Towards an integral version of rational homotopy theory

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RATIONAL HOMOTOPY THEORY

- Homotopy theory is the study of topological spaces up to homotopy equivalence.
- Typical examples of homotopy type invariants of a space X are the homology groups $H_n(X;\mathbb{Z})$ and the homotopy groups $\pi_n(X)$.
- The question whether one can reconstruct the homotopy type of a space from homological data goes back to the beginnings of Algebraic Topology.
- Poincaré realized that homology is not enough: $H_1(X;\mathbb{Z})$ only records the abelianization of $\pi_1(X)$.
- Even for simply-connected spaces, homology by itself fails to detect the Hopf map, $S^3 \to S^2$.
- But one can reconstitute all the higher homotopy groups of S^n, modulo torsion, from the de Rham algebra of differential forms.

- As founded by Quillen and Sullivan, rational homotopy theory is the study of rational homotopy types of spaces.
- Instead of considering the groups $H_n(X;\mathbb{Z})$ and $\pi_n(X)$, one considers the groups $H_n(X;\mathbb{Q})$ and $\pi_n(X)\otimes\mathbb{Q}$ (for $n \geqslant 2$).
- These objects are $\mathbb{Q}$-vector spaces, and hence the torsion information is lost, yet this is compensated by the fact that computations are easier in this setting.
- To every space X, Sullivan attached in a functorial way a commutative differential graded algebra over $\mathbb{Q}$, denoted $A_{\text{PL}}(X)$. This cdga is constructed from piecewise polynomial rational forms.
- It is weakly equivalent (through dgas) with the cochain algebra $(C^*(X;\mathbb{Q}), d)$ so that, under the resulting identification $H^*(A_{\text{PL}}(X)) \cong H^*(X;\mathbb{Q})$, the induced homomorphisms in cohomology correspond.

- Given a connected $\mathbb{Q}$-cdga A, Sullivan constructed a *minimal model* for it, $\rho\colon \mathcal{M}(A) \to A$, where ρ is a quasi-isomorphism and $\mathcal{M}(A)$ is a cdga obtained by iterated Hirsch extensions, starting from $\mathbb{Q}$, so that its differential is decomposable.
- These properties uniquely characterize the minimal model of A, up to isomorphism.
- The q-minimal models $\mathcal{M}_q(A)$ are generated by elements of degrees at most q, and the structural morphisms $\rho_q\colon \mathcal{M}_q(A) \to A$ are only q-quasi-isomorphisms.
- A minimal model for a connected space X, denoted $\mathcal{M}(X)$, is a minimal model for $A_{\text{PL}}(X)$. The isomorphism type of $\mathcal{M}(X)$ is uniquely defined by the rational homotopy type of X.
- if G is a finitely generated group, the 1-minimal model $\mathcal{M}_1(G) = \mathcal{M}_1(K(G,1))$ determines and is determined by the Malcev completion $G_{\mathbb{Q}} := \varprojlim_n (G/\gamma_n(G) \otimes \mathbb{Q})$.

- In previous work,[1] we combined properties of the Steenrod cup-one products of cochains and those of binomial rings to define the algebraic categories of binomial cup-one differential graded algebras over $R = \mathbb{Z}$ and over $R = \mathbb{Z}_p$, for p a prime.
- In current work,[2] we define the 1-minimal model $\rho\colon \mathcal{M}_1(A) \to A$ for such a dga (A, d), and prove some of its key properties:
  - $\mathcal{M}_1(A)$ is a free binomial cup-one dga, unique up to isomorphism.
  - $\mathcal{M}_1(A)$ determines a pronilpotent group, $G(A)$, which only depends on the 1-quasi-isomorphism type of A.
- This allows us to distinguish spaces with isomorphic (torsion-free) cohomology rings that share the same rational 1-minimal model, yet whose integral 1-minimal models are not isomorphic.

[1] R.D. Porter and A.I. Suciu, *Differential graded algebras, Steenrod cup-one products, binomial operations, and Massey products*, Topology Appl. **313** (2022), Paper No. 107987, 37 pp.

[2] R.D. Porter and A.I. Suciu, *Cup-one algebras and 1-minimal models*, 74 pp., preprint June 2023, arXiv:2306.11849.

CUP-ONE ALGEBRAS

- Let X be Δ-set; that is, a sequence of sets $X = \{X_n\}_{n\geqslant 0}$ and maps $d_i\colon X_n \to X_{n-1}$ $(0 \leqslant i \leqslant n)$ such that $d_i d_j = d_{j-1} d_i$ for all $i < j$.
- Its geometric realization, $|X|$, may be viewed either as a special kind of CW-complex, or a generalized simplicial complex.
- Let $A = (C^*(X; R), d)$ be the cellular cochain complex of $|X|$ with coefficients in a commutative ring R. This is a differential graded R-algebra, with multiplication given by the cup-product.
- In 1947, Steenrod introduced a sequence of operations, $\cup_i\colon A^p \otimes_R A^q \to A^{p+q-i}$, starting with $\cup_0 = \cup$.
- We focus on the $\cup_1$-product on A^1, which is tied to the differential and the cup product via the Steenrod and Hirsch identities,

$$d(a \cup_1 b) = -a \cup b - b \cup a + da \cup_1 b - a \cup_1 db,$$
$$(a \cup b) \cup_1 c = a \cup (b \cup_1 c) + (a \cup_1 c) \cup b.$$

- A *cup-one differential graded algebra* is an R-dga (A, d) with a map $\cup_1 \colon A^1 \otimes_R A^1 \to A^1$ that gives $R \oplus A^1$ the structure of a commutative ring and satisfies the Hirsch identity and

$$d(a \cup_1 b) = -a \cup b - b \cup a + da \cup_1 b + db \cup_1 a - da \circ db, \quad (*)$$

for all $a, b \in A^1$ with da, db equal to sums of cup products, with $\circ$ bilinear and $(a_1 \cup a_2) \circ (b_1 \cup b_2) = (a_1 \cup_1 b_1) \cup (a_2 \cup_1 b_2)$.

- Note that the role of the cup-one product in a (non-commutative) dga (A, d) is to ensure that $H^*(A)$ is commutative.

- A commutative ring A is called a *binomial ring* if A is torsion-free as a $\mathbb{Z}$-module, and has the property that the elements $\binom{a}{n} := a(a-1)\cdots(a-n+1)/n!$ lie in A for every $a \in A$ and $n > 0$.

- An analogous notion holds for $\mathbb{Z}_p$-algebras.

- These objects come equipped with maps $\zeta_n \colon A \to A$, $a \mapsto \binom{a}{n}$, defined for all $n > 0$ over $\mathbb{Z}$, and only for $n < p$ over $\mathbb{Z}_p$.

- A cup-one dga (A, d) over $R = \mathbb{Z}$ or $\mathbb{Z}_p$ is called a *binomial cup-one algebra* if A^0, with multiplication $A^0 \otimes_R A^0 \to A^0$ given by the cup-product, is a binomial R-algebra, and the R-submodule $R \oplus A^1 \subset A^{\leqslant 1}$, with multiplication $A^1 \otimes_R A^1 \to A^1$ given by the cup-one product, is an R-binomial algebra.

- The main motivating example is the cochain algebra of a space.

- Given a Δ-set X, the cellular cochain algebra $C = (C^*(X; R), d)$ is a binomial cup-one dga, with
  - $\cup_1 \colon C^1 \otimes_R C^1 \to C^1$ given by $(a \cup_1 b)(e) = a(e) \cdot b(e)$.
  - $\circ = \cup_2 \colon C^2 \otimes_R C^2 \to C^2$ given by $(v \circ w)(s) = v(s) \cdot w(s)$.
  - $\zeta_n(a)(e) = \binom{a(e)}{n}$ when $R = \mathbb{Z}$ and analogously for $R = \mathbb{Z}_p$.

FREE BINOMIAL CUP-ONE ALGEBRAS

- When $R = \mathbb{Z}$, consider the ring $\mathrm{Int}(\mathbb{Z}^{\boldsymbol{X}}) = \{q \in \mathbb{Q}[\boldsymbol{X}] \mid q(\mathbb{Z}^{\boldsymbol{X}}) \subseteq \mathbb{Z}\}$ of integrally-valued polynomials with variables in a set $\boldsymbol{X}$.
- This is a binomial ring, generated by the polynomials $\binom{\boldsymbol{X}}{\mathbf{n}} = \prod_{x \in \boldsymbol{X}} \binom{x}{\mathbf{n}_x}$ with $\mathbf{n}_x \in \mathbb{Z}_{\geqslant 0}$.
- We define the *free binomial cup-one graded algebra*, $\mathrm{T} = \mathrm{T}_R^*(\boldsymbol{X})$, to be the tensor algebra on $\mathfrak{m}_{\boldsymbol{X}}$, the maximal ideal at 0 in $\mathrm{Int}(\mathbb{Z}^{\boldsymbol{X}})$.
- When $R = \mathbb{Z}_p$, an analogous definition applies.
- In either case, we have R-linear maps
  - $\cup_1 \colon \mathrm{T}^1 \otimes \mathrm{T}^1 \to \mathrm{T}^1$, given by $a \cup_1 b = ab$;
  - $\circ \colon \mathrm{T}^2 \otimes \mathrm{T}^2 \to \mathrm{T}^2$, given by $(a_1 \otimes a_2) \circ (b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$.

THEOREM

Let $d\colon \mathrm{T}_R(\boldsymbol{X}) \to \mathrm{T}_R(\boldsymbol{X})$ be a degree-one map satisfying the $\cup_1$–d formula (*) and the Leibniz rule. Then $d^2(x) = 0$ for all $x \in \boldsymbol{X}$ if and only if $d^2(u) = 0$ for all $u \in \mathrm{T}_R(\boldsymbol{X})$, in which case $(\mathrm{T}_R(\boldsymbol{X}), d)$ is a binomial cup-one dga.

- In particular, the map sending each $x \in \boldsymbol{X}$ to 0 extends to a differential $d_0\colon \mathrm{T}_R(\boldsymbol{X}) \to \mathrm{T}_R(\boldsymbol{X})$, making $(\mathbf{T}_R(\boldsymbol{X}), d_{\mathbf{0}})$ into a binomial cup-one dga.
- $d_{\mathbf{0}}$ is compatible with the binomial structure on $\mathrm{T}_R(\boldsymbol{X})$:

$$d_{\mathbf{0}}(\zeta_n(x)) = -\sum_{\ell=1}^{n-1} \zeta_\ell(x) \otimes \zeta_{n-\ell}(x),$$

for all $n \geqslant 1$ when $R = \mathbb{Z}$ and for $1 \leqslant n \leqslant p-1$ when $R = \mathbb{Z}_p$.

- More generally, given a set map $\tau\colon \boldsymbol{X} \to \mathrm{T}^2_R(\boldsymbol{X})$, we define a binary operation, $\mu_\tau\colon M \times M \to M$, on the R-module $M = M(\boldsymbol{X}, R)$ of all functions from $\boldsymbol{X}$ to the ring $R = \mathbb{Z}$ or $\mathbb{Z}_p$.
- Letting $\Delta^{(2)}(M_\tau)$ be the 2-dim Δ-set associated to the magma $M_\tau = (M, \mu_\tau)$, we define a degree-preserving, R-linear map,

$$\psi = \psi_{\boldsymbol{X},\mu}\colon\ \mathrm{T}^{\leqslant 2}_R(\boldsymbol{X}) \to C^*(\Delta^{(2)}(M_\tau); R).$$

- This map is a monomorphism which commutes with cup products, cup-one products, and the $\circ$ maps.
- Using the embedding ψ, we show that there is a unique extension of τ to an R-linear map $d_\tau\colon\ \mathrm{T}_R(\boldsymbol{X}) \to \mathrm{T}_R(\boldsymbol{X})$ that satisfies the Leibniz rule and the $\cup_1$–d formula.

- We focus on the case when M_τ is a semigroup (we then say τ is *admissible*), and consider the associated cell complex, $\Delta(M_\tau)$.
- In that case, the map ψ extends uniquely to an inclusion $\psi\colon (\mathrm{T}(\boldsymbol{X}), d_\tau) \hookrightarrow C^*(\Delta(M_\tau); R)$ that satisfies $\psi \circ d_\tau = d_\Delta \circ \psi$.
- It then follows that d_τ^2 is the zero map. To summarize:

THEOREM

If the map $\tau\colon \boldsymbol{X} \to \mathrm{T}^2(\boldsymbol{X})$ is admissible, then $d_\tau^2 \equiv 0$ and the map $\mathrm{T}^1(\boldsymbol{X}) \to C^1(\Delta(M_\tau); R)$ given by $q \mapsto (\mathbf{a} \mapsto q(\mathbf{a}))$ extends uniquely to a monomorphism $\psi\colon (\mathrm{T}(\boldsymbol{X}), d_\tau) \hookrightarrow (C^*(\Delta(M_\tau); R), d_\Delta)$ of binomial cup-one dgas.

Hirsch extensions

- Hirsch extensions of free binomial $\cup_1$-dgas are the basic building blocks for constructing 1-minimal models.
- An inclusion $i\colon (\mathrm{T}_R(\boldsymbol{X}), d) \to (\mathrm{T}_R(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d})$ is called a *Hirsch extension* if $\bar{d}(y)$ is a cocycle in $\mathrm{T}^2_R(\boldsymbol{X})$ for all $y \in \boldsymbol{Y}$.
- There is a bijection between maps of sets from $\boldsymbol{Y}$ to cocycles in $\mathrm{T}^2_R(\boldsymbol{X})$ and Hirsch extensions of this sort.
- Assume that $\boldsymbol{X} = \bigcup_{i\geqslant 1} \boldsymbol{X}_i$ with each $\boldsymbol{X}_i$ a finite set and $\boldsymbol{X}_1 \neq \emptyset$. Write $\boldsymbol{X}^n = \boldsymbol{X}_1 \cup \cdots \cup \boldsymbol{X}_n$.
- A dga $\mathrm{T} = (\mathrm{T}_R(\boldsymbol{X}), d)$ is called a *colimit of Hirsch extensions* if d restricts to differentials d_n on $\mathrm{T}_R(\boldsymbol{X}^n)$ such that $d_1|_{\boldsymbol{X}_1} = 0$ and each dga $(\mathrm{T}_R(\boldsymbol{X}^{n+1}), d_{n+1})$ is a Hirsch extension of $(\mathrm{T}_R(\boldsymbol{X}^n), d_n)$.

- To each colimit of Hirsch extensions, T, we associate a pronilpotent group, G_{T}, together with a $\cup_1$-dga map, $\psi_{\mathsf{T}}\colon \mathsf{T} \to C^*(B(G_{\mathsf{T}}); R)$, which induces an iso on H^1.
- If T is a finite sequence of Hirsch extensions over $R = \mathbb{Z}$, there is a nilmanifold $N(\mathsf{T})$ with $\pi_1(N(\mathsf{T})) = G_{\mathsf{T}}$. Moreover, every nilmanifold is of the form $N(\mathsf{T})$, for some T.

EXAMPLE

For an integer $k \geqslant 1$, let $\mathsf{T}(k) = (\mathsf{T}_{\mathbb{Z}}(x_1, x_2, x_{1,2}), d)$, with $dx_i = 0$ and $dx_{1,2} = -kx_1 \otimes x_2$. Then $G_{\mathsf{T}(k)}$ is the Heisenberg group of upper triangular matrices of the form

$$\begin{pmatrix} 1 & a_1 & a_{1,2}/k \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

with $a_1, a_2, a_{1,2} \in \mathbb{Z}$.

THEOREM

Let $\mathsf{T} = (\mathsf{T}_R(\boldsymbol{X}), d)$ *be a colimit of Hirsch extensions, let* $G = G_{\mathsf{T}}$ *be the corresponding pronilpotent group, and assume the map* $\psi_{\mathsf{T}} \colon \mathsf{T}_R(\boldsymbol{X}) \to C^*(BG; R)$ *is a quasi-isomorphism. Moreover, let* $\pi \colon B\overline{G} \to BG$ *be the fibration corresponding to a central extension of groups,* $0 \to F \to \overline{G} \to G \to 1$, *with* F *a finitely generated, free* R*-module. Then*

- *There is a Hirsch extension* $i \colon \mathsf{T} \hookrightarrow \overline{\mathsf{T}} = (\mathsf{T}_R(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d})$ *such that* $\overline{G} = G_{\overline{\mathsf{T}}}$.
- *The diagram below commutes*

$$\begin{array}{ccc} \overline{\mathsf{T}} & \xrightarrow{\psi_{\overline{\mathsf{T}}}} & C^*(B\overline{G}; R) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle \pi^*} \\ \mathsf{T} & \xrightarrow{\psi_{\mathsf{T}}} & C^*(BG; R)\,. \end{array}$$

- *The map* $\psi_{\overline{\mathsf{T}}}$ *is a quasi-isomorphism.*

1-MINIMAL MODELS

- Let (A, d) be a binomial cup-one R-dga.
- A 1*-minimal model* for A is a colimit, $\mathcal{M} = (\mathrm{T}_R(\boldsymbol{X}), d)$, of Hirsch extensions $\mathcal{M}_n = (\mathrm{T}_R(\boldsymbol{X}^n), d_n)$, together with morphisms $\rho_n \colon \mathcal{M}_n \to A$ compatible with the Hirsch extensions $\mathcal{M}_n \hookrightarrow \mathcal{M}_{n+1}$.
- The map $H^1(\rho_1) \colon H^1(\mathcal{M}_1) \to H^1(A)$ is required to be an isomorphism; in particular, $\boldsymbol{X}_1$ corresponds to a basis for $H^1(A)$.
- For $n \geqslant 1$, the set $\boldsymbol{X}_{n+1}$ is a basis for the free submodule $\ker(H^2(\rho_n)) \subset H^2(\mathcal{M}_n)$ given by the cohomology classes of the 2-cocycles $d_{n+1}(x)$ with $x \in \boldsymbol{X}_{n+1}$.

LEMMA

Let (A, d_A) and $(A', d_{A'})$ be binomial cup-one R-dgas over $R = \mathbb{Z}$ or $\mathbb{Z}_p$, let $f\colon A \to A'$ be a surjective 1-quasi-isomorphism, and let $\varphi\colon (\mathrm{T}_R(\boldsymbol{X}), d) \to (A', d_{A'})$ be a morphism. There is then a lift $\widehat{\varphi}$,

$$\begin{array}{ccc} & & A \\ & \overset{\widehat{\varphi}}{\nearrow} & \big\downarrow f \\ \mathrm{T}_R(\boldsymbol{X}) & \xrightarrow{\ \varphi\ } & A'. \end{array}$$

THEOREM

Let (A, d_A) be a binomial cup-one R-dga. Assume $H^0(A) = R$ and $H^1(A)$ is a finitely generated, free R-module. Then,

- *There is a 1-minimal model, $\mathcal{M} = (\mathrm{T}_R(\boldsymbol{X}), d)$, and a structural morphism, $\rho\colon \mathcal{M} \to A$, that is a 1-quasi-isomorphism.*
- *Given 1-minimal models, $\rho\colon \mathcal{M} \to A$ and $\rho'\colon \mathcal{M}' \to A$, there is an isomorphism $f\colon \mathcal{M} \to \mathcal{M}'$ and a dga homotopy $\Phi\colon \mathcal{M} \to A \otimes_R C^*([0,1]; R)$ from ρ to $\rho' \circ f$.*

- In the case when (A, d_A) admits an augmentation, that is, a dga morphism $\varepsilon\colon A \to R$, the isomorphism f is unique (in the category of augmented dgas).
- More precisely, A has an augmented 1-minimal model, $\mathcal{M}$, such that the structural morphism $\rho\colon \mathcal{M} \to A$ is an augmentation-preserving 1-quasi-isomorphism.
- Moreover, given augmented 1-minimal models, $\rho\colon \mathcal{M} \to A$ and $\rho'\colon \mathcal{M}' \to A$, there is a *unique* augmentation-preserving isomorphism $f\colon \mathcal{M} \to \mathcal{M}'$ such that ρ is augmentation-preserving homotopic to $\rho' \circ f$.

THEOREM

Let Y be a connected topological space with $H^1(Y;\mathbb{Z})$ finitely generated. Then the 1-minimal model for $C^*(Y;\mathbb{Z})$ tensored with $\mathbb{Q}$ is weakly equivalent as a dga to Sullivan's 1-minimal model for $A_{\mathrm{PL}}(Y)$.

n-STEP EQUIVALENCE

- If a morphism $\varphi\colon A \to A'$ induces an isomorphism in $H^{\leqslant 2}$, then for each $n \geqslant 1$ there is an isomorphism $f_n\colon \mathcal{M}_n \to \mathcal{M}'_n$ such that

$$\begin{array}{ccc} H^2(\mathcal{M}_n) & \xrightarrow{H^2(f_n)} & H^2(\mathcal{M}'_n) \\ {\scriptstyle H^2(\rho_n)}\downarrow & & \downarrow{\scriptstyle H^2(\rho'_n)} \\ H^2(A) & \xrightarrow{H^2(\varphi)} & H^2(A') \,. \end{array}$$

- We say that A and A' are *n-step equivalent* if there are isomorphisms $f_n\colon \mathcal{M}_n \to \mathcal{M}'_n$ and $e_n\colon H^2(A) \to H^2(A')$ such that the diagram commutes with $H^2(\varphi)$ replaced by e_n.
- If A and A' are n-step equivalent, then the cokernels of the homomorphisms $H^2(\rho_n)$ and $H^2(\rho'_n)$ are isomorphic, and hence have isomorphic torsion subgroups.
- Given a space X with n-th step in the 1-minimal model given by $(\mathcal{M}_n, \rho_n)$, we define $\kappa_n(X) := \operatorname{Tors}(\operatorname{coker} H^2(\rho_n))$.

THEOREM

Let X and X' be two connected Δ-complexes with first and second integral cohomology groups finitely generated. Then,

- *If $\pi_1(X) \cong \pi_1(X')$, then $\kappa_n(X) \cong \kappa'_n(X)$ for all $n \geqslant 1$.*
- *If $\kappa_n(X) \ncong \kappa'_n(X)$ for some $n \geqslant 1$, then the cochain algebras $C^*(X;\mathbb{Z})$ and $C^*(X';\mathbb{Z})$ are not n-step equivalent.*

- We apply this result to a sequence of links in the three-sphere, $\{L(n)\}_{n\geqslant 1}$, the first term of which is the Borromean rings.
- Set $X(n) = S^3 \backslash L(n)$. We show that $\kappa_2(X(n)) = \mathbb{Z}_n \oplus \mathbb{Z}_n$.
- Hence, $X(n)$ and $X(m)$ are not 2-step equivalent for $n \neq m$.
- On the other hand, $A_{\text{PL}}(X(n))$ and $A_{\text{PL}}(X(m))$ are 2-step equivalent for all $n, m > 0$.

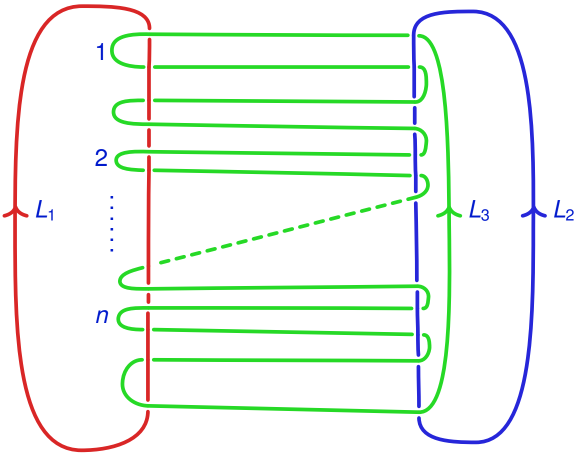

FIGURE: Generalized Borromean Rings