Lower central series and Alexander invariants in group extensions

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N-series

- An N-*series* for a group G is a descending filtration $G = K_1 \geqslant \cdots \geqslant K_n \geqslant \cdots$ such that $[K_m, K_n] \subseteq K_{m+n}$, $\forall m, n \geqslant 1$.
- In particular, $\kappa = \{K_n\}_{n\geqslant 1}$ is a *central series*, i.e., $[G, K_n] \subseteq K_{n+1}$.
- Thus, it is also a *normal series*, i.e., $K_n \lhd G$.
- Consequently, each quotient K_n/K_{n+1} lies in the center of G/K_{n+1}, and thus is an abelian group.
- If all those quotients are torsion-free, κ is called an N_0-*series*.
- *Associated graded Lie algebra*:

$$\operatorname{gr}^{\kappa}(G) = \bigoplus_{n\geqslant 1} K_n/K_{n+1},$$

with addition induced by $\cdot\colon G \times G \to G$, and Lie bracket $[\ ,\ ]\colon \operatorname{gr}_m \times \operatorname{gr}_n \to \operatorname{gr}_{m+n}$ induced by $[x, y] := xyx^{-1}y^{-1}$.

Lower central series

- The *lower central series*, $\gamma(G) = \{\gamma_n(G)\}_{n\geqslant 1}$ is defined inductively by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{n+1}(G) = [G, \gamma_n(G)]$.
- It is an N-series, and the fastest descending central series for G.
- If $\varphi\colon G \to H$ is a homomorphism, then $\varphi(\gamma_n(G)) \subseteq \gamma_n(H)$.
- $\operatorname{gr}(G) := \operatorname{gr}^{\gamma}(G)$ is generated by $\operatorname{gr}_1(G) = G_{\text{ab}}$.
- If $b_1(G) < \infty$, the *LCS ranks* of G are $\phi_n(G) := \dim_{\mathbb{Q}} \operatorname{gr}_n(G) \otimes \mathbb{Q}$.
- For each N-series κ, there is a morphism $\operatorname{gr}(G) \to \operatorname{gr}^{\kappa}(G)$.
- $\Gamma_n := G/\gamma_n(G)$ is the maximal $(n-1)$-step nilpotent quotient of G.
- $G/\gamma_2(F) = G_{\text{ab}}$, while $G/\gamma_3(G) \leftrightarrow H^{\leqslant 2}(G, \mathbb{Z})$.
- G is residually nilpotent $\iff \gamma_\omega(G) := \bigcap_{n\geqslant 1} \gamma_n(G)$ is trivial.

Split exact sequences

- A short exact sequence of groups,

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1 \qquad (*)$$

  yields representations $\varphi\colon Q \to \mathrm{Out}(K)$ and $\bar{\varphi}\colon Q \to \mathrm{Aut}(K_{\mathrm{ab}})$.

- If (*) admits a splitting, $\sigma\colon Q \to G$, then $G = K \rtimes_{\varphi} Q$, where $\varphi\colon Q \to \mathrm{Aut}(K)$, $x \mapsto \text{conjugation by } \sigma(x)$.

- (*) is *ab-exact* if $0 \longrightarrow K_{\mathrm{ab}} \xrightarrow{\iota_{\mathrm{ab}}} G_{\mathrm{ab}} \xrightarrow{\pi_{\mathrm{ab}}} Q_{\mathrm{ab}} \longrightarrow 0$ is also exact; equivalently, Q acts trivially on K_{ab} and ι_{ab} is injective.

THEOREM (FALK-RANDELL 1985/88)

Let $G = K \rtimes_{\varphi} Q$. If Q acts trivially on K_{ab}, then

- $\gamma_n(G) = \gamma_n(K) \rtimes_{\varphi} \gamma_n(Q)$, *for all* $n \geqslant 1$.
- $\mathrm{gr}(G) = \mathrm{gr}(K) \rtimes_{\tilde{\varphi}} \mathrm{gr}(Q)$, *where* $\tilde{\varphi}\colon \mathrm{gr}(Q) \to \mathrm{Der}(\mathrm{gr}(K))$.
- *If K and Q are residually nilpotent, then G is residually nilpotent.*

- For a split extension $G = K \rtimes_{\varphi} Q$, Guaschi and de Miranda e Pereiro define a sequence $L = \{L_n\}_{n\geqslant 1}$ of subgroups of K by

$$L_1 = K, \quad L_{n+1} = \langle [K, L_n], [K, \gamma_n(Q)], [L_n, Q] \rangle.$$

THEOREM (GUASCHI–PEREIRO 2020)

- $\varphi\colon Q \to \operatorname{Aut}(K)$ *restricts to* $\varphi\colon \gamma_n(Q) \to \operatorname{Aut}(L_n)$.
- $\gamma_n(G) = L_n \rtimes_{\varphi} \gamma_n(Q)$.

LEMMA

L is an N-series for K.

THEOREM

$\operatorname{gr}(G) = \operatorname{gr}^L(K) \rtimes_{\tilde{\varphi}} \operatorname{gr}(Q)$, *where* $\tilde{\varphi}\colon \operatorname{gr}(Q) \to \operatorname{Der}(\operatorname{gr}(K))$.

REMARK

If Q acts trivially on K_{ab}, then $L = \gamma(K)$. So these results generalize those of Falk and Randell.

Isolators

- The *isolator* in G of a subset $S \subseteq G$ is the subset

$$\sqrt{S} := \sqrt[G]{S} = \{g \in G \mid g^m \in S \text{ for some } m \in \mathbb{N}\}$$

- Clearly, $S \subseteq \sqrt{S}$ and $\sqrt{\sqrt{S}} = \sqrt{S}$. Also, if $\varphi\colon G \to H$ is a homomorphism, and $\varphi(S) \subseteq T$, then $\varphi(\sqrt[G]{S}) \subseteq \sqrt[H]{T}$.
- The isolator of a subgroup of G need not be a subgroup; for instance, $\sqrt[G]{\{1\}} = \operatorname{Tors}(G)$, which is not a subgroup in general (although it is if G is nilpotent).
- If $N \lhd G$ is a normal subgroup, then $\sqrt[G]{N} = \pi^{-1}(\operatorname{Tors}(G/N))$, where $\pi\colon G \twoheadrightarrow G/N$, and so $\sqrt[G]{N}/N \cong \operatorname{Tors}(G/N)$.

PROPOSITION (MASSUYEAU 2007)

Suppose $\kappa = \{K_n\}_{n\geqslant 1}$ is an N-series for G. Then $\sqrt{\kappa} := \{\sqrt{K_n}\}_{n\geqslant 1}$ is an N_0-series for G.

The rational lower central series

- The *rational lower central series*, $\gamma^{\mathbb{Q}}(G)$, is defined by $\gamma_1^{\mathbb{Q}}(G) = G$ and $\gamma_{n+1}^{\mathbb{Q}}(G) = \sqrt{[G, \gamma_n^{\mathbb{Q}}(G)]}$. (Stallings, 1965)
- $\gamma_n^{\mathbb{Q}}(G) = \sqrt{\gamma_n(G)}$ for all $n \geqslant 1$.
- Hence, $\gamma^{\mathbb{Q}}(G)$ is an N_0-series (since $\gamma(G)$ is an N-series).
- $G/\gamma_n^{\mathbb{Q}}(G) = \Gamma_n / \operatorname{Tors}(\Gamma_n)$ is the maximal torsion-free $(n-1)$-step nilpotent quotient of G; in particular, $G/\gamma_2^{\mathbb{Q}}(G) = G_{\text{abf}}$.
- Associated graded Lie algebra: $\operatorname{gr}^{\mathbb{Q}}(G) = \bigoplus_{n \geqslant 1} \gamma_n^{\mathbb{Q}}(G)/\gamma_{n+1}^{\mathbb{Q}}(G)$.
- G is residually torsion-free nilpotent (RTFN) iff $\gamma_\omega^{\mathbb{Q}}(G) = \{1\}$.

PROPOSITION (BASS & LUBOTZKY 1994)

- $\operatorname{gr}(G) \to \operatorname{gr}^{\mathbb{Q}}(G)$ *has torsion kernel and cokernel in each degree.*
- $\operatorname{gr}(G) \otimes \mathbb{Q} \to \operatorname{gr}^{\mathbb{Q}}(G) \otimes \mathbb{Q}$ *is an isomorphism.*

- Thus, if $b_1(G) < \infty$, then $\phi_n^{\mathbb{Q}}(G) = \phi_n(G)$

Split extensions

- Let $G = K \rtimes_{\varphi} Q$. Since L is an N-series, $\sqrt{L}$ is an N_0-series for K.

THEOREM

- $\varphi\colon Q \to \operatorname{Aut}(K)$ *restricts to* $\varphi\colon \sqrt[Q]{\gamma_n(Q)} \to \operatorname{Aut}\big(\sqrt[K]{L_n}\big)$.
- $\sqrt[G]{\gamma_n(G)} = \sqrt[K]{L_n} \rtimes_{\varphi} \sqrt[Q]{\gamma_n(Q)}$.
- $\operatorname{gr}^{\mathbb{Q}}(G) \cong \operatorname{gr}^{\sqrt{L}}(K) \rtimes_{\tilde{\varphi}} \operatorname{gr}^{\mathbb{Q}}(Q)$.

THEOREM

Suppose Q *acts trivially on* $K_{\text{abf}} := H_1(K,\mathbb{Z})/\operatorname{Tors}$. *Then*

- $\sqrt[K]{L_n} = \sqrt[K]{\gamma_n(K)}$ *for all* n.
- $\sqrt[G]{\gamma_n(G)} = \sqrt[K]{\gamma_n(K)} \rtimes_{\varphi} \sqrt[Q]{\gamma_n(Q)}$.
- $\operatorname{gr}^{\mathbb{Q}}(Q) \cong \operatorname{gr}^{\mathbb{Q}}(K) \rtimes_{\tilde{\varphi}} \operatorname{gr}^{\mathbb{Q}}(Q)$.

COROLLARY

Let $G = K \rtimes Q$ *be a split extension of RTFN groups. If* Q *acts trivially on* K_{abf}, *then* G *is also RTFN.*

Alexander invariants and Chen ranks

- The *Chen Lie algebra* of G is $\operatorname{gr}(G/G'')$, where $G'' = (G')'$.
- If $b_1(G) < \infty$, the *Chen ranks* of G are defined as $\theta_n(G) := \dim_{\mathbb{Q}} \operatorname{gr}_n(G/G'') \otimes \mathbb{Q}$.
- $\theta_n(G) \leqslant \phi_n(G)$, with equality for $n \leqslant 3$.
- *Alexander invariant*: $B(G) := G'/G''$, viewed as a $\mathbb{Z}[G_{\text{ab}}]$-module via $gG' \cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$.
- (Massey) $I^n B(G) = \gamma_{n+2}(G/G'')$, where I is the augmentation ideal of $\mathbb{Z}[G_{\text{ab}}]$, and hence $\operatorname{gr}_n(B(G)) \cong \operatorname{gr}_{n+2}(G/G'')$, for all $n \geqslant 0$.
- If $b_1(G) < \infty$, then $\operatorname{Hilb}(\operatorname{gr}(B(G) \otimes \mathbb{Q}), t) = \sum_{n\geqslant 0} \theta_{n+2}(G) t^n$.

Theorem

Suppose $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *is an* ab*-exact sequence of groups, and* Q *is abelian. Then,*

- *The induced map on Alexander invariants,* $B(\iota)\colon B(K) \to B(G)$*, factors through a* $\mathbb{Z}[K_{\mathrm{ab}}]$*-linear isomorphism,* $B(K) \to B(G)_{\iota}$.
- *If* G_{ab} *is finitely generated, then* $\theta_n(K) \leqslant \theta_n(G)$ *for all* $n \geqslant 1$.
- *If the sequence is split exact, then* ι *induces isomorphisms of graded Lie algebras,*

$$\mathrm{gr}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G) \text{ and } \mathrm{gr}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G/G'').$$

  Consequently, if $b_1(G) < \infty$*, then* $\phi_n(K) = \phi_n(G)$ *and* $\theta_n(K) = \theta_n(G)$ *for all* $n \geqslant 2$.

The rational Alexander invariant

- Let $B_{\mathbb{Q}}(G) := G'_{\mathbb{Q}}/G''_{\mathbb{Q}}$, viewed as a module over $\mathbb{Z}G_{\mathrm{abf}}$, where $G''_{\mathbb{Q}} = (G'_{\mathbb{Q}})'_{\mathbb{Q}} = \sqrt{[G'_{\mathbb{Q}}, G'_{\mathbb{Q}}]}$.
- $I^n(B_{\mathbb{Q}}(G) \otimes \mathbb{Q}) = \gamma^{\mathbb{Q}}_{n+2}(G/G''_{\mathbb{Q}}) \otimes \mathbb{Q}$, where $I = I_{\mathbb{Q}}(G_{\mathrm{abf}})$.
- Hence, $\mathrm{gr}_n(B_{\mathbb{Q}}(G) \otimes \mathbb{Q}) \cong \mathrm{gr}_{n+2}(G/G''_{\mathbb{Q}}) \otimes \mathbb{Q}$, for all $n \geqslant 0$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ be an abf-exact sequence and suppose Q is torsion-free abelian. Then,

- *The map ι induces a $\mathbb{Z}[K_{\mathrm{abf}}]$-linear isomorphism, $B_{\mathbb{Q}}(K) \to B_{\mathbb{Q}}(G)_{\iota}$.*
- *If G_{abf} is finitely generated, then $\theta_n(K) \leqslant \theta_n(G)$ for all $n \geqslant 1$.*
- *If the sequence is split exact, then ι induces isos of graded Lie algebras, $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K) \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G)$ and $\mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(K/K'') \xrightarrow{\simeq} \mathrm{gr}^{\mathbb{Q}}_{\geqslant 2}(G/G'')$.*
  - *Consequently, if $b_1(G) < \infty$, then $\phi_n(K) = \phi_n(G)$ and $\theta_n(K) = \theta_n(G)$ for all $n \geqslant 2$.*

Characteristic varieties

- Let G be a finitely generated group. Then $\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*)$ is an algebraic group, with identity 1 the trivial character, $g \mapsto 1$.
- Clearly, $\mathbb{T}_G = \mathbb{T}_{G_{\mathrm{ab}}}$ and $\mathbb{T}^0_G = \mathbb{T}_{G_{\mathrm{abf}}}$.
- Characteristic varieties: $\mathcal{V}_k(G) := \{\rho \in \mathbb{T}_G \mid \dim H^1(G, \mathbb{C}_\rho) \geqslant k\}$.
- Set $\mathcal{W}_k(G) := \mathcal{V}_k(G) \cap \mathbb{T}^0_G$.
- For each $k \geqslant 1$, we have

$$\mathcal{V}_k(G) = V\big(\operatorname{ann}(\textstyle\bigwedge^k B(G) \otimes \mathbb{C})\big)$$

$$\mathcal{W}_k(G) = V\big(\operatorname{ann}(\textstyle\bigwedge^k B_{\mathbb{Q}}(G) \otimes \mathbb{C})\big),$$

at least away from $1 \in \mathbb{T}^0_G$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be an exact sequence of f.g. groups.*

- *If the sequence is* ab*-exact and* Q *is abelian, then the map* $\iota^*\colon \mathbb{T}_G \twoheadrightarrow \mathbb{T}_K$ *restricts to maps* $\iota^*\colon \mathcal{V}_k(G) \to \mathcal{V}_k(K)$ *for all* $k \geqslant 1$*; furthermore,* $\iota^*\colon \mathcal{V}_1(G) \to \mathcal{V}_1(K)$ *is a surjection.*
- *If the sequence is* abf*-exact and* Q *is torsion-free abelian, then the map* $\iota^*\colon \mathbb{T}^0_G \twoheadrightarrow \mathbb{T}^0_K$ *restricts to maps* $\iota^*\colon \mathcal{W}_k(G) \to \mathcal{W}_k(K)$ *for all* $k \geqslant 1$*; furthermore,* $\iota^*\colon \mathcal{W}_1(G) \to \mathcal{W}_1(K)$ *is a surjection.*

Holonomy Lie algebra

- Assume G_{abf} is finitely generated, and let $\mathbb{L} = \mathrm{Lie}(G_{\mathrm{abf}})$ be the free Lie algebra on G_{abf}, so that $\mathbb{L}_1 = G_{\mathrm{abf}}$ and $\mathbb{L}_2 = G_{\mathrm{abf}} \wedge G_{\mathrm{abf}}$.
- The *holonomy Lie algebra* of G is $\mathfrak{h}(G) := \mathrm{Lie}(G_{\mathrm{abf}})/(\mathrm{im}(\cup_G^{\vee}))$, where $\cup_G^{\vee}\colon H^2(G)^{\vee} \to (H^1(G) \wedge H^1(G))^{\vee} \cong G_{\mathrm{abf}} \wedge G_{\mathrm{abf}}$.
- There is a natural epimorphism $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$, which induces epimorphisms $\mathfrak{h}(G)/\mathfrak{h}(G)'' \twoheadrightarrow \mathrm{gr}(G/G'')$.
- Let $\bar{\theta}_n(G) := \mathrm{rank}\,(\mathfrak{h}(G)/\mathfrak{h}(G)'')_n$. Then: $\bar{\theta}_n(G) \geqslant \theta_n(G)$, $\forall n \geqslant 1$.
- If $b_1(G) < \infty$, we may also define $\mathfrak{h}(G;\mathbb{Q})$. If G_{abf} is finitely generated, $\mathfrak{h}(G;\mathbb{Q}) = \mathfrak{h}(G) \otimes \mathbb{Q}$.
- The *infinitesimal Alexander invariant* is $\mathfrak{B}(G) := \mathfrak{h}(G)'/\mathfrak{h}(G)''$, viewed as a graded module over $\mathrm{Sym}(G_{\mathrm{abf}})$ via $g \cdot \bar{x} = \overline{[g,x]}$ for $g \in \mathfrak{h}/\mathfrak{h}' = G_{\mathrm{abf}}$ and $x \in \mathfrak{h}'$.
- If $b_1(G) < \infty$, then $\bar{\theta}_n(G) = \dim_{\mathbb{Q}} \mathfrak{B}_{n-2}(G;\mathbb{Q})$, for all $n \geqslant 2$.

Resonance varieties

- Let G be a group with $b_1(G) < \infty$. Let $H^* = H^*(G; \mathbb{C})$.
- For each $a \in H^1$, left-multiplication by a yields a cochain complex,

$$(H, \delta_a)\colon \; H^0 \xrightarrow{\delta_a^0} H^1 \xrightarrow{\delta_a^1} H^2.$$

- The *resonance varieties* of G:

$$\mathcal{R}_k(G) \coloneqq \{a \in H^1 \mid \dim_{\mathbb{C}} H^1(H, \delta_a) \geqslant k\}.$$

- They are homogeneous algebraic subvarieties of the affine space $H^1 \cong \mathbb{C}^{b_1(G)}$. Note: $0 \in \mathcal{R}_k(G)$ iff $b_1(G) \geqslant k$.
- $\mathcal{R}_k(G)$ contains every isotropic subspace of H^1 of dimension $\leqslant k+1$; moreover, $\mathcal{R}_1(G)$ is the union of all isotropic planes in H^1.
- $\mathcal{R}_k(G) = V(\operatorname{ann}(\bigwedge^k \mathfrak{B}(G; \mathbb{C}))$, away from 0.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ be an exact sequence of f.g. groups. Suppose that either

- *The sequence is split exact, $\operatorname{gr}(G)$ is quadratic, Q is abelian, and Q acts trivially on $H_1(K, \mathbb{Q})$.*
- *The sequence if ab-exact, G and K are 1-formal, and Q is abelian.*
- *The sequence if abf-exact, G and K are 1-formal, and Q is torsion-free abelian.*

Then $\iota^*\colon H^1(G, \mathbb{C}) \twoheadrightarrow H^1(K, \mathbb{C})$ restricts to maps $\iota^*\colon \mathcal{R}_k(G) \twoheadrightarrow \mathcal{R}_k(K)$ for all $k \geqslant 1$; furthermore, $\iota^*\colon \mathcal{R}_1(G) \twoheadrightarrow \mathcal{R}_1(K)$ is surjective.

Corollary

With hypothesis as above, suppose that $\mathcal{R}_1(G) \subseteq \{0\}$. Then

- $\mathcal{R}_1(K) \subseteq \{0\}$.
- $\bar{\theta}_n(K) \leqslant \bar{\theta}_n(G)$ *for all* $n \geqslant 1$.
- $\bar{\theta}_n(G) = 0$ *for* $n \gg 0$ *and* $\bar{\theta}_n(K) = 0$ *for* $n \gg 0$.

Right-angled Artin groups

- Let $G_\Gamma = \langle v \in V : [v,w] = 1 \text{ if } \{v,w\} \in E \rangle$ be the RAAG associated to a finite (simple) graph $\Gamma = (V,E)$.
- There is a finite $K(G_\Gamma, 1)$ which is formal; thus, G_Γ is 1-formal.
- $H^*(G_\Gamma, \mathbb{Z})$ is the *exterior Stanley–Reisner ring*,
$$\bigwedge(v^* : v \in V)/(v^* w^* : \{v,w\} \notin E).$$
- (Papadima–S. 2006) $\mathfrak{h}(G_\Gamma) = \text{Lie}(V)/([v,w] = 0 \text{ if } \{v,w\} \in E)$ and $\mathfrak{h}(G_\Gamma) \xrightarrow{\simeq} \text{gr}(G_\Gamma)$.
- (Duchamp–Krob 1992, PS06) Each group $\text{gr}_n(G_\Gamma)$ is torsion-free, of rank ϕ_n given by
$$\prod_{n=1}^{\infty} (1 - t^n)^{\phi_n} = P_\Gamma(-t),$$
where $P_\Gamma(t) = \sum_{k \geqslant 0} f_k(\Gamma) t^k$ is the clique polynomial of Γ, with $f_k(\Gamma) = \#\{k\text{-cliques in } \Gamma\}$.

- $\mathfrak{h}_\Gamma/\mathfrak{h}''_\Gamma \xrightarrow{\simeq} \operatorname{gr}(G_\Gamma/G''_\Gamma)$.

- The graded pieces of $\operatorname{gr}(G_\Gamma/G''_\Gamma)$ are torsion-free, with ranks θ_n given by

$$\sum_{n=2}^{\infty} \theta_n t^n = Q_\Gamma\left(\frac{t}{1-t}\right),$$

where $Q_\Gamma(t) = \sum_{j\geqslant 2} c_j(\Gamma) t^j$ is the "cut polynomial" of Γ, with

$$c_j(\Gamma) = \sum_{W\subset V\colon |W|=j} \tilde{b}_0(\Gamma_W).$$

- $\mathcal{R}_1(G_\Gamma)$ is the union of the coordinate subspaces $\mathbb{C}^W \subset \mathbb{C}^V$ for which the induced subgraph Γ_W is disconnected.

- $\mathcal{V}_1(G_\Gamma)$ is the union of the coordinate subtori $(\mathbb{C}^*)^W \subset (\mathbb{C}^*)^V$ for which the induced subgraph Γ_W is disconnected.

Bestvina–Brady groups

- The *Bestvina–Brady group* associated to Γ is defined as $N_\Gamma = \ker(\pi\colon G_\Gamma \to \mathbb{Z})$, where $\pi(v) = 1$, for each $v \in V(\Gamma)$.
- (Meier–Van Wyck 1995) N_Γ is finitely generated iff Γ is connected.
- (Bestvina–Brady 1997) N_Γ is finitely presented iff the flag complex Δ_Γ is simply connected.
- (BB97) A counterexample to either the Eilenberg–Ganea conjecture or the Whitehead asphericity conjecture can be constructed from these groups.
- The cohomology ring $H^*(N_\Gamma, \mathbb{Z})$ was computed in (Papadima–S. 2007) and (Leary–Saadetoğlu 2011).

Theorem (Papadima–S. 2007/2009, S. 2021)

Suppose Γ is connected. Then

- $1 \to N_\Gamma \xrightarrow{\iota} G_\Gamma \xrightarrow{\pi} \mathbb{Z} \to 1$ *is a split,* $\mathtt{ab}$*-exact sequence.*
- $\operatorname{gr}_{\geqslant 2}(N_\Gamma) \cong \operatorname{gr}_{\geqslant 2}(G_\Gamma)$.
- $\operatorname{gr}_{\geqslant 2}(N_\Gamma/N_\Gamma'') \cong \operatorname{gr}_{\geqslant 2}(G_\Gamma/G_\Gamma'')$.
- $\phi_k(N_\Gamma) = \phi_k(G_\Gamma)$ *and* $\theta_k(N_\Gamma) = \theta_k(G_\Gamma)$ *for all* $k \geqslant 2$.
- *The map* $\iota^*\colon H^1(G_\Gamma, \mathbb{C}^*) \twoheadrightarrow H^1(N_\Gamma, \mathbb{C}^*)$ *restricts to a surjection,* $\iota^*\colon \mathcal{V}_1(G_\Gamma) \twoheadrightarrow \mathcal{V}_1(N_\Gamma)$.
- *The map* $\iota^*\colon H^1(G_\Gamma, \mathbb{C}) \twoheadrightarrow H^1(N_\Gamma, \mathbb{C})$ *restricts to a surjection,* $\iota^*\colon \mathcal{R}_1(G_\Gamma) \twoheadrightarrow \mathcal{R}_1(N_\Gamma)$.

The complement of a hyperplane arrangement

- Let $\mathcal{A}$ be a central arrangement of m hyperplanes in $\mathbb{C}^d$. For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$; set $f = \prod_{H \in \mathcal{A}} \alpha_H$.
- The complement, $M(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}} H$, is a Stein manifold, and so it has the homotopy type of a (connected) d-dimensional CW-complex.
- In fact, $M = M(\mathcal{A})$ has a minimal cell structure. Consequently, $H_*(M, \mathbb{Z})$ is torsion-free (and finitely generated).
- In particular, $H_1(M, \mathbb{Z}) = \mathbb{Z}^m$, generated by meridians $\{x_H\}_{H \in \mathcal{A}}$.
- The cohomology ring $H^*(M, \mathbb{Z})$ is determined solely by the intersection lattice, $L(\mathcal{A})$.
- M is $\mathbb{Q}$-formal, but not $\mathbb{Z}_p$-formal, in general.

Fundamental groups of arrangements

- For an arrangement $\mathcal{A}$, the group $G = \pi_1(M(\mathcal{A}))$ admits a finite presentation, with generators $\{x_H\}_{H\in\mathcal{A}}$ and commutator-relators.
- $\mathcal{V}_k(M)$ is a finite union of torsion-translated subtori of $\mathbb{T}_G = (\mathbb{C}^*)^m$.
- $G/\gamma_2(G)$ and $G/\gamma_3(G)$ are determined by $L_{\leqslant 2}(\mathcal{A})$.
- $G/\gamma_4(G)$—and thus G—is not necessarily determined by $L_{\leqslant 2}(\mathcal{A})$.
- [Porter–S. 2020] Suppose $\mathcal{A}$ is decomposable, i.e., $\mathfrak{h}_3(G) \cong \bigoplus_{X\in L_2(A)} \mathfrak{h}_3\big(F_{|X|-1}\big)$. Then *all* nilpotent quotients are combinatorially determined.
- Since M is formal, G is 1-formal, i.e., its pronilpotent completion, $\mathfrak{m}(G)$, is quadratic.
- Hence, $\mathrm{gr}(G)\otimes\mathbb{Q} = \mathrm{gr}(\mathfrak{m}(G))$ is determined by $L_{\leqslant 2}(\mathcal{A})$.

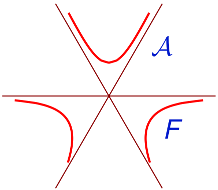
- The holonomy Lie algebra of $G = G(\mathcal{A})$ is determined by $L_{\leqslant 2}(\mathcal{A})$,

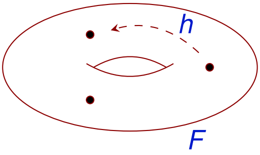
$$\mathfrak{h}(G) = \mathrm{Lie}(x_H : H \in \mathcal{A}) \Big/ \mathrm{ideal}\left\{ \Big[x_H, \sum_{K \in \mathcal{A},\, K \supset Y} x_K \Big] \ : \ \substack{H \in \mathcal{A}, Y \in L_2(\mathcal{A}) \\ H \supset Y} \right\}.$$

- Then $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$ (since G is 1-formal).
- An explicit combinatorial formula is lacking in general for the LCS ranks $\phi_n(G)$, although such formulas are known when
  - $\mathcal{A}$ is supersolvable $\Rightarrow H^*(M, \mathbb{Q})$ is Koszul
  - $\mathcal{A}$ is decomposable
  - $\mathcal{A}$ is a graphic arrangement

  and in some more cases just for $\phi_3(G)$.
- $\mathrm{gr}_n(G)$ may have torsion (at least for $n \geqslant 4$), but the torsion is not necessarily determined by $L_{\leqslant 2}(\mathcal{A})$.
- The map $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism [Porter–S.], but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- (Papadima–S. 2004) The Chen ranks $\theta_n(G)$ are determined by $L_{\leqslant 2}(\mathcal{A})$.

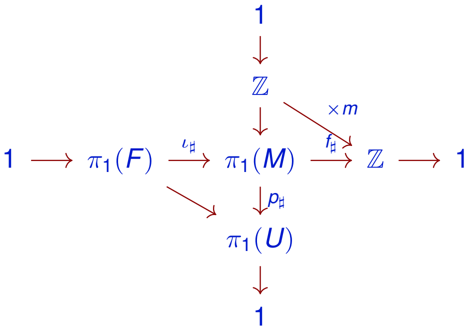
The Milnor fibration





- The map $f\colon \mathbb{C}^d \to \mathbb{C}$ restricts to a smooth fibration, $f\colon M \to \mathbb{C}^*$, called the *Milnor fibration* of $\mathcal{A}$.
- The *Milnor fiber* is $F(\mathcal{A}) := f^{-1}(1)$. The monodromy, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/m} z$, where $m = |\mathcal{A}|$.
- F is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $d-1$ (connected if $d > 1$).
- MHS on F may not be pure; $\pi_1(F)$ may be non-1-formal [Zuber].
- $H_1(F,\mathbb{Z})$ may have torsion [Yoshinaga].

- F is the regular, $\mathbb{Z}_m$-cover of $U = \mathbb{P}(M)$, classified by the epimorphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_m$, $x_H \mapsto 1$.
- To study $\pi_1(F)$, we may assume w.l.o.g. that $d = 3$.
- Let $\iota\colon F \hookrightarrow M$ be the inclusion. Induced maps on π_1:



- $b_1(F) \geqslant m - 1$, and may be computed from $\mathcal{V}^1_k(U)$. Combinatorial formulas are known in some cases (e.g., if $\mathbb{P}(\mathcal{A})$ has only double or triple points [Papadima–S. 2017]), but not in general.

Trivial algebraic monodromy

Theorem (S. 2021)

Suppose $h_*\colon H_1(F;\mathbb{Z}) \to H_1(F;\mathbb{Z})$ *is the identity. Then*

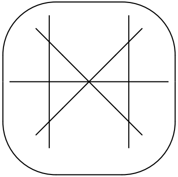
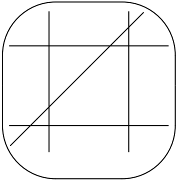
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)) \cong \operatorname{gr}_{\geqslant 2}(G)$.
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \cong \operatorname{gr}_{\geqslant 2}(G/G'')$.

Theorem (S. 2021)

Suppose $h_*\colon H_1(F,\mathbb{Q}) \to H_1(F,\mathbb{Q})$ *is the identity. Then*

- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)) \otimes \mathbb{Q} \cong \operatorname{gr}_{\geqslant 2}(G) \otimes \mathbb{Q}$.
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \otimes \mathbb{Q} \cong \operatorname{gr}_{\geqslant 2}(G/G'') \otimes \mathbb{Q}$.
- $\phi_k(\pi_1(F)) = \phi_k(G)$ *and* $\theta_k(\pi_1(F)) = \theta_k(G)$ *for all* $k \geqslant 2$.

Falk's pair of arrangements

$\mathcal{A}$   $\mathcal{A}'$

- Both $\mathcal{A}$ and $\mathcal{A}'$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$. Nevertheless, $M(\mathcal{A}) \simeq M(\mathcal{A}')$.
- $\mathcal{V}_1(M)$ and $\mathcal{V}_1(M')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^6$, corresponding to the triple points; $\mathcal{V}_2(M) = \mathcal{V}_2(M') = \{1\}$.
- Both Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- $\mathcal{V}_1(F)$ and $\mathcal{V}_1(F')$ consist of two 2-dimensional subtori of $(\mathbb{C}^*)^5$.
- (S. 2017) $\pi_1(F) \not\cong \pi_1(F')$.
- The difference is picked by the depth-2 characteristic varieties: $\mathcal{V}_2(F) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2(F') = \{1\}$

Yoshinaga's icosidodecahedral arrangement

- The icosidodecahedron is the convex hull of 30 vertices given by the even permutations of $(0, 0, \pm 1)$ and $\frac{1}{2}(\pm 1, \pm\phi, \pm\phi^2)$, where $\phi = (1 + \sqrt{5})/2$.
- It gives rise to an arrangement of 16 hyperplanes in $\mathbb{R}^3$, whose complexification is the icosidodecahedral arrangement $\mathcal{A}$ in $\mathbb{C}^3$.
- $M(\mathcal{A})$ is a $K(G, 1)$.
- $H_1(F, \mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. Thus, the algebraic monodromy of the Milnor fibration is trivial over $\mathbb{Q}$ and $\mathbb{Z}_p$ ($p > 2$), but not over $\mathbb{Z}$.
- Hence, $\operatorname{gr}(\pi_1(F)) \cong \operatorname{gr}(\pi_1(U))$, away from the prime 2. Moreover,
  - $\operatorname{gr}_1(\pi_1(F)) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$
  - $\operatorname{gr}_2(\pi_1(F)) = \mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$
  - $\operatorname{gr}_3(\pi_1(F)) = \mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$
  - $\operatorname{gr}_4(\pi_1(F)) = \mathbb{Z}^{1,405} \oplus \mathbb{Z}_2^{?}$ and $\mathfrak{h}_4(\pi_1(F)) = \mathbb{Z}^{1,405} \oplus \mathbb{Z}_2^{20}$.

References

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- Alexander I. Suciu, *Alexander invariants and cohomology jump loci in group extensions*, Annali della Scuola Normale Superiore di Pisa (doi), arXiv:2107.05148.