it follows from [23, Theorem C] that

$$b_q(Y) \leq b_q(X) + \dim_{\mathbb{Z}_2} H^q(H^{\bullet}(X,\mathbb{Z}_2),\delta_\alpha). \tag{60}$$

When X is the (projectivized) complement of a hyperplane arrangement, an explicit formula was proposed in [26, Conjecture 1.9], expressing the first Betti number of the

Milnor fiber of the arrangement in terms of the resonance varieties $\mathcal{R}^1_s(X,\mathbb{Z}_p)$, for $p = 2$ and 3. At the prime $p = 2$, the conjecture is equivalent to the inequality (60) holding as equality in degree $q = 1$ for the 2-fold cover $Y \to X$ corresponding to the class $\alpha \in H^1(X,\mathbb{Z}_2)$ which evaluates to 1 on each meridian. As shown in [**13**, Corollary 2.5], though, that equality holds if and only if $H_1(Y,\mathbb{Z}_2)$ has no 2-torsion. Therefore, the formula conjectured in [**26**] fails at the prime $p = 2$ precisely when $H_1(Y,\mathbb{Z}_2)$ has non-trivial 2-torsion, as is the case in the example mentioned in Remark 8.6.

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