# CUP-ONE ALGEBRAS AND 1-MINIMAL MODELS 

RICHARD D. PORTER ${ }^{1}$ AND ALEXANDER I. SUCIU ${ }^{1,2}$


#### Abstract

In previous work we introduced the notion of binomial cup-one algebras, which are differential graded algebras endowed with Steenrod $\cup_{1}$-products and compatible binomial operations. Given such an $R$-dga, $\left(A, d_{A}\right)$, defined over the $\operatorname{ring} R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ (for $p$ a prime), with $H^{0}(A)=R$ and with $H^{1}(A)$ a finitely generated, free $R$-module, we show that $A$ admits a functorially defined 1-minimal model, $\rho:(\mathcal{M}(A), d) \rightarrow\left(A, d_{A}\right)$, which is unique up to isomorphism. Furthermore, we associate to this model a pronilpotent group, $G(A)$, which only depends on the 1-quasi-isomorphism type of $A$. These constructions, which refine classical notions from rational homotopy theory, allow us to distinguish spaces with isomorphic (torsion-free) cohomology rings that share the same rational 1-minimal model, yet whose integral 1-minimal models are not isomorphic.


## Contents

1. Introduction ..... 2
2. Delta-sets, magmas, and cochain algebras ..... 8
3. Differential graded algebras and homotopies ..... 13
4. The Steenrod $\cup_{i}$-products ..... 18
5. Binomial cup-one differential graded algebras ..... 23
6. Free binomial $\cup_{1}$-differential graded algebras ..... 28
7. Differentials defined by admissible maps ..... 33
8. Hirsch extensions ..... 40
9. The existence of 1-minimal models ..... 48

[^0]10. Uniqueness and functoriality of 1-minimal models.......................... . . 54
11. Compatibility of integer and rational 1-minimal models ..................... . . 64
12. Distinguishing homotopy types via 1-minimal models........................ . 67

References........................................................................................... 73

## 1. Introduction

1.1. Overview. In previous work we combined properties of the Steenrod cup-one products of cochains and binomial rings in the cochain complex of a space to define the algebraic categories of binomial cup-one differential graded algebras over the integers and over $\mathbb{Z}_{p}$ for $p$ a prime. In this paper we define the 1 -minimal model for such a dga $(A, d)$, and prove its key properties; namely, the 1 -minimal model of $A$ is a free binomial cup-one dga unique up to isomorphism, the 1-minimal model determines a group, $G(A)$, also unique up to isomorphism. If the group is nilpotent, then the cohomology of the 1-minimal model is isomorphic to the cohomology of the group.

Since the cochains of a space with integer or with $\mathbb{Z}_{p}$ coefficients are binomial $\cup_{1}$-dgas, it follows that invariants of the 1-minimal model give homotopy type invariants of spaces in the case where $A=C^{*}(X ; R)$ with $R$ equal to $\mathbb{Z}$ or $\mathbb{Z}_{p}$. As an application, we define in Section 12 one such invariant, called $n$-step equivalence, and exhibit a family of spaces that can be distinguished using the 1 -minimal model over $\mathbb{Z}$, where the corresponding approach in rational homotopy theory fails to distinguish those spaces.
The reader familiar with rational homotopy theory ([6, 7, 9]), will note that the 1-minimal models here over $\mathbb{Z}$ and over $\mathbb{Z}_{p}$ can be viewed as analogues of Sullivan's 1-minimal model over the rationals. Hence, techniques from rational homotopy theory have the potential to serve as a starting point for developing techniques that yield stronger results over $\mathbb{Z}$ and $\mathbb{Z}_{p}$, as shown by the example in Section 12.3.
1.2. Cochain algebras and Steenrod $\cup_{i}$-products. We start by considering a $\Delta$-set $X$, that is, a sequence of sets $X=\left\{X_{n}\right\}_{n \geq 0}$ and maps $d_{i}: X_{n} \rightarrow X_{n-1}$ for each $0 \leq i \leq n$ such that $d_{i} d_{j}=d_{j-1} d_{i}$ for all $i<j$. The geometric realization of $X$, denoted $|X|$, may be viewed either as a special kind of CW-complex, or a generalized simplicial complex.

Let $A=\left(C^{*}(X ; R), d\right)$ be the cellular cochain complex of $|X|$, with coefficients in a commutative ring $R$. Then $A$ is, in fact, a differential graded $R$-algebra, with multiplication given by the cup-product of cochains. In [22], Steenrod introduced a whole sequence of operations, $\cup_{i}: A^{p} \otimes_{R} A^{q} \rightarrow A^{p+q-i}$, starting with $\cup_{0}=\cup$, the usual cup-product. We are mainly interested here in the additional structure on the cochain algebra provided by the $\cup_{1}$-product, which is tied to the differential and the cup product via the Steenrod [22] and

Hirsch [13] identities,

$$
\begin{align*}
& d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b-a \cup_{1} d b,  \tag{1.1}\\
& (a \cup b) \cup_{1} c=a \cup\left(b \cup_{1} c\right)+\left(a \cup_{1} c\right) \cup b . \tag{1.2}
\end{align*}
$$

for all $a, b, c \in A^{1}$.
1.3. Binomial cup-one dgas. We defined in [18] several categories of differential graded algebras (over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ ) with some extra structure, coming from either the cup-one products, or from a binomial ring structure, or both, bound together by suitable compatibility conditions. In Sections 4.3 and 4.4 we recall these notions, with some mild modifications to better fit the present context.
A cup-one differential graded algebra is an $R$-dga ( $A, d$ ) with a cup-one product map $\cup_{1}: A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$ that gives $R \oplus A^{1}$ the structure of a commutative ring and satisfies the Hirsch identity, as well as the following " $\cup_{1}-d$ formula,"

$$
\begin{equation*}
d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b+d b \cup_{1} a-d a \circ d b \tag{1.3}
\end{equation*}
$$

for all $a, b \in A^{1}$ with $d a, d b$ equal to sums of cup products, where the map $\circ$ is bilinear and satisfies $\left(a_{1} \cup a_{2}\right) \circ\left(b_{1} \cup b_{2}\right)=\left(a_{1} \cup_{1} b_{1}\right) \cup\left(a_{2} \cup_{1} b_{2}\right)$. The significance of the $\cup_{1}-d$ formula is that it expresses the differentials of cup-one products of elements in $A^{1}$ in terms of the differentials of the factors and cup-one products of elements in $A^{1}$, as opposed to formula (1.1), which involves cup-one products of elements in $A^{2}$ with elements in $A^{1}$. Moreover, as shown in [18], if $A=C^{*}(X ; R)$ is a cochain algebra, Steenrod's formula (1.1) restricts to formula (1.3) for elements $a, b \in A^{1}$ with $d a$ and $d b$ equal to sums of cup products.
In Section 5, we add extra structure to these algebras. A commutative ring $A$ is called a binomial ring if $A$ is torsion-free as a $\mathbb{Z}$-module, and has the property that the elements $\binom{a}{n}:=a(a-1) \cdots(a-n+1) / n!$ lie in $A$ for every $a \in A$ and every $n>0$. An analogous notion holds for $\mathbb{Z}_{p}$-algebras. These objects come equipped with maps $\zeta_{n}: A \rightarrow A$, $a \mapsto\binom{a}{n}$, defined for all $n>0$ over $\mathbb{Z}$, and only for $n<p$ over $\mathbb{Z}_{p}$.
Now consider a cup-one dga $(A, d)$ over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. Such an object is called a binomial cup-one algebra if $A^{0}$, with multiplication $A^{0} \otimes_{R} A^{0} \rightarrow A^{0}$ given by the cup-product, is a binomial $R$-algebra, and the $R$-submodule $R \oplus A^{1} \subset A^{\leq 1}$, with multiplication $A^{1} \otimes_{R} A^{1} \rightarrow$ $A^{1}$ given by the cup-one product, is an $R$-binomial algebra.
Our main motivating example is the cochain algebra of a space. In Theorem 5.13 we show that, for any $\Delta$-set $X$, the cellular cochain algebra $C=\left(C^{*}(X ; R), d\right)$ is a binomial cup-one dga, with maps $C^{1} \otimes_{R} C^{1} \rightarrow C^{1}$ given by $\left(a \cup_{1} b\right)(e)=a(e) \cdot b(e)$ for all 1-simplices $e$ and with the $\circ$ map equal to Steenrod's $U_{2}$ product, with $\circ=\cup_{2}: C^{2} \otimes_{R} C^{2} \rightarrow C^{2}$ given by $(v \circ w)(s)=v(s) \cdot w(s)$ for all 2-simplices $s$, and with binomial maps given by $\zeta_{n}(a)(e)=\binom{a(e)}{n}$ when $R=\mathbb{Z}$ and analogously for $R=\mathbb{Z}_{p}$.
1.4. Free binomial cup-one dgas. These structures allow us to define in Section 6 the free binomial cup-one graded algebra, $\mathrm{T}=\mathrm{T}_{R}^{*}(\boldsymbol{X})$, on a set $\boldsymbol{X}$. When $R=\mathbb{Z}$, the starting point is the $\operatorname{ring} \operatorname{Int}\left(\mathbb{Z}^{X}\right)=\left\{q \in \mathbb{Q}[X] \mid q\left(\mathbb{Z}^{X}\right) \subseteq \mathbb{Z}\right\}$ of integrally-valued polynomials with variables in $\boldsymbol{X}$. This is a binomial ring, generated by the polynomials $\binom{X}{\mathbf{n}}=\prod_{x \in X}\binom{x}{\mathbf{n}_{x}}$ with $\mathbf{n}_{x}$ equal to the non-negative integers. We define $T$ to be the tensor algebra on $\mathfrak{m}_{X}$, the maximal ideal at $0 \operatorname{in} \operatorname{Int}\left(\mathbb{Z}^{X}\right)$. When $R=\mathbb{Z}_{p}$, an analogous definition applies, with suitable modifications. In either case, we have $R$-linear maps $U_{1}: \mathrm{T}^{1} \otimes \mathrm{~T}^{1} \rightarrow \mathrm{~T}^{1}$, given by $a \cup_{1} b=a b$ and $\circ: \mathrm{T}^{2} \otimes \mathrm{~T}^{2} \rightarrow \mathrm{~T}^{2}$ given by $\left(a_{1} \otimes a_{2}\right) \circ\left(b_{1} \otimes b_{2}\right)=\left(a_{1} b_{1}\right) \otimes\left(a_{2} b_{2}\right)$. Using the binomial structure on $\mathrm{T}^{1}$, we show that the map sending each $x \in X$ to 0 extends to a linear map $d_{0}: \mathrm{T}^{1} \rightarrow \mathrm{~T}^{2}$. In turn, $d_{0}$ extends to a differential on the whole tensor algebra by the graded Leibniz rule, and $\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{0}\right)$ is then a binomial cup-one dga. The key result that allows us to construct the differential $d_{0}$ is Theorem 6.7, which reads as follows.

Theorem 1.1. Let $d: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X})$ be a degree-one map satisfying the $\cup_{1}-d$ formula and the Leibniz rule. Then $d^{2}(x)=0$ for all $x \in \boldsymbol{X}$ if and only if $d^{2}(u)=0$ for all $u \in \mathrm{~T}_{R}(\boldsymbol{X})$, in which case $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ is a binomial cup-one dga.

Taking $d(x)=0$ for all $x \in 0$ yields the differential $d_{0}$ from above. As shown in [18], this differential is compatible with the binomial structure on $\mathrm{T}_{R}(\boldsymbol{X})$; more precisely,

$$
\begin{equation*}
d_{0}\left(\zeta_{n}(x)\right)=-\sum_{\ell=1}^{n-1} \zeta_{\ell}(x) \otimes \zeta_{n-\ell}(x) \tag{1.4}
\end{equation*}
$$

for all $x \in \boldsymbol{X}$, and for all $n \geq 1$ when $R=\mathbb{Z}$ and for $1 \leq n \leq p-1$ when $R=\mathbb{Z}_{p}$. As an application of the methods developed here, we give in Theorem 7.10 a quicker, more conceptual proof of this result.
1.5. Differentials defined by admissible maps. A key thread in our paper involves the correspondence (described in Section 2) between a magma, that is, a set $M$ with a binary operation $\mu: M \times M \rightarrow M$, and a certain 2-dimensional $\Delta$-set, $\Delta^{(2)}(M)$. In the case when $\mu$ is associative, that is, $(M, \mu)$ is a semigroup, this 2 -complex extends to an infinitedimensional cell complex $\Delta(M)$, whose $n$-simplices are given by ordered $n$-tuples of elements in $M$. Moreover, if $(M, \mu)$ is a group, then $\Delta(M)$ is the cell complex of the bar construction applied to $M$; that is, an Eilenberg-MacLane classifying space for $M$.

Properties of the cellular cochain algebras $\left(C^{*}(\Delta(M) ; R), d_{\Delta}\right)$ are used in Section 7 to derive properties of differential graded algebras in our category of binomial $\cup_{1}$-dgas, as follows. Given a set $\boldsymbol{X}$ and a set map $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$, we start by defining a binary operation, $\mu_{\tau}: M \times M \rightarrow M$, on the $R$-module $M=M(\boldsymbol{X}, R)$ of all functions from $\boldsymbol{X}$ to the ring $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. Letting $\Delta^{(2)}\left(M_{\tau}\right)$ be the 2-dimensional $\Delta$-set associated to the
magma $M_{\tau}=\left(M, \mu_{\tau}\right)$, we define a degree-preserving, $R$-linear map,

$$
\begin{equation*}
\psi=\psi_{X, \mu}: \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \longrightarrow C^{*}\left(\Delta^{(2)}\left(M_{\tau}\right) ; R\right) . \tag{1.5}
\end{equation*}
$$

This map sends $1 \in \mathrm{~T}_{R}^{0}(\boldsymbol{X})=R$ to the unit 0 -cochain; a polynomial $q \in \mathrm{~T}_{R}^{1}(\boldsymbol{X})=\mathfrak{m}_{X}$ to the 1 -cochain whose value on a 1-simplex $\mathbf{a}: \boldsymbol{X} \rightarrow R$ is $q(\mathbf{a})$; and a tensor $q \otimes q^{\prime} \in \mathrm{T}_{R}^{2}(\boldsymbol{X})$ to the 2 -cochain whose value on a 2 -simplex ( $\left.\mathbf{a}, \mathbf{a}^{\prime}\right)$ is $q(\mathbf{a}) \cdot q^{\prime}\left(\mathbf{a}^{\prime}\right)$. We then show in Lemma 7.1 that the map $\psi$ is monomorphism that commutes with cup products, cup-one products, and the o maps. Using the embedding $\psi$, we show in Theorem 7.3 that there is a unique extension of the map $\tau: \boldsymbol{X} \rightarrow \mathrm{T}_{R}^{2}(\boldsymbol{X})$ to an $R$-linear map $d_{\tau}: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X})$ that satisfies the Leibniz rule and the $\cup_{1}-d$ formula.

To make further headway, we focus on the case when $M_{\tau}$ is a semigroup (we then say $\tau$ is admissible), and consider the associated cell complex, $\Delta\left(M_{\tau}\right)$. We then show in Theorem 7.6 that the map $\psi$ extends uniquely to an inclusion $\psi:\left(\mathrm{T}(\boldsymbol{X}), d_{\tau}\right) \hookrightarrow C^{*}\left(\Delta\left(M_{\tau}\right) ; R\right)$ that satisfies $\psi \circ d_{\tau}=d_{\Delta} \circ \psi$, from which it follows that $d_{\tau}^{2}$ is the zero map. These results can be summarized as follows.

Theorem 1.2. If the map $\tau: \boldsymbol{X} \rightarrow \mathrm{T}^{2}(\boldsymbol{X})$ is admissible, then $d_{\tau}^{2} \equiv 0$ and the map $\mathrm{T}^{1}(\boldsymbol{X}) \rightarrow C^{1}\left(\Delta\left(M_{\tau}\right) ; R\right)$ given by $q \mapsto(\mathbf{a} \mapsto q(\mathbf{a}))$ extends uniquely to a monomorphism $\psi_{X}:\left(\mathrm{T}(\boldsymbol{X}), d_{\tau}\right) \hookrightarrow\left(C^{*}\left(\Delta\left(M_{\tau}\right) ; R\right), d_{\Delta}\right)$ of binomial cup-one dgas.
1.6. Hirsch extensions. In Section 8 we continue laying out the groundwork for the construction of 1-minimal models for $\cup_{1}$-dgas over the ring $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. The first step in the construction is a free binomial $\cup_{1}$-dga of the form $\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{\mathbf{0}}\right)$ with $\boldsymbol{X}$ a finite set of $n$ elements. In this case, the map $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$ is the zero map, and the corresponding $R$-module, $M=M_{\tau}$, is isomorphic to $R^{n}$. Using a spectral sequence argument, we prove in Theorem 8.9 that the map $\psi_{\boldsymbol{X}}:\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{\mathbf{0}}\right) \rightarrow C^{*}\left(B\left(R^{n}\right) ; R\right)$ induces an isomorphism on cohomology.

As in rational homotopy theory, Hirsch extensions of free binomial $\cup_{1}$-dgas are the basic building blocks for constructing 1-minimal models. An inclusion $i:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow$ ( $\left.\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ is called a Hirsch extension if $\bar{d}(y)$ is a cocycle in $\mathrm{T}_{R}^{2}(\boldsymbol{X})$ for all $y \in \boldsymbol{Y}$. As shown in Theorem 8.2, there is a bijection between maps of sets from $Y$ to cocycles in $\mathrm{T}_{R}^{2}(\boldsymbol{X})$ and Hirsch extensions of this sort.

Assume now that $\boldsymbol{X}=\bigcup_{i \geq 1} \boldsymbol{X}_{i}$ with each $\boldsymbol{X}_{i}$ a finite set and $\boldsymbol{X}_{1} \neq \emptyset$. An $R$-dga $\mathrm{T}=$ ( $\left.\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ is called a colimit of Hirsch extensions if the differential $d$ restricts to differentials $d_{n}$ on $\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right)$ such that $d_{1}(x)=0$ for all $x \in \boldsymbol{X}_{1}$ and each dga $\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n+1}\right), d_{n+1}\right)$ is a Hirsch extension of $\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right), d_{n}\right)$. To such a colimit of Hirsch extensions, T, we associate in Lemma 8.11 a pronilpotent group, $G_{\mathrm{T}}$, together with a $\cup_{1}$-dga map, $\psi_{\mathrm{T}}: \mathrm{T} \rightarrow$ $C^{*}\left(B\left(G_{\mathrm{T}}\right) ; R\right)$, which induces an isomorphism on $H^{1}$. Moreover, as shown in Theorem 8.12, if $\psi_{\mathrm{T}}$ is a quasi-isomorphism and $0 \rightarrow F \rightarrow \bar{G} \xrightarrow{\pi} G \rightarrow 1$ is a central extension of groups with $F$ a finitely generated, free $R$-module, then there is a Hirsch extension
$i: \mathrm{T} \hookrightarrow \overline{\mathrm{T}}$ such that $\bar{G}=G_{\bar{T}}$, the diagram

commutes, and the map $\psi_{\overline{\mathrm{T}}}$ is also a quasi-isomorphism.
In work in progress [19], we build on this correspondence between colimits of Hirsch extensions and sequences of central extensions of groups. Results include the relationship between finite colimits of Hirsch extensions and nilmanifolds. For the cochain algebra $A=C^{*}(Y ; R)$ of a path-connected space $Y$, we describe a concrete relationship between the group $G_{\mathrm{T}}$ and the fundamental group $\pi_{1}(Y)$, where T is a colimit of Hirsch extensions together with a 1-quasi-isomorphism $\rho: \mathrm{T} \rightarrow C^{*}(Y ; R)$; that is, ( $\left.\mathrm{T}, \rho\right)$ is a 1-minimal model for $C^{*}(Y ; R)$.
1.7. 1-minimal models. In Sections 9 and 10 (which form the core of this work), we develop these ideas into a theory of 1-minimal models over a ring $R$ equal to $\mathbb{Z}$ or $\mathbb{Z}_{p}$.
A key technical tool is provided by the following lifting criterion (Theorem 9.4). Let $f: A \rightarrow A^{\prime}$ be a surjective 1-quasi-isomorphism between binomial cup-one $R$-dgas and let $\varphi: \mathrm{T} \rightarrow A^{\prime}$ be a morphism from a colimit of Hirsch extensions to $A^{\prime}$. There is then a morphism $\widehat{\varphi}: \mathrm{T} \rightarrow A$ such that $f \circ \widehat{\varphi}=\varphi$.
Now let $(A, d)$ be a binomial cup-one $R$-dga. A 1-minimal model for $A$ is a colimit of Hirsch extensions $\mathcal{M}_{n}=\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right), d_{n}\right)$, together with morphisms $\rho_{n}: \mathcal{M}_{n} \rightarrow A$ compatible with the Hirsch extensions of $\mathcal{M}_{n}$ into $\mathcal{M}_{n+1}$. Additionally, the map $H^{1}\left(\rho_{1}\right): H^{1}\left(\mathcal{M}_{1}\right) \rightarrow$ $H^{1}(A)$ is required to be an isomorphism; in particular, $\boldsymbol{X}_{1}$ corresponds to a basis for $H^{1}(A)$. For $n \geq 1$, the set $\boldsymbol{X}_{n+1}$ is a basis for the free submodule $\operatorname{ker}\left(H^{2}\left(\rho_{n}\right)\right) \subset H^{2}\left(\mathcal{M}_{n}\right)$ given by the cohomology classes of the 2-cocycles $d_{n+1}(x)$ with $x \in X_{n+1}$.

In Theorems 9.8 and 10.3 we show that every binomial cup-one dga admits (under some mild finiteness assumptions) a 1-minimal model, unique up to isomorphism. These results may be summarized, as follows.

Theorem 1.3. Let $\left(A, d_{A}\right)$ be a binomial cup-one dga over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, with $p$ a prime. Assume $H^{0}(A)=R$ and $H^{1}(A)$ is a finitely generated, free $R$-module. Then,
(1) There is a 1-minimal model, $\mathcal{M}=\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$, for $A$, and a structural morphism, $\rho: \mathcal{M} \rightarrow A$, that is a 1-quasi-isomorphism.
(2) Given 1-minimal models, $\rho: \mathcal{M} \rightarrow A$ and $\rho^{\prime}: \mathcal{M}^{\prime} \rightarrow A$, there is an isomorphism $f: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ and a dga homotopy $\Phi: \mathcal{M} \rightarrow A \otimes_{R} C^{*}([0,1] ; R)$ from $\rho$ to $\rho^{\prime} \circ f$.

In the case when $\left(A, d_{A}\right)$ admits an augmentation, that is, a dga morphism $\varepsilon: A \rightarrow R$, the isomorphism $f$ from above is unique (in the category of augmented dgas). More precisely, we prove in Theorems 9.10 and 10.8 that $A$ has an augmented 1-minimal model, $\mathcal{M}$, such that the structural morphism is an augmentation-preserving 1-quasiisomorphism. Moreover, given augmented 1-minimal models, $\rho: \mathcal{M} \rightarrow A$ and $\rho^{\prime}: \mathcal{M}^{\prime} \rightarrow$ $A$, there is a unique augmentation-preserving isomorphism $f: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ such that $\rho$ is augmentation-preserving homotopic to $\rho^{\prime} \circ f$.
1.8. Compatibility of integer and rational 1-minimal models. In Section 11 we show that the integer 1-minimal model of a space $Y$ tensored with the rationals is weakly equivalent as a dga to the 1 -minimal model for $Y$ in rational homotopy theory.

The algebra of polynomial forms with rational coefficients on a standard simplex was used by Sullivan in [25] to define the algebra $A_{\mathrm{PL}}(Y)$ of compatible polynomial forms on the singular simplices of a space $Y$; this algebra is a commutative dga over the rationals. The properties of the 1-minimal model of a cdga over $\mathbb{Q}$ are analogous to-and in fact are the motivation for-the properties we use to define the 1-minimal model for a binomial cup-one dga over $\mathbb{Z}$ or $\mathbb{Z}_{p}$.
In addition to the Sullivan algebra $A_{\mathrm{PL}}(Y)$ and the singular cochain algebra $C^{*}(Y ; \mathbb{Q})$, there is a dga over the rationals $C A(Y)$ with the property proved in [6] that for topological spaces $Y$, there are natural quasi-isomorphisms $C^{*}(Y ; \mathbb{Q}) \rightarrow C A(Y) \leftarrow A_{\mathrm{PL}}(Y)$. Consequently, $A_{\mathrm{PL}}(Y)$ is weakly equivalent (as a dga) to $C^{*}(Y ; \mathbb{Q})$. The following result, Theorem 11.4, shows that weak equivalence extends to 1-minimal models.
Theorem 1.4. Let $Y$ be a connected topological space with $H^{1}(Y ; \mathbb{Z})$ finitely generated. Then the 1-minimal model for $C^{*}(Y ; \mathbb{Z})$ tensored with the rationals is weakly equivalent as a differential graded algebra to the 1-minimal model in rational homotopy theory for $A_{\text {PL }}(Y)$.

As we shall see next, although the 1-minimal model for $C^{*}(Y ; \mathbb{Z})$ is weakly equivalent over $\mathbb{Q}$ to the (rational) 1-minimal model for $A_{\mathrm{PL}}(Y)$, the integral version does contain more refined information than its rational version.
1.9. $n$-step equivalence and triple Massey products. Given a positive integer $n$, we define in Section 12 the relation of $n$-step equivalence on the set of augmented binomial cup-one dgas $(A, d)$ over $\mathbb{Z}$ for which $H^{0}(A)=\mathbb{Z}, H^{1}(A)$ is finitely generated and torsionfree, and $H^{2}(A)$ is finitely generated. We then construct an infinite family of spaces for which elements in the family can be distinguished using the 1 -minimal model over $\mathbb{Z}$, but the same approach in rational homotopy theory fails to distinguish among the spaces in the family.
The definition of $n$-step equivalence is motivated as follows. If a morphism $\varphi: A \rightarrow A^{\prime}$ induces isomorphisms of cohomology groups in degrees up to 2 , then for each $n \geq 1$ there
is an isomorphism of the $n$ th-step in the respective 1-minimal models, $f_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$, such that the following diagram commutes


Note that the horizontal arrows in (1.7) are isomorphisms. We say that $A$ and $A^{\prime}$ are $n$ step equivalent if there are isomorphisms $f_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$ and $e_{n}: H^{2}(A) \rightarrow H^{2}\left(A^{\prime}\right)$ such that the diagram (1.7) commutes with $H^{2}(\varphi)$ replaced by $e_{n}$.

If $A$ and $A^{\prime}$ are $n$-step equivalent, then the cokernels of the homomorphisms $H^{2}\left(\rho_{n}\right)$ and $H^{2}\left(\rho_{n}^{\prime}\right)$ are isomorphic, and hence have isomorphic torsion subgroups. Given a space $X$ with $n$ th-step in the 1 -minimal model given by $\left(\mathcal{M}_{n}, \rho_{n}\right)$, we define $\kappa_{n}(X)=$ Tors(coker $H^{2}\left(\rho_{n}\right)$ ). The following result (proved in Theorem 12.4) relates the invariant $\kappa_{n}(X)$ of the $n$-step equivalence class of $C^{*}(X ; \mathbb{Z})$ to the fundamental group of $X$.
Theorem 1.5. Let $X$ and $X^{\prime}$ be two connected $\Delta$-complexes with first and second integral cohomology groups finitely generated. Then,
(1) If $\pi_{1}(X) \cong \pi_{1}\left(X^{\prime}\right)$, then $\kappa_{n}(X) \cong \kappa_{n}^{\prime}(X)$ for all $n \geq 1$.
(2) If $\kappa_{n}(X) \not \equiv \kappa_{n}^{\prime}(X)$ for some $n \geq 1$, then the cochain algebras $C^{*}(X ; \mathbb{Z})$ and $C^{*}\left(X^{\prime} ; \mathbb{Z}\right)$ are not $n$-step equivalent.

We apply this result to an infinite family of links in the three-sphere, $\{L(n)\}_{n \geq 1}$, the first term of which is the well-known Borromean rings. Set $X(n)$ equal to the complement of $L(n)$ in $S^{3}$. In Proposition 12.5, we show that $\kappa_{2}(X(n))=\mathbb{Z}_{n} \oplus \mathbb{Z}_{n}$, so by part (2) of Theorem 1.5, $X(n)$ and $X(m)$ are not 2 -step equivalent for $n \neq m$. We also show that the Sullivan algebras $A_{\mathrm{PL}}(X(n))$ and $A_{\mathrm{PL}}(X(m))$ are 2-step equivalent for all positive integers $n$ and $m$.
In the proof of Proposition 12.5, the cokernel of $H^{2}\left(\rho_{2}\right)$ is given by triple Massey products of cohomology classes in $H^{1}(X(n) ; \mathbb{Z})$. This framework provides the context for defining restricted Massey products, which is a particular case of a more general construction that will be developed in [20]. The theory of generalized Massey products continues the program initiated in [17] and is being developed more fully in [19, 20], along with further applications to topological spaces, including complements of complex hyperplane arrangements.

## 2. Delta-sets, magmas, and cochain algebras

2.1. $\Delta$-sets and $\Delta$-complexes. We start the section by reviewing the notion of a $\Delta$ complex, in the sense of Rourke and Sanderson [21]; see also Hatcher [12] and Friedman
[8]. We will view such a complex as the geometric realization of the corresponding $\Delta$-set, cf. [8].

An (abstract) $n$-simplex $\Delta^{n}$ is simply a finite ordered set, $(0,1, \ldots, n)$. The face maps $d_{i}: \Delta^{n} \rightarrow \Delta^{n-1}$, given by omitting the $i$-th element in the set, satisfy $d_{i} d_{j}=d_{j-1} d_{i}$ whenever $0 \leq i<j \leq n$. The geometric realization of the simplex, $\left|\Delta^{n}\right|$, is the convex hull of $n+1$ affinely independent vectors in $\mathbb{R}^{n+1}$, endowed with the subspace topology; the face maps induce continuous maps, $d_{i}:\left|\Delta^{n}\right| \rightarrow|\Delta|^{n-1}$.
More generally, a $\Delta$-set consists of a sequence of sets $X=\left\{X_{n}\right\}_{n \geq 0}$ and maps $d_{i}: X_{n} \rightarrow$ $X_{n-1}$ for each $0 \leq i \leq n$ such that $d_{i} d_{j}=d_{j-1} d_{i}$ whenever $i<j$. This is the generalization of the notion of ordered (abstract) simplicial complex, where the sets $X_{n}$ are the sets of $n$-simplices and the maps $d_{i}$ are the face maps. We refer to $X^{(n)}=\left\{X_{i}\right\}_{i=0}^{n}$ as the $n$-skeleton of the $\Delta$-set, and say that $X^{(n)}$ has dimension (at most) $n$.

The geometric realization of a $\Delta$-set $X$ is the topological space

$$
\begin{equation*}
|X|=\coprod_{n \geq 0} X_{n} \times\left|\Delta^{n}\right| / \sim, \tag{2.1}
\end{equation*}
$$

where $\sim$ is the equivalence relation generated by $\left(x, d^{i}(p)\right) \sim\left(d_{i}(x), p\right)$ for $x \in X_{n+1}$, $p \in\left|\Delta^{n}\right|$, and $0 \leq i \leq n$, where $d^{i}:\left|\Delta^{n}\right| \rightarrow\left|\Delta^{n+1}\right|$ is the inclusion of the $i$-th face. Such a space is called a $\Delta$-complex, and can be viewed either as a special kind of CW-complex, or a generalized simplicial complex.

The assignment $X \leadsto|X|$ is functorial: if $f: X \rightarrow Y$ is a map of $\Delta$-sets (i.e., $f$ is a family of maps $f_{n}: X_{n} \rightarrow Y_{n}$ commuting with the face maps), there is an obvious realization, $|f|:|X| \rightarrow|Y|$, and this is a (continuous) map of $\Delta$-complexes.
The chain complex of a $\Delta$-set $X$, denoted $C_{*}(X ; \mathbb{Z})$, coincides with the simplicial chain complex of its geometric realization: for each $n \geq 0$, the chain group $C_{n}(X)$ is the free abelian group on $X_{n}$, while the boundary maps $\partial_{n}: C_{n}(X) \rightarrow C_{n-1}(X)$ are the linear maps given by $\partial_{n}=\sum_{i}(-1)^{i} d_{i}$. If $B$ is an abelian group, the chain complex of $X$ with coefficients in $B$ is defined as $C_{*}(X ; B)=C_{*}(X ; \mathbb{Z}) \otimes B$. The cochain complex $C^{*}(X ; B)$ is defined by setting $C^{n}(X ; B)=\operatorname{Hom}\left(C_{n}(X), B\right)$ and dualizing the differentials. We denote by $H_{*}(X ; B)$ and $H^{*}(X ; B)$, respectively, the homology groups of these complexes.
2.2. From binary operations to $\Delta$-sets. Let $M=(M, \mu)$ be a magma, that is, a set $M$ equipped with a binary operation, $\mu: M \times M \rightarrow M$, commonly written as $\left(a_{1}, a_{2}\right) \mapsto$ $a_{1} a_{2}$. These data determine a 2 -dimensional $\Delta$-set, denoted $\Delta^{(2)}(M)$, whose geometric realization can be described as follows. There is a single vertex; each element $a \in M$ gives a 1 -simplex, and to each ordered pair of 1 -simplices, $a_{1}$ and $a_{2}$, we assign a 2simplex, $\left(a_{1}, a_{2}\right)$, with front face $a_{1}$, back face $a_{2}$, and third face equal to $a_{1} a_{2}$.

This construction is functorial, in the following sense. Suppose $h:(M, \mu) \rightarrow\left(M^{\prime}, \mu^{\prime}\right)$ is a morphism of magmas, that is, $\mu^{\prime}\left(h\left(a_{1}\right), h\left(a_{2}\right)\right)=h\left(\mu\left(a_{1}, a_{2}\right)\right)$ for all $a_{1}, a_{2} \in M$.


Figure 1.
Then $h$ determines in a straightforward manner a simplicial map between the respective $\Delta$-complexes, $\Delta(h): \Delta^{(2)}\left(M^{\prime}, \mu^{\prime}\right) \rightarrow \Delta^{(2)}(M, \mu)$, so that $\Delta(h \circ g)=\Delta(h) \circ \Delta(g)$.
Now suppose $M=(M, \mu)$ is a semigroup, that is, the operation $\mu$ on the magma $M$ is associative. Then this construction can be pushed through in all dimensions.

Lemma 2.1. Let $M$ be a semigroup, and let $\Delta^{(2)}(M)$ be the 2-dimensional $\Delta$-set determined by the underlying magma. Then $\Delta^{(2)}(M)$ is the 2 -skeleton of a $\Delta$-set, $\Delta(M)$, whose $n$-simplices are given by ordered $n$-tuples of elements in $M$.

Proof. We let $\Delta_{0}(M)$ be a singleton, and $\Delta_{n}(M)=M^{n}$, the cartesian product of $n$ copies of $M$. The face maps $d_{i}: M^{n} \rightarrow M^{n-1}$ send an ordered $n$-tuple $\left(a_{1}, \ldots, a_{n}\right) \in M^{n}$ to $\left(a_{2}, \ldots, a_{n}\right)$ if $i=0$, to $\left(a_{1}, \ldots, a_{n}\right)$ if $i=n$, and to $\left(a_{1}, \ldots, a_{i} a_{i+1}, \ldots, a_{n}\right)$, otherwise.

Now, if the operation $\mu$ is associative, then, as indicated in Figure 1, the 2-dimensional $\Delta$ complex corresponding to $\Delta^{(2)}(M)$ extends to a 3-dimensional $\Delta$-set, whose 3-simplices are ordered triples, $\left(a_{1}, a_{2}, a_{3}\right)$, of elements in $A$. More generally, a routine computation shows that $d_{i} d_{j}=d_{j-1} d_{i}$ for all $i<j$, and thus $\Delta(M)$ is indeed a $\Delta$-set, with 2 -skeleton equal to $\Delta^{(2)}(M)$.

Given a semigroup $M$, we define a $\Delta$-set $S(M)$, as follows. We let $S_{n}(M)$ equal to the set of all functions, $f$, from the 1 -simplices of the abstract $n$-simplex $\Delta^{n}$ to $M$ with the property that

$$
\begin{equation*}
f(i, \ell)=f(i, j) \cdot f(j, \ell) \quad \text { for all } 0 \leq i<j<\ell \leq k \tag{2.2}
\end{equation*}
$$

The face maps $d_{i}: S_{n}(M) \rightarrow S_{n-1}(M)$ are given by the restriction of $f$ to the faces of $\Delta^{n}$.
Lemma 2.2. The $\Delta$-set $S(M)$ coincides with $\Delta(M)$.

Proof. From the associativity of the multiplication in $M$ it follows that an arbitrary map from the 1 -simplices of $\Delta^{n}$ of the form $(i, i+1)$ to $M$ extends uniquely to a map from all 1 -simplices of $\Delta^{n}$ to $M$ that satisfies equation (2.2). This gives a bijection between the elements in $S_{n}(M)$ and the $n$-tuples, $\left(a_{1}, \ldots, a_{n}\right)$, of elements in $M$. It is readily seen that this bijection is compatible with the respective face maps.
Remark 2.3. Of particular importance is the case when $M$ is a monoid, that is, a semigroup with multiplication $\mu: M \times M \rightarrow M$ and two-sided identity $e$. Then $\Delta(M)$ is the bar construction on $M$ : the corresponding $\Delta$-complex, $B(M)=|\Delta(M)|$, has a single 0 cell, and an $n$-cell $\left[g_{1}|\ldots| g_{n}\right]$ for each $n$-tuple $\left(g_{1}, \ldots, g_{n}\right) \in M^{n}$. The chain complex $C_{*}(B(M) ; \mathbb{Z})$ yields a resolution by free $\mathbb{Z}[M]$-modules of the group $\mathbb{Z}$, viewed as a trivial module over the monoid-ring $\mathbb{Z}[M]$. Finally, if $M=G$ is a group, then $B(G)$ is an Eilenberg-MacLane classifying space $K(G, 1)$; see [14, Ch. 10] and also [1, 3, 4].
2.3. Cocycles and $\Delta$-complexes. Let $M=(M, \mu)$ be a magma, with multiplication $\mu: M \times M \rightarrow M$ written as $\mu\left(a_{1}, a_{2}\right)=a_{1} a_{2}$, and let $B$ be an abelian group, together with a map of sets $v: M \times M \rightarrow B$, written $\left(a_{1}, a_{2}\right) \mapsto a_{1} * a_{2}$. Defining a map $\eta:(M \times B) \times(M \times B) \rightarrow M \times B$ by

$$
\begin{equation*}
\eta\left(\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right)\right)=\left(a_{1} a_{2}, b_{1}+b_{2}+a_{1} * a_{2}\right) \tag{2.3}
\end{equation*}
$$

for all $a_{i} \in M$ and $b_{i} \in B$ turns the set $M \times B$ into a magma which we call the extension of $(M, \mu)$ by $v$. If $(M, \mu)$ has a two-sided identity, $e$, and if $\eta$ satisfies $\eta\left(\left(a, b_{1}\right),\left(e, b_{2}\right)\right)=$ $\eta\left(\left(e, b_{2}\right),\left(a, b_{1}\right)\right)$ and $\eta\left(\left(e, b_{1}\right),\left(e, b_{2}\right)\right)=\left(e, b_{1}+b_{2}\right)$ for all $a \in M$ and $b_{1}, b_{2} \in B$, then the extension is called a central extension.

From the correspondence between 2-simplices in $|\Delta(M)|$ and ordered pairs of elements in $M$, it follows that $v$ may be viewed as an element in $C^{2}(\Delta(M) ; B)$. If the magma $(M, \mu)$ has a two-sided identity $e$, we say that $v$ is normalized if $v(a, e)=v(e, a)=0$ for all $a \in M$. In the case when $(M, \mu)$ is a monoid (that is, the operation $\mu$ is associative and has a two-sided identity $e$ ), a cochain $\xi \in C^{k}(\Delta(M) ; B)$ is called normalized if $\xi\left(a_{1}, \ldots, a_{k}\right)=0$ whenever $a_{i}=e$ for some $i$.

The following lemma gives conditions on $(M, \mu)$ and $v$ for the extension of a semigroup to be a semigroup, the extension of a monoid to be a monoid, and for the extension of a group to be a group. Note that a monoid is a semigroup with identity.
Lemma 2.4. Given a magma $(M, \mu)$ and a map $v: M \times M \rightarrow B$, the extension $E=$ $(M \times B, \eta)$ of $M$ by the abelian group $B$ as defined above has the following properties.
(1) Suppose $(M, \mu)$ has a two-sided identity $e$ and $v$ is a normalized cochain. Then $E$ is a central extension.
(2) Suppose $(M, \mu)$ is a semigroup. Then $(E, \eta)$ is a semigroup if and only if $v$ is a cocycle in $C^{2}(\Delta(M) ; B)$.
(3) Suppose $(M, \mu)$ is a monoid and $v$ is a normalized cocycle in $Z^{2}(\Delta(M) ; B)$. Then $(E, \eta)$ is a monoid.
(4) Suppose $(M, \mu)$ is a group and $v$ is a normalized cocycle in $Z^{2}(\Delta(M) ; B)$. Then $(E, \eta)$ is a group.

Proof. Part (1) follows directly from formula (2.3) expressing $\eta$ in terms of $v$ and using the assumption that $v$ is a normalized cochain.

To prove part (2), note that from equation (2.3), it follows that

$$
\begin{aligned}
& \left(a_{1}, b_{1}\right)\left[\left(a_{2}, b_{2}\right)\left(a_{3}, b_{3}\right)\right]=\left(a_{1}\left(a_{2} a_{3}\right), b_{1}+b_{2}+b_{3}+a_{2} * a_{3}+a_{1} *\left(a_{2} a_{3}\right)\right) \\
& {\left[\left(a_{1}, b_{1}\right)\left(a_{2}, b_{2}\right)\right]\left(a_{3}, b_{3}\right)=\left(\left(a_{1} a_{2}\right) a_{3}, b_{1}+b_{2}+b_{3}+a_{1} * a_{2}+\left(a_{1} a_{2}\right) * a_{3}\right) .}
\end{aligned}
$$

Hence, the binary operation $\eta$ on $M \times B$ is associative if and only if

$$
\begin{equation*}
a_{2} * a_{3}+a_{1} *\left(a_{2} a_{3}\right)=a_{1} * a_{2}+\left(a_{1} a_{2}\right) * a_{3} \tag{2.4}
\end{equation*}
$$

for all $a_{i} \in M$. Let us view the map $v: M \times M \rightarrow B$ as a $B$-valued 2-cochain on $\Delta(M)$. To find a formula for the coboundary of $v$, we use Figure 1 to identify an arbitrary triple $\left(a_{1}, a_{2}, a_{3}\right) \in M^{3}$ with the standard 3 -simplex with vertices $0,1,2$, and 3 and then to identify ordered pairs of elements in $M$ with 2-simplices. This gives the following

$$
\begin{aligned}
\delta v\left(a_{1}, a_{2}, a_{3}\right) & =v([1,2,3])-v([0,2,3])+v([0,1,3])-v([0,1,2]) \\
& =v\left(a_{2}, a_{3}\right)-v\left(a_{1} a_{2}, a_{3}\right)+v\left(a_{1}, a_{2} a_{3}\right)-v\left(a_{1}, a_{2}\right) \\
& =a_{2} * a_{3}-\left(a_{1} a_{2}\right) * a_{3}+a_{1} *\left(a_{2} a_{3}\right)-a_{1} * a_{2} .
\end{aligned}
$$

Hence, the identity (2.4) is satisfied precisely when $v$ is a cocycle, and the proof of part (2) is complete.

To prove part (3), note that from part (2) we know that $E=(M \times B, \eta)$ is a semigroup, so it suffices to show that $E$ has an identity. From equation (2.3) and using the assumption that $v$ is a normalized cochain, we have

$$
(a, b)(e, 0)=(a e, b+0+a * e)=(a, b),
$$

for all $(a, b) \in M \times B$, where we used the assumption that $a * e=0$ for all $a \in M$. A similar argument shows that $(e, 0)(a, b)=(a, b)$ for all $(a, b) \in M \times B$, and the proof of part (3) is complete.

To prove part (4), note that from parts (2) and (3) it follows that $E$ is a monoid so it suffices to show that every element in $E$ has a two-sided inverse. It is well known that if $E$ is a monoid and each element $a$ has a right inverse $a_{r}$, then $a_{r}$ is also a left inverse of $a$, and hence $E$ is a group. So we only need to show that every element in $E$ has a right inverse. Let $(a, b) \in M \times B$; then

$$
(a, b)\left(a^{-1},-b-a * a^{-1}\right)=\left(a a^{-1}, b+(-b)-a * a^{-1}+a * a^{-1}\right)=(e, 0)
$$

and the proof of part (4) is complete.

## 3. Differential graded algebras and homotopies

3.1. Differential graded algebras. Throughout this section, we work over a fixed coefficient ring $R$, assumed to be commutative and with unit 1 . We start with some basic definitions.

Definition 3.1. A graded algebra over $R$ is an $R$-algebra $A$ such that the underlying $R$ module is a direct sum of $R$-modules, $A=\bigoplus_{i \geq 0} A^{i}$, and such that the product $A \otimes_{R} A \rightarrow A$ sends $A^{i} \otimes_{R} A^{j}$ to $A^{i+j}$.

We refer to the multiplication maps $\cup: A^{i} \otimes_{R} A^{j} \rightarrow A^{i+j}$, given by $\cup(a \otimes b)=a \cup b$ as the cup-product maps; we also refer to the elements of $A^{i}$ as $i$-cochains. A morphism of graded algebras is a map of $R$-algebras preserving degrees.

Definition 3.2. A differential graded algebra over $R$ (for short, a dga) is a graded $R$ algebra $A=\bigoplus_{i \geq 0} A^{i}$ endowed with a degree 1 map, $d: A \rightarrow A$, satisfying $d^{2}=0$ and the graded Leibniz rule,

$$
\begin{equation*}
d(a \cup b)=d a \cup b+(-1)^{|a|} a \cup d b \tag{3.1}
\end{equation*}
$$

for all homogenous elements $a, b \in A$, where $|a|$ is the degree of $a$.
We denote by $[a] \in H^{i}(A)$ the cohomology class of a cocycle $a \in Z^{i}(A)$. As usual, the graded $R$-module $H^{*}(A)$ inherits an algebra structure from $A$.
Observe that $A^{0}$ is a subring of $A$ and the structure map $R \rightarrow A$ sends the unit $1 \in R$ to the unit of $A$, which we will also denote by 1 , and which necessarily has degree 0 . Consequently, $R$ may be viewed as a subring of $A^{0}$, and the graded pieces $A^{i}$ may be viewed as $A^{0}$-modules. We say that $A$ is connected if the structure map $R \rightarrow A$ maps $R$ isomorphically to $A^{0}$. We say that $A$ is graded commutative (for short, $A$ is a cdga) if $a b=(-1)^{|a||b|} b a$ for all homogeneous elements $a, b \in A$.
A morphism of differential graded $R$-algebras is an $R$-linear map $\varphi: A \rightarrow B$ between two dgas which preserves the grading and commutes with the respective differentials and products. The induced map in cohomology, $\varphi^{*}: H^{*}(A) \rightarrow H^{*}(B),[a] \mapsto[\varphi(a)]$, is a morphism of graded $R$-algebras. The map $\varphi$ is called a quasi-isomorphism if $\varphi^{*}$ is an isomorphism. Two dgas are called weakly equivalent if there is a zig-zag of quasiisomorphisms connecting one to the other; plainly, this is an equivalence relation among dgas. A dga $(A, d)$ is said to be formal if it is weakly equivalent to its cohomology algebra, $H^{*}(A)$, endowed with the zero differential.
All these notions have partial analogues. Fix an integer $q \geq 1$. A dga map $\varphi: A \rightarrow$ $B$ is a $q$-quasi-isomorphism if the induced homomorphism, $\varphi^{*}: H^{i}(A) \rightarrow H^{i}(B)$, is an isomorphism for $i \leq q$ and a monomorphism for $i=q+1$. Two dgas are called $q$ equivalent if they may be connected by a zig-zag of $q$-quasi-isomorphisms. Finally, a dga $(A, d)$ is $q$-formal if it is $q$-equivalent to $\left(H^{*}(A), d=0\right)$.

Remark 3.3. For later use, let us note the following. Suppose $\varphi: A \rightarrow B$ is a surjective $q$-quasi-isomorphism. From the long exact sequence in cohomology induced by the exact sequence of cochain complexes $0 \longrightarrow \operatorname{ker}(\varphi) \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$, it then follows that $H^{i}(\operatorname{ker}(\varphi))=0$ for $i \leq q+1$.

If $\left(A, d_{A}\right)$ and $\left(B, d_{B}\right)$ are two dgas, then the tensor product of the underlying graded $R$ modules, $A \otimes_{R} B$, acquires a dga structure, with multiplication and differential given on homogeneous elements by $(a \otimes b) \cdot\left(a^{\prime} \otimes b^{\prime}\right)=(-1)^{\left|a^{\prime}\right|| |} a a^{\prime} \otimes b b^{\prime}$ and $d_{A \otimes_{R} B}(a \otimes b)=$ $d_{A}(a) \otimes b+(-1)^{|a|} a \otimes d_{B}(b)$. The direct product $A \times B$ also has a natural structure of a dga, with $(a, b) \cdot\left(a^{\prime}, b^{\prime}\right)=a a^{\prime} \otimes b b^{\prime}$ and $d_{A \times B}(a, b)=\left(d_{A}(a), d_{B}(b)\right)$. If $\varphi: A \rightarrow B$ and $\varphi^{\prime}: A^{\prime} \rightarrow B$ are two dga maps, then their fiber product, denoted $A \times_{B} A^{\prime}$, is the sub-dga of $A \times B$ consisting of all pairs $(a, b)$ with $\varphi(a)=\varphi^{\prime}(b)$.
3.2. Cochain algebras. The motivating example for us is the singular cochain algebra $C^{*}(X ; R)$ on a space $X$, with coefficients in a commutative ring $R$. This is an $R$-dga, with differentials given by the usual coboundary maps, and with multiplication given by the cup product. We will be mostly interested in the case when $X$ is a simplicial complex, or, more generally, a $\Delta$-complex (see $[21,12,8]$ ). We will view such a complex as the geometric realization of the corresponding abstract simplicial complex or $\Delta$-set, respectively, and we will use the simplicial cochain algebra of $X$, still to be denoted by $C^{*}(X ; R)$. Let us note that the structure map $R \rightarrow C^{0}(X ; R)$ sends an element $r \in R$ to the cochain whose value on every vertex is $r$.

Example 3.4. Let $I$ be the closed interval [0,1], viewed as a simplicial complex in the usual way, and let $C=C^{*}(I ; R)$ be its cochain algebra over $R$. Then $C^{0}=R \oplus R$ with generators $t_{0}, t_{1}$ corresponding to the endpoints 0 and 1 , and $C^{1}=R$ with generator $u$. The differential $d: C^{0} \rightarrow C^{1}$ is given by $d t_{0}=-u$ and $d t_{1}=u$, while the multiplication is given on generators by $t_{i} t_{j}=\delta_{i j} t_{i}, t_{0} u=u t_{1}=u$, and $u t_{0}=t_{1} u=0$. Note that the cocycle $t_{0}+t_{1}$ is the unit of $C$. Furthermore, $H^{*}(C)=R$, concentrated in degree 0 .

Example 3.5. Now let $G$ be a group. Recall from Section 2.2 that in the $\Delta$-complex $B(G)$ for the bar construction on $G$, there is one 0 -cell, one 1 -cell [ $g$ ] for each $g \in G$, and one 2 -cell for each ordered pair $\left[g_{1} \mid g_{2}\right]$ of elements in $G$. Thus, the 1 -cochains are functions $f: G \rightarrow R$ and the 2-cochains are functions from $G \times G$ to $R$. The cup product and differential are as follows:

$$
\begin{align*}
(f \cup h)\left(\left[g_{1} \mid g_{2}\right]\right) & =f\left(g_{1}\right) \cdot h\left(g_{2}\right),  \tag{3.2}\\
(d f)\left(\left[g_{1} \mid g_{2}\right]\right) & =f\left(\left[g_{1}\right]\right)+f\left(\left[g_{2}\right]\right)-f\left(\left[g_{1} \cdot g_{2}\right]\right),
\end{align*}
$$

where • denotes the product in $R$ or $G$ depending on the context.
Following J.H.C. Whitehead [26], we say that two maps of spaces, $f, g: X \rightarrow Y$, are $n$ homotopic if $f \circ h \simeq g \circ h$, for every map $h: K \rightarrow X$ from a CW-complex $K$ of dimension
at most $n$. A map $f: X \rightarrow Y$ is an $n$-homotopy equivalence (for some $n \geq 1$ ) if it admits an $n$-homotopy inverse. If such a map $f$ exists, we say that $X$ and $Y$ have the same $n$ homotopy type. Two CW-complexes, $X$ and $Y$, are said to be of the same $n$-type if their $n$-skeleta have the same ( $n-1$ )-homotopy type. Any two connected CW-complexes have the same 1-type, and they have the same 2-type if and only if their fundamental groups are isomorphic.

A (cellular) map $f: X \rightarrow Y$ between two CW-complexes induces a morphism of dgas, $f^{\sharp}: C^{*}(Y ; R) \rightarrow C^{*}(X ; R)$, between the respective cochain algebras, and thus a morphism, $f^{*}: H^{*}(Y ; R) \rightarrow H^{*}(X ; R)$, between their cohomology algebras. If $f$ is a homotopy equivalence, then $f^{\sharp}$ is a quasi-isomorphism of $R$-dgas. The next result, which develops ideas from [26], was proved in [18].

Theorem 3.6 ([18]). If $X$ and $Y$ are $C W$-complexes of the same n-type, then the cochain algebras $C^{*}(X ; R)$ and $C^{*}(Y ; R)$ are $(n-1)$-equivalent. In particular, if $\pi_{1}(X) \cong \pi_{1}(Y)$, then $C^{*}(X ; R)$ and $C^{*}(Y ; R)$ are 1-equivalent.

The $(n-1)$-equivalence between the $n$-skeleta of $X$ and $Y$ takes a special form, which we now recall, for it will be needed in the proof of Theorem 12.4. By [26, Theorem 6], there is a homotopy equivalence, $f$, from $\bar{X}^{(n)}=X^{(n)} \vee \bigvee_{i \in I} S_{i}^{n}$ to $\bar{Y}^{(n)}=Y^{(n)} \vee \bigvee_{j \in J} S_{j}^{n}$, for some indexing sets $I$ and $J$. Let $q_{X}: \bar{X}^{(n)} \rightarrow X^{(n)}$ and $q_{Y}: \bar{Y}^{(n)} \rightarrow Y^{(n)}$ be the maps that collapse the wedges of $n$-spheres to the basepoint of the wedge, and consider the induced morphisms on cochain algebras,

$$
\begin{equation*}
C^{*}\left(X^{(n)} ; R\right) \xrightarrow{q_{X}^{\nexists}} C^{*}\left(\bar{X}^{(n)} ; R\right) \stackrel{f^{\sharp}}{\longleftarrow} C^{*}\left(\bar{Y}^{(n)} ; R\right) \stackrel{q_{Y}^{\sharp}}{\longleftarrow} C^{*}\left(Y^{(n)} ; R\right) \tag{3.3}
\end{equation*}
$$

The map $f^{\sharp}$ is a quasi-isomorphism, while $q_{X}^{\sharp}$ and $q_{Y}^{\sharp}$ are $(n-1)$-quasi-isomorphisms; thus, (3.3) is the desired $(n-1)$-equivalence between the $n$-skeleta of $X$ and $Y$.
3.3. Homotopy invariance. Let $C^{*}(I ; R)$ be the cochain algebra of the interval $I$, as described in Example 3.4, and let $\eta_{0}, \eta_{1}: C^{*}(I ; R) \rightarrow R$ denote the $R$-linear maps induced by restriction to the endpoints of $I$; that is to say, $\eta_{i}\left(t_{j}\right)=\delta_{i j}$, and $\eta_{i}(u)=0$.

Definition 3.7. Two dga maps, $\varphi_{0}, \varphi_{1}: A \rightarrow B$, are said to be homotopic (denoted $\varphi_{0} \simeq$ $\varphi_{1}$ ) if there is a dga map $\Phi: A \rightarrow B \otimes_{R} C^{*}(I ; R)$ such that the following diagram commutes for $i=0,1$ :


From the commutativity of the diagram (3.4) it follows that a homotopy $\Phi$ is given on elements $a \in A^{i}$ by

$$
\begin{equation*}
\Phi(a)=\varphi_{0}(a) t_{0}+\varphi_{1}(a) t_{1}-c(a) u \tag{3.5}
\end{equation*}
$$

for some $c(a) \in B^{i-1}$. In particular, if $a \in A^{0}$, then $c(a)=0$, and so $\Phi(a)=\varphi_{0}(a) t_{0}+$ $\varphi_{1}(a) t_{1}$.

Theorem 3.8. Homotopic dga maps induce the same map on cohomology:

$$
\varphi_{0} \simeq \varphi_{1} \Rightarrow \varphi_{0}^{*}=\varphi_{1}^{*} .
$$

Proof. We proceed in a manner similar to the proof of [6, Proposition 12.8(i)] (see also [11, Remark 5.10.3]). Define a linear map $h: A \rightarrow B$ of degree -1 by

$$
\begin{equation*}
\Phi(a)=\varphi_{0}(a) t_{0}+\varphi_{1}(a) t_{1}-(-1)^{|a|} h(a) u \tag{3.6}
\end{equation*}
$$

for every homogeneous element $a \in A$. Then:

$$
\begin{aligned}
& d \Phi(a)=d \varphi_{0}(a) t_{0}-(-1)^{|a|} \varphi_{0}(a) u+d \varphi_{1}(a) t_{1}+(-1)^{|a|} \varphi_{1}(a) u-(-1)^{|a|} d h(a) u, \\
& \Phi(d a)=\varphi_{0}(d a) t_{0}+\varphi_{1}(d a) t_{1}-(-1)^{|a|+1} h(d a) u .
\end{aligned}
$$

Since the maps $\varphi_{0}, \varphi_{1}$, and $\Phi$ commute with the differentials, we infer that $\varphi_{1}-\varphi_{0}=$ $d h+h d$, and the claim follows.
3.4. Augmented dgas and the wedge sum. Let $\left(A, d_{A}\right)$ be a differential graded algebra over a unital commutative ring $R$. Let us view the ground ring $R$ as a dga concentrated in degree 0 and with differential $d=0$. An augmentation for $A$, then, is a dga-map, $\varepsilon_{A}: A \rightarrow R$. We call the triple $\left(A, d_{A}, \varepsilon_{A}\right)$ an augmented dga. A morphism in this category is a dga map, $\varphi:\left(A, d_{A}\right) \rightarrow\left(B, d_{B}\right)$, such that $\varepsilon_{B} \circ \varphi=\varepsilon_{A}$.

Recall that $A$ is connected if the structure map $\sigma_{A}: R \rightarrow A^{0}$ is an isomorphism of rings; in this case we assume the augmentation $\varepsilon_{A}: A \rightarrow R$ then restricts to an isomorphism from $A^{0}$ to $R$. The composition $\varepsilon_{A} \circ \sigma_{A}: R \rightarrow R$ then is an isomorphism of rings, and hence, is the identity map. Thus, if $A$ is connected, it has a unique augmentation map. Moreover, if $\varphi: A \rightarrow B$ is an augmentation-preserving morphism between connected $R$-dgas, the map $\varphi^{0}: A^{0} \rightarrow B^{0}$ may be identified with $\operatorname{id}_{R}$. In general, though, a dga may admit many augmentations.
If $A$ and $B$ are two augmented dgas, we denote by $A \vee B=A \times_{R} B$ the fiber product of the augmentation maps $\varepsilon_{A}: A \rightarrow R$ and $\varepsilon_{B}: B \rightarrow R$. Note that $(A \vee B)^{0}$ is the kernel of the map $\left(\varepsilon_{A},-\varepsilon_{B}\right): A^{0} \oplus B^{0} \rightarrow R$, while $(A \vee B)^{i}=A^{i} \oplus B^{i}$ for $i>0$.
The motivation for these definitions comes from topology. Let $X$ be a topological space, and let $C^{*}(X ; R)$ be its singular cochain algebra. Choosing a basepoint $x_{0} \in X$ yields an augmentation, $\varepsilon_{0}: C^{*}(X ; R) \rightarrow R$, which sends a 0 -cochain $\xi$ to its evaluation $\xi\left(x_{0}\right) \in$ $R$ and any cochain of higher degree to 0 . If $f:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ is a pointed map, then the induced morphism of cochain algebras, $f^{*}: C^{*}(Y ; R) \rightarrow C^{*}(X ; R)$, preserves the
respective augmentations. Finally, if $X \vee Y$ is the wedge sum of two pointed spaces, then $C^{*}(X \vee Y ; R)$ is isomorphic to $C^{*}(X ; R) \vee C^{*}(Y ; R)$.

Example 3.9. Let $C^{*}(I ; R)$ be the cochain algebra of the unit interval $I=[0,1]$ as in Example 3.4, and let $x_{0}=0$. Then $\varepsilon_{0}\left(t_{0}\right)=1$, while $\varepsilon_{0}\left(t_{1}\right)=\varepsilon_{0}(u)=0$.
3.5. Augmentation-preserving homotopies. Two augmented dga maps, $\varphi_{0}, \varphi_{1}: A \rightarrow$ $B$, are said to be augmentation-preserving homotopic if there is a homotopy $\Phi: A \rightarrow$ $B \otimes_{R} C^{*}(I ; R)$ between them such that the following diagram commutes,

where the diagonal map is the structure map for the $R$-algebra $C^{*}(I ; R)$, which sends 1 to $t_{0}+t_{1}$. As noted in (3.5), the homotopy $\Phi$ is given on elements $a \in A^{i}$ by $\Phi(a)=$ $\varphi_{0}(a) t_{0}+\varphi_{1}(a) t_{1}-c(a) u$, for some $c(a) \in B^{i-1}$. The commutativity of (3.7) implies that $\varepsilon_{B}(c(a))=0$. When both $A$ and $B$ are connected and $A$ is generated in degree 1 , augmentation-preserving homotopies take a very special form, which we describe next.

Lemma 3.10. Let $A$ and $B$ be augmented dgas such that $A$ and $B$ are connected and $A$ is generated as a graded $R$-algebra by $A^{1}$. Let $\varphi_{0}$ and $\varphi_{1}$ be augmentation-preserving morphisms of dgas and let $\Phi: A \rightarrow B \otimes_{R} C^{*}(I ; R)$ be an augmentation-preserving homotopy between $\varphi_{0}$ and $\varphi_{1}$. Then, for all $a \in A$,

$$
\begin{gather*}
\Phi(a)=\varphi_{0}(a) t_{0}+\varphi_{1}(a) t_{1}  \tag{3.8}\\
\varphi_{0}(a)=\varphi_{1}(a) \tag{3.9}
\end{gather*}
$$

Proof. We start with equation (3.8). For each $a \in A^{i}$, write as before $\Phi(a)=\varphi_{0}(a) t_{0}+$ $\varphi_{1}(a) t_{1}-c(a) u$, for some $c(a) \in B^{i-1}$. When $a \in A^{0}$, we necessarily have $c(a)=0$. When $a \in A^{1}$, we have that $c(a) \in B^{0}$ and $\varepsilon_{B}(c(a))=0$; since $B$ is connected, it follows that $c(a)=0$. Therefore, (3.8) holds for all $a \in A^{0} \oplus A^{1}$.

Recall that $t_{i} t_{j}=\delta_{i j} t_{i}$. Since $\varphi_{0}, \varphi_{1}$, and $\Phi$ are all maps of graded algebras, $A$ is generated in degree 1 , and (3.8) holds for all $a \in A^{1}$, it follows that equation (3.8) holds for all $a \in A$.

We now turn to equation (3.9). Using (3.8), we have that,

$$
\begin{aligned}
\left(\Phi \circ d_{A}\right)(a) & =\varphi_{0}\left(d_{A} a\right) t_{0}+\varphi_{1}\left(d_{A} a\right) t_{1} \\
\left(d_{B \otimes_{R} C^{*}(I ; R)} \circ \Phi\right)(a) & =d_{B \otimes_{R} C^{*}(I ; R)}\left(\varphi_{0}(a) t_{0}+\varphi_{1}(a) t_{1}\right) \\
& =\varphi_{0}\left(d_{A} a\right) t_{0}+\varphi_{1}\left(d_{A} a\right) t_{1}-\left(\varphi_{0}(a)-\varphi_{1}(a)\right) u
\end{aligned}
$$

for every $a \in A$. Since $\Phi$ is a map of dgas, it follows that $\varphi_{0}(a)-\varphi_{1}(a)=0$ for all $a \in A$, and the proof is complete.

## 4. The Steenrod $\cup_{i}$-Products

4.1. The $\cup_{i}$ operations. We now enrich the notion of a differential graded algebra with extra structure, motivated by properties of the cochain algebra of a space, as laid out in the foundational paper of Steenrod [22], and further developed by Hirsch in [13].
Let $X$ be a $\Delta$-complex, and let $A=\left(C^{*}(X ; R), d\right)$ be its cochain algebra with coefficients in a commutative ring $R$, with multiplication given by the cup product $\cup: A^{p} \otimes_{R} A^{q} \rightarrow A^{p+q}$. This $R$-dga comes endowed with $R$-linear maps, $\cup_{i}: A^{p} \otimes_{R} A^{q} \rightarrow A^{p+q-i}$, which coincide with the usual cup product when $i=0$, vanish if either $p<i$ or $q<i$, and satisfy

$$
\begin{align*}
& d\left(a \cup_{i} b\right)=(-1)^{|a|+|| |-i} a \cup_{i-1} b+(-1)^{|a||b|+|a|+|b|} b \cup_{i-1} a+d a \cup_{i} b+(-1)^{|a|} a \cup_{i} d b  \tag{4.1}\\
& (a \cup b) \cup_{1} c=a \cup\left(b \cup_{1} c\right)+(-1)^{|b|(c \mid-1)}\left(a \cup_{1} c\right) \cup b \tag{4.2}
\end{align*}
$$

for all homogeneous elements $a, b, c \in A$. We shall refer to (4.1) as the "Steenrod identities" and to (4.2), with the cup product to the left of the cup-one product, as the "Hirsch identity".

Steenrod's $\cup_{i}$ operations enjoy the following naturality property. Suppose $f: X \rightarrow Y$ is a map of $\Delta$-complexes which preserves the ordering of the vertices of simplices. Then, by [22, Theorem 3.1], the induced map on cochains, $f^{*}: C^{*}(Y ; R) \rightarrow C^{*}(X ; R)$, is a morphism of differential graded algebras that commutes with the $\cup_{i}$ products.

Steenrod's $\cup_{i}$ products also occur in the theory of non-commutative differential forms, as developed by M. Karoubi, N. Battikh, and A. Abbassi. We refer to our prior work [18] for an overview of this subject and detailed references.
4.2. Cup and cup-one operations on 1-cochains. Henceforth, we will focus on the aforementioned operations on cochains in low degrees. Let ( $A, d$ ) be an $R$-dga and assume we have an $R$-linear map $\cup_{1}: A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$ that satisfies the Steenrod and Hirsch identities (4.1) and (4.2), that is,

$$
\begin{align*}
& d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b-a \cup_{1} d b,  \tag{4.3}\\
& (a \cup b) \cup_{1} c=a \cup\left(b \cup_{1} c\right)+\left(a \cup_{1} c\right) \cup b, \tag{4.4}
\end{align*}
$$

for all $a, b, c \in A^{1}$. In particular, if $a, b \in Z^{1}(A)$ are 1-cocyles, we then have

$$
\begin{equation*}
d\left(a \cup_{1} b\right)=-(a \cup b+b \cup a) \tag{4.5}
\end{equation*}
$$

Under these assumptions, the operation $\cup_{1}: Z^{1}(A) \otimes_{R} Z^{1}(A) \rightarrow A^{2}$ provides an explicit witness for the non-commutativity of the multiplication map $\cup: Z^{1}(A) \otimes_{R} Z^{1}(A) \rightarrow Z^{2}(A)$ and shows that $u v=-v u$ for elements $u, v \in H^{1}(A)$.
Now let $X$ be a $\Delta$-complex and let $C^{*}(X ; R)$ be its cochain algebra with coefficients in a commutative ring $R$. By [22, Theorem 2.1], $u \cup_{1} v=0$ if either $u$ or $v$ is a 0 -cochain. Formulas for computing the cup products and cup-one products for 1 -cochains $u, v \in$ $C^{1}(X ; R)$ are as follows:

$$
\begin{align*}
& (u \cup v)(s)=u\left(e_{1}\right) \cdot v\left(e_{2}\right), \\
& \left(u \cup_{1} v\right)(e)=u(e) \cdot v(e), \tag{4.6}
\end{align*}
$$

where in the first formula $s$ is a 2 -simplex with front face $e_{1}$ and back face $e_{2}$, while in the second formula $e$ is a 1-simplex, and • denotes the product in $R$. In particular, the $\cup_{1}-$ product on $C^{1}(X ; R)$ is both associative and commutative, and thus defines an $R$-algebra structure on $C^{\leq 1}(X ; R)$.

Example 4.1. For the cochain algebra $C=C^{*}(I ; R)$ from Example 3.4, the cup-one product $C^{1} \otimes_{R} C^{1} \rightarrow C^{1}$ is given by $u \cup_{1} u=u$.

Example 4.2. Now let $G$ be a group, and let $C^{*}(B(G) ; R)$ be the cochain algebra of the bar construction on $G$, as described in Example 3.5. The $\cup_{1}$-product on $C^{1}(B(G) ; R)$ is given by $\left(f \cup_{1} h\right)([g])=f(g) \cdot h(g)$.
4.3. Graded algebras with cup-one products. Given a graded $R$-algebra $A$, we let $D^{2}(A)$ denote the decomposables in $A^{2}$; that is, the $R$-submodule of $A^{2}$ spanned by all elements of the form $a \cup b$, with $a, b \in A^{1}$.

Definition 4.3. A graded algebra with cup-one products is a graded $R$-algebra $A$ with a cup-one product map, $\cup_{1}: A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$, which gives the $R$-submodule $R \oplus A^{1} \subset A^{0} \oplus A^{1}$ the structure of a commutative ring, and a cup-one product map $\cup_{1}: D^{2}(A) \otimes_{R} A^{1} \rightarrow A^{2}$ that satisfies the Hirsch identity (4.4).

A morphism of graded algebras with cup-one products is a map $\varphi: A \rightarrow B$ between two such objects which is a map of graded algebras and commutes with cup-one products; that is, $\varphi\left(a_{1} \cup_{1} a_{2}\right)=\varphi\left(a_{1}\right) \cup_{1} \varphi\left(a_{2}\right)$, for all $a_{1}, a_{2} \in A$.

Lemma 4.4. Let $A$ and $B$ be two graded $R$-algebras with cup-one products. Then the tensor product $A \otimes_{R} B$ is again a graded algebra with cup-one products.

Proof. We extend the $\cup_{1}$-products on $A^{1}$ and $B^{1}$ to a $\cup_{1}$-product on $\left(A \otimes_{R} B\right)^{1}=\left(A^{1} \otimes_{R}\right.$ $\left.B^{0}\right) \oplus\left(A^{0} \otimes_{R} B^{1}\right)$ by setting

$$
\begin{align*}
& \left(a_{1} \otimes b_{0}\right) \cup_{1}\left(a_{1}^{\prime} \otimes b_{0}^{\prime}\right)=\left(a_{1} \cup_{1} a_{1}^{\prime}\right) \otimes b_{0} b_{0}^{\prime} \\
& \left(a_{0} \otimes b_{1}\right) \cup_{1}\left(a_{0}^{\prime} \otimes b_{1}^{\prime}\right)=a_{0} a_{0}^{\prime} \otimes\left(b_{1} \cup_{1} b_{1}^{\prime}\right)  \tag{4.7}\\
& \left(a_{1} \otimes b_{0}\right) \cup_{1}\left(a_{0}^{\prime} \otimes b_{1}^{\prime}\right)=\left(a_{0} \otimes b_{1}\right) \cup_{1}\left(a_{1}^{\prime} \otimes b_{0}^{\prime}\right)=0
\end{align*}
$$

for all $a_{i}, a_{i}^{\prime} \in A^{i}$ and $b_{i}, b_{i}^{\prime} \in B^{i}(i=0,1)$ and extending linearly to $\left(A \otimes_{R} B\right)^{1}$. Since the $\cup_{1}$-product on $A^{1}$ and $B^{1}$ and the multiplication on $A^{0}$ and $B^{0}$ are all commutative, it follows that the $\cup_{1}$-product on $\left(A \otimes_{R} B\right)^{1}$ is also commutative. Since the Hirsch identity (4.4) holds for both $A$ and $B$, it also holds for $A \otimes_{R} B$; for instance,

$$
\begin{aligned}
\left(a_{1} \otimes b_{0} \cup a_{1}^{\prime} \otimes b_{0}^{\prime}\right) \cup_{1}\left(a_{1}^{\prime \prime} \otimes b_{0}^{\prime \prime}\right)= & \left(\left(a_{1} \cup a_{1}^{\prime}\right) \cup_{1} a_{1}^{\prime \prime}\right) \otimes b_{0} b_{0}^{\prime} b_{0}^{\prime \prime} \\
= & \left(\left(a_{1} \cup_{1} a_{1}^{\prime \prime}\right) \cup a_{1}^{\prime}+a_{1} \cup\left(a_{1}^{\prime} \cup_{1} a_{1}^{\prime \prime}\right)\right) \otimes b_{0} b_{0}^{\prime} b_{0}^{\prime \prime} \\
= & \left(a_{1} \otimes b_{0} \cup_{1} a_{1}^{\prime \prime} \otimes b_{0}^{\prime \prime}\right) \cup a_{1}^{\prime} \otimes b_{1}^{\prime}+ \\
& a_{1} \otimes b_{0} \cup\left(a_{1}^{\prime} \otimes b_{0}^{\prime} \cup_{1} a_{1}^{\prime \prime} \otimes b_{0}^{\prime \prime}\right),
\end{aligned}
$$

and similarly for the other types of $\cup$ and $\cup_{1}$ products. This completes the proof.
4.4. Cup-one differential graded algebras. In this section we make a definition that will play an important role in our investigation. We begin with some motivation. Note that if a dga is generated by a set of elements $\left\{x_{i}\right\}_{i \in J}$ in degree 1, then the Leibniz rule gives a formula for the differential of any product of the $x_{i}$ as a sum of cup products of the $x_{i}$ and $d x_{i}$. Hence, the differential on the algebra is completely determined by the differentials of the generators $x_{i}$.

This raises the question of whether there is a formula for the differential of cup-one products of the generators $x_{i}$ that allows one to write $d\left(x_{i} \cup_{1} x_{j}\right)$ as a sum of cup products of 1 -cochains. If so, then it follows that if a dga is generated by elements $x_{i}$ in degree one and by iterated cup-one products of the $x_{i}$, then the differential on the algebra is completely determined by the differentials of the $x_{i}$.

The next definition answers this question by giving as part of hypothesis (iv) a formula for the differential of a cup-one product of 1-cochains that, along with the Hirsch identity, allows one to write the differential of cup-one products of 1-cochains as a sum of cup products. This definition is a slight modification of a notion introduced in [18], better adapted to the current context by including the additional hypothesis (iii).

Definition 4.5. A differential graded $R$-algebra $(A, d)$ is called a cup-one differential graded algebra if the following conditions hold.
(i) $A$ is a graded $R$-algebra with cup-one products.
(ii) There is an $R$-linear map $\circ: D^{2}(A) \otimes_{R} D^{2}(A) \rightarrow D^{2}(A)$ such that

$$
\begin{equation*}
(u \cup v) \circ(w \cup z)=\left(u \cup_{1} w\right) \cup\left(v \cup_{1} z\right) \tag{4.8}
\end{equation*}
$$

for all $u, v, w, z \in A^{1}$.
(iii) The differential $d$ and the $\cup$ and $\cup_{1}$ products satisfy the identity

$$
\begin{equation*}
a \cup_{1} d c=a \cup c-c \cup a \tag{4.9}
\end{equation*}
$$

for all $a \in A^{1}$ and $c \in A^{0}$.
(iv) The differential $d$ satisfies the " $\cup_{1}-d$ formula,"

$$
\begin{equation*}
d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b+d b \cup_{1} a-d a \circ d b, \tag{4.10}
\end{equation*}
$$

for all $a, b \in A^{1}$ with $d a, d b \in D^{2}(A)$.
Remark 4.6. Formula (4.10) comes from Steenrod's definition of the $\cup_{i}$ products in a cochain algebra $A=C^{*}(X ; R)$, as follows. By equation (4.3), we have that $d\left(a \cup_{1} b\right)=$ $-a \cup b-b \cup a+d a \cup_{1} b-a \cup_{1} d b$ for all $a, b \in A^{1}$. If $d a$ is decomposable, then $d a \cup_{1} b$ can be written as a sum of cup products using the Hirsch formula (4.4). This leaves the problem of writing $a \cup_{1} d b$ as a sum of cup products. By a direct computation using Steenrod's definition of the cup-one product $\cup_{1}: A^{1} \otimes_{R} A^{2} \rightarrow A^{2}$, it follows that

$$
\begin{equation*}
a \cup_{1}\left(b_{1} \cup b_{2}\right)=d a \circ\left(b_{1} \cup b_{2}\right)-\left(b_{1} \cup b_{2}\right) \cup_{1} a, \tag{4.11}
\end{equation*}
$$

where we assume $d a$ is decomposable and $\circ$ is given by (4.8). This then gives the $\cup_{1}-d$ formula, equation (4.10).

Our motivation for Definition 4.5 arises from the cochain algebras of $\Delta$-complexes. As shown in [18, Theorem 4.4], such algebras are indeed $\cup_{1}$-algebras. We briefly review this result, with the necessary modifications for our context here.

Theorem 4.7 ([18]). Let $X$ be a non-empty $\Delta$-complex, and let $R$ be a unital commutative ring. The the cochain algebra $\left(C^{*}(X ; R), \delta\right)$ is a cup-one dga.

Proof. As we saw in Sections 3.2 and 4.2, the cellular cochain algebra $C=\left(C^{*}(X ; R), \delta\right)$ is a graded algebra with cup-one products. Moreover, it is a differential graded algebra, and the Steenrod identities (4.1) hold in full generality.
Setting $\left(c_{1} \circ c_{2}\right)(s)=c_{1}(s) \cdot c_{2}(s)$ for any 2-cochains $c_{1}, c_{2}$ and any 2-simplex $s$ defines an $R$-linear map $\circ: C^{2} \otimes_{R} C^{2} \rightarrow C^{2}$. It follows straight from the definitions of the $\cup$ - and $\cup_{1}$-products that

$$
\begin{equation*}
(u \cup v)(s) \cdot(w \cup z)(s)=\left(\left(u \cup_{1} w\right) \cup\left(v \cup_{1} z\right)\right)(s), \tag{4.12}
\end{equation*}
$$

for all 1-cochains $u, v, w, z$. Thus, the restriction of the o-map to decomposable elements yields a map, ○: $D^{2}(C) \otimes_{R} D^{2}(C) \rightarrow D^{2}(C)$, which clearly obeys formula (4.8). It is now straightforward to verify that the simplicial differential $d$ satisfies formula (4.10).

It remains to check that formula (4.9) holds. Given a 1 -cochain $u$, a 0 -cochain $c$, and a 1 -simplex $e$ with endpoints $v_{0}$ and $v_{1}$, we have

$$
\begin{align*}
\left(u \cup_{1} \delta c\right)(e) & =u(e) \cdot(\delta c)(e) \\
& =u(e) \cdot c\left(v_{1}-v_{0}\right)  \tag{4.13}\\
& =u(e) \cdot c\left(v_{1}\right)-c\left(v_{0}\right) \cdot u(e) \\
& =(u \cup c)(e)-(c \cup u)(e),
\end{align*}
$$

and this completes the proof.
Remark 4.8. Comparing the definition of the map ०: $C^{2} \otimes_{R} C^{2} \rightarrow C^{2}$ given in the above proof to that of Steenrod's map $\cup_{2}: C^{2} \otimes_{R} C^{2} \rightarrow C^{2}$, we readily see that these two maps coincide. Moreover, as shown in Remark 4.6, in this case the $\cup_{1}-d$ formula (4.10) is a consequence of Steenrod's formula (4.3).
4.5. Tensor products of $\cup_{1}$-dgas. We conclude this section with a result showing that the category of $\cup_{1}$-dgas is closed under taking tensor products.

Proposition 4.9. If $\left(A, d_{A}\right)$ and $\left(B, d_{B}\right)$ are cup-one differential graded algebras, then the tensor product $\left(A \otimes_{R} B, d_{A \otimes B}\right)$ is again a cup-one differential graded algebra.

Proof. By Lemma 4.4, $A \otimes_{R} B$ is a graded algebra with cup-one products. The $\circ$ operations on $D^{2}(A)$ and $D^{2}(B)$ extend to a binary operation, $\circ: D^{2}\left(A \otimes_{R} B\right) \otimes_{R} D^{2}\left(A \otimes_{R} B\right) \rightarrow$ $D^{2}\left(A \otimes_{R} B\right)$, by letting

$$
\begin{align*}
& {\left[\left(a_{1} \otimes b_{0}\right) \cup\left(a_{1}^{\prime} \otimes b_{0}^{\prime}\right)\right] \circ\left[\left(a_{1}^{\prime \prime} \otimes b_{0}^{\prime \prime}\right) \cup\left(a_{1}^{\prime \prime \prime} \otimes b_{0}^{\prime \prime \prime}\right)\right]=} \\
&  \tag{4.14}\\
& {\left[\left(a_{1} \cup a_{1}^{\prime}\right) \circ\left(a_{1}^{\prime \prime} \cup a_{1}^{\prime \prime \prime}\right)\right] \otimes b_{0} b_{0}^{\prime} b_{0}^{\prime \prime} b_{0}^{\prime \prime \prime}}
\end{align*}
$$

and so on. Using (4.7), it is readily verified that equation (4.8) holds for $\left(A \otimes_{R} B, d_{A \otimes B}\right)$.
Next, we verify that equation (4.9) holds:

$$
\begin{align*}
\left(a_{1} \otimes b_{0}\right) \cup_{1} d_{A \otimes B}\left(a_{0}^{\prime} \otimes b_{0}^{\prime}\right) & =\left(a_{1} \otimes b_{0}\right) \cup_{1}\left(d_{A}\left(a_{0}^{\prime}\right) b_{0}^{\prime}+a_{0}^{\prime} d_{B}\left(b_{0}^{\prime}\right)\right) \\
& =\left(a_{1} \cup_{1} d_{A}\left(a_{0}^{\prime}\right)\right) \otimes b_{0} b_{0}^{\prime} \\
& =\left(a_{1} \cup a_{0}^{\prime}-a_{0}^{\prime} \cup a_{1}\right) \otimes b_{0} b_{0}^{\prime}  \tag{4.15}\\
& =\left(a_{1} \otimes b_{0}\right) \cup\left(a_{0}^{\prime} \otimes b_{0}^{\prime}\right)-\left(a_{0}^{\prime} \otimes b_{0}^{\prime}\right) \cup\left(a_{1} \otimes b_{0}\right),
\end{align*}
$$

and similarly for the other cases.
The last step is to verify that equation (4.10) holds for the tensor product of $A$ and $B$. First let $a \in A^{1}$ such that $d_{A}(a) \in D^{2}(A)$, and let $b \in B^{0}$. It is readily seen that $d_{A \otimes B}(a \otimes b) \in$ $D^{2}\left(A \otimes_{R} B\right)$; for instance, if $d_{A}(a)=u \cup v$ for some $u, v \in A^{1}$, then

$$
\begin{equation*}
d_{A \otimes B}(a \otimes b)=d_{A} a \otimes b-a \otimes d_{B} b=(u \otimes 1) \cup(v \otimes b)-(a \otimes 1) \cup\left(1 \otimes d_{B} b\right) . \tag{4.16}
\end{equation*}
$$

Now also let $a^{\prime}$ be an element in $A^{1}$ such that $d_{A}\left(a^{\prime}\right) \in D^{2}(A)$. and let $b^{\prime} \in B^{0}$. Using equations (4.8) and (4.7), we find that

$$
\begin{align*}
\left(d_{A}(a) \otimes b\right) \circ\left(a^{\prime} \otimes d_{B}\left(b^{\prime}\right)\right) & =((u \cup v) \otimes b) \circ\left(\left(a^{\prime} \otimes 1\right) \cup\left(1 \otimes d_{B}\left(b^{\prime}\right)\right)\right) \\
& =\left(u \cup_{1}\left(a^{\prime} \otimes b\right)\right) \cup\left((v \otimes 1) \cup 1\left(1 \otimes d_{B}\left(b^{\prime}\right)\right)\right) \\
& =\left(\left(u \cup_{1} a^{\prime}\right) \otimes b\right) \cup 0  \tag{4.17}\\
& =0,
\end{align*}
$$

and similarly $\left(a \otimes d_{B}(b)\right) \circ\left(d_{A}\left(a^{\prime}\right) \otimes b^{\prime}\right)=0$. Furthermore, using equations (4.8) and (4.9), we get

$$
\begin{align*}
\left(a \otimes d_{B}(b)\right) \circ\left(a^{\prime} \otimes d_{B}\left(b^{\prime}\right)\right) & =\left((a \otimes 1) \cup\left(1 \otimes d_{B}(b)\right)\right) \circ\left(\left(a^{\prime} \otimes 1\right) \cup\left(1 \otimes d_{B}\left(b^{\prime}\right)\right)\right) \\
& =\left((a \otimes 1) \cup_{1}\left(a^{\prime} \otimes 1\right)\right) \cup\left(\left(1 \otimes d_{B}(b)\right) \cup_{1}\left(1 \otimes d_{B}\left(b^{\prime}\right)\right)\right) \\
& =\left(a \cup_{1} a^{\prime}\right) \otimes\left(d_{B}\left(b^{\prime}\right) \cup_{1} d_{B}(b)\right)  \tag{4.18}\\
& =\left(a \cup_{1} a^{\prime}\right) \otimes\left(d_{B}\left(b^{\prime}\right) b-b d_{B}\left(b^{\prime}\right)\right) .
\end{align*}
$$

Finally, using formula (4.10) for $A$ as well as equations (4.17) and (4.18) we find that

$$
d_{A \otimes B}\left[(a \otimes b) \cup_{1}\left(a^{\prime} \otimes b^{\prime}\right)\right]=d_{A \otimes B}\left[\left(a \cup_{1} a^{\prime}\right) \otimes b b^{\prime}\right]=d_{A}\left(a \cup_{1} a^{\prime}\right) \otimes d_{B}\left(b b^{\prime}\right)
$$

is equal to

$$
\begin{aligned}
& \left(-a \cup a^{\prime}-a^{\prime} \cup a+d_{A}(a) \cup_{1} a^{\prime}+d_{A}\left(a^{\prime}\right) \cup_{1} a-d_{A}(a) \circ d_{A}\left(a^{\prime}\right)\right) \otimes b b^{\prime}- \\
& \quad\left(a \cup_{1} a^{\prime}\right) \otimes\left(d_{B}(b) b^{\prime}+b d_{B}\left(b^{\prime}\right)\right) \\
& =-(a \otimes b) \cup\left(a^{\prime} \otimes b^{\prime}\right)-\left(a^{\prime} \otimes b^{\prime}\right) \cup(a \otimes b)+\left(d_{A}(a) \otimes b\right) \cup_{1}\left(a^{\prime} \otimes b^{\prime}\right)+ \\
& \quad\left(d_{A}\left(a^{\prime}\right) \otimes b^{\prime}\right) \cup_{1}(a \otimes b)-\left(a \cup_{1} a^{\prime}\right) \otimes\left(d_{B}(b) b^{\prime}+b d_{B}\left(b^{\prime}\right)\right)- \\
& \quad\left(d_{A}(a) \otimes b\right) \circ\left(d_{A}\left(a^{\prime}\right) \otimes b^{\prime}\right) \\
& =-(a \otimes b) \cup\left(a^{\prime} \otimes b^{\prime}\right)-\left(a^{\prime} \otimes b^{\prime}\right) \cup(a \otimes b)+d_{A \otimes B}(a \otimes b) \cup_{1}\left(a^{\prime} \otimes b^{\prime}\right)+ \\
& \quad d_{A \otimes B}\left(a^{\prime} \otimes b^{\prime}\right) \cup_{1}(a \otimes b)-d_{A \otimes B}(a \otimes b) \circ d_{A \otimes B}\left(a^{\prime} \otimes b^{\prime}\right) .
\end{aligned}
$$

This shows that (4.10) holds for elements in $A \otimes_{R} B$ of the form $(a \otimes b) \cup_{1}\left(a^{\prime} \otimes b^{\prime}\right)$ with $|a|=\left|a^{\prime}\right|=1$ and $|b|=\left|b^{\prime}\right|=0$. The case $|a|=\left|b^{\prime}\right|=1,|b|=\left|a^{\prime}\right|=0$ follows using similar computations to show that in this case the right side of equation (4.10) equals zero. The case $|a|=\left|a^{\prime}\right|=0,|b|=\left|b^{\prime}\right|=1$ follows by the same computations as in the first case with the elements in $A$ and $B$ interchanged.

## 5. Binomial cup-one differential graded algebras

In this section we begin by reviewing the definition and basic properties of binomial rings and then define $\mathbb{Z}_{p}$-binomial algebras for $p$ a prime. This leads to the definition of binomial cup-one differential graded algebras over the ring $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. A consequence of including the binomial algebra structure is that it then follows that the cohomology of the free binomial cup-one differential graded algebra on a single generator in degree

1 is isomorphic to the cohomology ring $H^{*}(K(R, 1) ; R)$ of the Eilenberg-MacLane space $K(R, 1)$ with $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ (see Theorem 8.9).
5.1. Binomial rings and $R$-valued polynomials. Following P. Hall [10] and J. Elliott [5], we say that a commutative ring $A$ is a binomial ring if the element

$$
\begin{equation*}
\binom{a}{n}:=a(a-1) \cdots(a-n+1) / n!\in A \otimes_{\mathbb{Z}} \mathbb{Q} \tag{5.1}
\end{equation*}
$$

lies in $A$ for every $a \in A$ and every $n>0$. Therefore we have maps $\zeta_{n}: A \rightarrow A, a \mapsto\binom{a}{n}$ for all $n \in \mathbb{N}$, with the convention that $\zeta_{0}(a)=1$ for all $a \in A$.
Let $(x)_{n}:=x(x-1) \cdots(x-n+1) \in \mathbb{Z}[x]$ be the "falling factorial" polynomial. Writing $(x)_{n}=\sum_{k=0}^{n} s(n, k) x^{k}$, the coefficients $s(n, k)$ of this polynomial are the Stirling numbers of the first kind. Now note that numerator of the fraction in (5.1) is obtained by evaluating the polynomial $(x)_{n}$ at the value $x=a$, and so we may also write

$$
\begin{equation*}
\zeta_{n}(a)=\frac{(a)_{n}}{n!} \tag{5.2}
\end{equation*}
$$

The next lemma follows straight from the definitions.
Lemma 5.1. Let $R$ be a binomial ring, and let $M$ be an $R$-module. Then the dual module, $M^{\vee}=\operatorname{Hom}_{R}(M, R)$, is a binomial ring with maps $\zeta_{n}: M^{\vee} \rightarrow M^{\vee}$ given by $\zeta_{n}(f)(m)=$ $\zeta_{n}(f(m))$ for all $f \in M^{\vee}$ and $m \in M$.

Now suppose $R$ is an integral domain, and let $K=\operatorname{Frac}(R)$ be its field of fractions. Let $K[\boldsymbol{X}]$ be the ring of polynomials in a set of formal variables $\boldsymbol{X}$, with coefficients in $K$, and let $R^{X}$ be the free $R$-module on the set $\boldsymbol{X}$. Following [2,5], we define the ring of $R$-valued polynomials (in the variables $X$ and with coefficients in $K$ ) as the subring

$$
\begin{equation*}
\operatorname{Int}\left(R^{X}\right):=\left\{p \in K[X] \mid p\left(R^{X}\right) \subseteq R\right\} . \tag{5.3}
\end{equation*}
$$

Assume now that the domain $R$ has characteristic 0 , that is, $R$ is torsion-free as an abelian group. Then $\operatorname{Int}\left(R^{X}\right)$ is a binomial ring, generated by the polynomials $\binom{X}{\mathbf{n}}:=\prod_{x \in X}\binom{x}{\mathbf{n}_{x}}$, for all multi-indices $\mathbf{n}=\left(\mathbf{n}_{x}\right)_{x \in X} \in \bigoplus_{X} \mathbb{Z}_{\geq 0}$. Moreover, $\operatorname{Int}\left(R^{X}\right)$ satisfies a type of universality property which makes it into the free binomial ring on variables in $\boldsymbol{X}$. As a consequence (at least when $R=\mathbb{Z}$ ), any binomial ring is a quotient of $\operatorname{Int}\left(R^{X}\right)$, for some set $\boldsymbol{X}$.

As shown in [5, Theorem 7.1], every torsion-free ring $A$ is contained in a smallest binomial ring, $\operatorname{Bin}(A)$, which is defined as the intersection of all binomial subrings of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ containing $A$. Alternatively,

$$
\begin{equation*}
\operatorname{Bin}(A)=\operatorname{Int}\left(\mathbb{Z}^{X_{A}}\right) / I_{A} \mathbb{Q}^{X_{A}} \cap \operatorname{Int}\left(\mathbb{Z}^{X_{A}}\right), \tag{5.4}
\end{equation*}
$$

where $\mathbb{Z}\left[\boldsymbol{X}_{A}\right]$ denotes the polynomial ring in variables indexed by the elements of $A$, and $I_{A}$ is the kernel of the canonical epimorphism $\mathbb{Z}\left[\boldsymbol{X}_{A}\right] \rightarrow A$. Moreover, if $A$ is generated as
a $\mathbb{Z}$-algebra by a collection of elements $\left\{a_{i}\right\}_{i \in J}$, then $\operatorname{Bin}(A)$ is the $\mathbb{Z}$-subalgebra of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by all elements of the form $\zeta_{k}\left(a_{i}\right)=\binom{a_{i}}{k}$ with $i \in J$ and $k \geq 0$. Finally, it is readily seen that $\operatorname{Bin}(\mathbb{Z}[X])=\operatorname{Int}\left(\mathbb{Z}^{X}\right)$.
5.2. Products of binomial polynomials. Let $I: X \rightarrow \mathbb{Z}_{\geq 0}$ be a function which takes only finitely many non-zero values; in other words, the support of the function, $\operatorname{supp}(I):=$ $\{x \in X \mid I(x) \neq 0\}$, is a finite subset of $\boldsymbol{X}$. Given a coefficient ring $R$, we associate to such a function an $R$-valued polynomial function, as follows.

First let $\mathbf{x}=\left\{x_{1}, \ldots, x_{n}\right\}$ be a finite subset of $\boldsymbol{X}$ that contains $\operatorname{supp}(I)$. Then the indexing function $I: X \rightarrow \mathbb{Z}_{\geq 0}$ takes values $I\left(x_{k}\right)=i_{k}$ for $k=1, \ldots, n$, and 0 otherwise, and so it may be identified with the $n$-tuple $\left(i_{1}, \ldots, i_{n}\right) \in\left(\mathbb{Z}_{\geq 0}\right)^{n}$. We define a polynomial, $\zeta_{I}(\mathbf{x})$, in the variables $x_{1}, \ldots, x_{n}$, by

$$
\begin{equation*}
\zeta_{I}(\mathbf{x})=\prod_{k=1}^{n} \zeta_{i_{k}}\left(x_{k}\right) \tag{5.5}
\end{equation*}
$$

Clearly, $\zeta_{I}(\mathbf{x})$ is an $R$-valued polynomial $\operatorname{int} \operatorname{Int}\left(R^{\mathbf{x}}\right) \subset \operatorname{Int}\left(R^{X}\right)$. That is to say, given an $n$-tuple $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in R^{\mathbf{x}}$, the evaluation $\zeta_{I}(\mathbf{a}):=\zeta_{I}(\mathbf{x})(\mathbf{a})$ of the polynomial $\zeta_{I}(\mathbf{x})$ at $x_{k}=a_{k}$ is an element of $R$.

We now define an $R$-valued polynomial $\zeta_{I} \in \operatorname{Int}\left(R^{X}\right)$ by setting $\zeta_{I}=\zeta_{I}(\mathbf{x})$, for some finite set of variables $\mathbf{x}$ with $\mathbf{x} \supseteq \operatorname{supp}(I)$. Since $\zeta_{0}(a)=1$ for all $a \in R$, this definition is independent of the choice of $\mathbf{x}$. Given any $\mathbf{a} \in R^{X}$, we have a well-defined evaluation $\zeta_{I}(\mathbf{a}) \in R$; in fact, we do have such an evaluation for any function $\mathbf{a}: X \rightarrow R$, again since $I$ has finite support. For instance, if $I=\mathbf{0}$ is the function that takes only the value 0 , then $\zeta_{0}$ is the constant polynomial 1 in the variables $\boldsymbol{X}$, and $\zeta_{0}(\mathbf{a})=1$, for any $\mathbf{a}: \boldsymbol{X} \rightarrow R$.

The above notions extend to an arbitrary binomial ring $A$. For instance, if $I=\left(i_{1}, \ldots, i_{n}\right)$ is an $n$-tuple of non-negative integers, the evaluation of the polynomial function $\zeta_{I}(\mathbf{x})$ from (5.5) at an $n$-tuple $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ of elements in $A$ is equal to $\zeta_{I}(\mathbf{a})=\prod_{k=1}^{n} \zeta_{i_{k}}\left(a_{k}\right)$. More generally, the evaluation $\zeta_{I}(\mathbf{a}) \in A$ is defined for any function $\mathbf{a}: X \rightarrow A$.
5.3. A basis for integer-valued polynomials. We restrict now to the case when $R=$ $\mathbb{Z}$. The next theorem provides a useful $\mathbb{Z}$-basis for the $\operatorname{ring} \operatorname{Int}\left(\mathbb{Z}^{X}\right)$ of integer-valued polynomials; for a proof, we refer to [2, Proposition XI.I.12] and [5, Lemma 2.2].

Theorem 5.2. The $\mathbb{Z}$-module $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ is free, with basis consisting of all polynomials of the form $\zeta_{I}$ with $I: X \rightarrow \mathbb{Z}_{\geq 0}$ a function with finite support.

Alternatively, one may take as a basis for $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ all polynomials $\zeta_{I}(\mathbf{x})$ with $\operatorname{supp}(I)=\mathbf{x}$, together with the constant polynomial $\zeta_{\mathbf{0}}$. We emphasize that in the products $\zeta_{I}(\mathbf{x})$, there is no repetition allowed among the variables comprising the set $\mathbf{x}$. For instance, the product $\zeta_{m}(x) \zeta_{n}(x)$ is not part of the aforementioned $\mathbb{Z}$-basis; rather, it may be expressed
as a linear combination of the binomials $\zeta_{m}(x), \ldots, \zeta_{m+n}(x)$. On the other hand, if $I$ and $J$ have disjoint supports, we have that $\zeta_{I} \cdot \zeta_{J}=\zeta_{I+J}$, and this polynomial is again part of the aforementioned basis for $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$.

As an application of the above theorem, we obtain the following universality property for free binomial rings.

Corollary 5.3 ([18]). Let $\boldsymbol{X}$ be a set, let A be a binomial ring, and let $\phi: \boldsymbol{X} \rightarrow$ A be a map of sets. There is then a unique extension of $\phi$ to a map $\tilde{\phi}: \operatorname{Int}\left(\mathbb{Z}^{X}\right) \rightarrow A$ of binomial rings.

A characterization of binomial rings in terms of integer-valued polynomials is given in the following theorem (see [5, Theorem 4.1] and [27, Theorem 5.34]).

Theorem 5.4. A ring $R$ is a binomial ring if and only if the following two conditions hold:
(1) $R$ is $\mathbb{Z}$-torsion-free.
(2) $R$ is the homomorphic image of a ring $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ of integer-valued polynomials, for some set $\boldsymbol{X}$.

Corollary 5.5 ([18]). Let $R_{1}$ and $R_{2}$ be binomial rings. Then the tensor product $R_{1} \otimes_{\mathbb{Z}} R_{2}$, with product $(a \otimes b) \cdot(c \otimes d)=a c \otimes b d$, is a binomial ring.
5.4. $\mathbb{Z}_{p}$-binomial algebras. Now fix a prime $p$, and let $\mathbb{Z}_{p}=\mathbb{Z} / p \mathbb{Z}$ be the field with $p$ elements. Let $A$ be a commutative $\mathbb{Z}_{p}$-algebra; we will assume that the structure map $\mathbb{Z}_{p} \rightarrow A$ which sends $1 \in \mathbb{Z}_{p}$ to the identity $1 \in A$ is injective. Note that the binomial operations $\zeta_{n}(a)=(a)_{n} / n$ ! with $a \in A$ are defined for $1 \leq n \leq p-1$, since $n$ ! is then a unit in $\mathbb{Z}_{p}$.

Example 5.6. Let $A=C^{*}\left(X ; \mathbb{Z}_{p}\right)$ be the cochain algebra over $\mathbb{Z}_{p}$ of a $\Delta$-complex $X$. For a cochain $a \in A^{1}$, we have that $(a)_{p}=0$, where the product is the $\cup_{1}$-product on $A^{1}$. To see this, let $e$ be any 1 -simplex in $X$; then the elements $a(e), a(e)-1, \ldots, a(e)-p+1$ are distinct elements in $\mathbb{Z}_{p}$. Since there are $p$ of these elements, one of the elements must be 0 and the property follows.

This motivates the following definition.
Definition 5.7. Let $A$ be a commutative $\mathbb{Z}_{p}$-algebra. We say that $A$ is a $\mathbb{Z}_{p}$-binomial algebra if $(a)_{p}=0$, for all $a \in A$.

Clearly, this condition is equivalent to $(a)_{n}=0$ for all integers $n \geq p$ and all $a \in A$. Note that in $\mathbb{Z}_{p}[x]$ we have the equality

$$
\begin{equation*}
(x)_{p}=x^{p}-x . \tag{5.6}
\end{equation*}
$$

Indeed, both polynomials are monic, of degree $p$, and both have the same set of $p$ distinct roots, namely $0,1, \ldots, p-1$. Therefore, a commutative $\mathbb{Z}_{p}$-algebra $A$ is a $\mathbb{Z}_{p}$-binomial algebra if and only if $a^{p}=a$, for all $a \in A$.
The next step is to derive properties of binomials in a $\mathbb{Z}_{p}$-binomial algebra analogous to those for a binomial ring over $\mathbb{Z}$. We start by defining the analog of $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$.
Given a set $\boldsymbol{X}$, we will denote by $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ the quotient of the free binomial algebra $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ by the ideal generated by the elements $\zeta_{n}(x)$ for $x \in X$ and $n \geq p$, tensored with $\mathbb{Z}_{p}$. The next result shows that, modulo the constant terms, $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ has $\mathbb{Z}_{p}$-basis given by products of the elements $\zeta_{i}(x)$ for $0<i<p$ and $x \in \boldsymbol{X}$. Recall from (5.5) that, for a finite subset $\mathbf{x}=\left\{x_{1}, \ldots, x_{n}\right\} \subset \boldsymbol{X}$ and a finitely supported function $I: X \rightarrow \mathbb{Z}_{\geq 0}$, we write $\zeta_{I}(\mathbf{x})=\prod_{k=1}^{n} \zeta_{I\left(x_{k}\right)}\left(x_{k}\right)$.
Lemma 5.8 ([18]). The ring $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ is a $\mathbb{Z}_{p}$-binomial algebra, with $\mathbb{Z}_{p}$-basis given by the $\mathbb{Z}_{p}$-valued polynomials $\zeta_{I}(\mathbf{x})$ with $I: X \rightarrow\{0, \ldots, p-1\}$.

Theorem 5.9 ([18]). Let A be a $\mathbb{Z}_{p}$-binomial algebra. There is then a bijection between maps of $\mathbb{Z}_{p}$-binomial algebras from $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ to $A$ and set maps from $\boldsymbol{X}$ to $A$.
Lemma 5.10. Let $A$ and $B$ be $\mathbb{Z}_{p}$-binomial algebras. Then the tensor product $A \otimes_{\mathbb{Z}_{p}} B$, with product $(a \otimes b) \cdot\left(a^{\prime} \otimes b^{\prime}\right)=a a^{\prime} \otimes b b^{\prime}$, is a $\mathbb{Z}_{p}$-binomial algebra.

Proof. Let $a \in A$ and $b \in B$, so that $a^{p}=a$ and $b^{p}=b$. Then $(a \otimes b)^{p}=a^{p} \otimes b^{p}=a \otimes b$, and the claim follows.
5.5. Binomial cup-one differential graded algebras. Following our previous work [18], we combine the notions of cup-one algebras and binomial algebras into a single package. Henceforth we will assume that our ground ring $R$ is equal to either $\mathbb{Z}$ or $\mathbb{Z}_{p}$ for some prime $p$.

Definition 5.11. A differential graded algebra $(A, d)$ over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ is called a binomial cup-one algebra if
(i) $A$ is a cup-one algebra.
(ii) $A^{0}$, with multiplication $A^{0} \otimes_{R} A^{0} \rightarrow A^{0}$ given by the cup-product, is an $R$-binomial algebra.
(iii) The $R$-submodule $R \oplus A^{1} \subset A^{\leq 1}$, with multiplication $A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$ given by the cup-one product, is an $R$-binomial algebra.

If $\varphi \in \operatorname{Hom}_{1}(A, B)$ is a morphism of binomial cup-one algebras, it follows from the definitions that $\varphi\left(\zeta_{k}(a)\right)=\zeta_{k}(\varphi(a))$, for all $k \geq 1$ and all $a \in A^{1}$.

Proposition 5.12. Let $\left(A, d_{A}\right)$ and $\left(B, d_{B}\right)$ be binomial cup-one algebras over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. Then the tensor product $\left(A \otimes_{R} B, d_{A \otimes B}\right)$ of the underlying dgas is again a binomial cup-one algebra.

Proof. Recall that $\left(A \otimes_{R} B\right)^{1}=\left(A^{1} \otimes_{R} B^{0}\right) \oplus\left(A^{0} \otimes_{R} B^{1}\right)$. The claim follows at once from Proposition 4.9, Corollary 5.5, and Lemma 5.10.
5.6. Binomial operations on cochains. In this section, we show that cochain complexes with coefficients in a binomial algebra are binomial cup-one algebras and give examples. The next result builds on work from [18].

Theorem 5.13. For any non-empty $\Delta$-complex $X$, the cochain algebra $C^{*}(X ; R)$, where $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, is a binomial $\cup_{1}$-dga.

Proof. First assume $R=\mathbb{Z}$. The cochain algebra $C^{*}(X ; \mathbb{Z})$ is then a cup-one dga by Theorem 4.7. The claim now follows from Lemma 5.1.
Now assume $R=\mathbb{Z}_{p}$. We define maps $\zeta_{n}^{X}: C^{1}\left(X ; \mathbb{Z}_{p}\right) \rightarrow C^{1}\left(X ; \mathbb{Z}_{p}\right)$ for $1 \leq n \leq p-1$, by setting $\zeta_{n}^{X}(f)(e):=(f(e))_{n} / n!$ for each 1-cochain $f \in C^{1}\left(X ; \mathbb{Z}_{p}\right)=\operatorname{Hom}\left(C_{1}\left(X ; \mathbb{Z}_{p}\right), \mathbb{Z}_{p}\right)$ and each 1-simplex $e$ in $X$. As noted in Example 5.6, we have that $(f)_{p}=0$. With this structure, it is readily verified that the cochain algebra $C^{*}\left(X ; \mathbb{Z}_{p}\right)$ is a $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga, and this completes the proof.

This theorem together with Proposition 5.12 yield the following corollary.
Corollary 5.14. Let A be a binomial $\cup_{1}$-dga over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. Then the tensor product $A \otimes_{R} C^{*}(I ; R)$ is again a binomial $\cup_{1}$-dga.

It is readily seen that the $\zeta$-maps enjoy the following naturality property: If $h: X \rightarrow Y$ is a map of $\Delta$-complexes, then each $\zeta_{k}$ commutes with the pullback of cochains, that is, $h^{*} \circ \zeta_{k}^{Y}=\zeta_{k}^{X} \circ h^{*}: C^{*}(Y ; R) \rightarrow C^{*}(X ; R)$.

Note that in the case $R=\mathbb{Z}$, the evaluation $\zeta_{k}(f)(e)$ is simply the binomial coefficient $\binom{f(e)}{k}$, for all $f \in \operatorname{Hom}\left(C_{1}(X ; R), R\right)$ and all $e \in C_{1}(X ; R)$.
Example 5.15. For the cochain algebra $C=C^{*}(I ; \mathbb{Z})$ from Example 3.4, the $\zeta_{k}$-maps are given by $\zeta_{k}(n u)=\binom{n}{k} u$. In particular, $\zeta_{k}(u)=0$ for $k \geq 2$.
Example 5.16. Let $G$ be a group, and let $C^{*}(B(G) ; R)$ be the cochain algebra of the bar construction on $G$, as described in Example 3.5 with coefficient ring $R$ a binomial algebra. Then the $\zeta_{k}$ maps on $C^{1}(B(G) ; R)$ are given by $\zeta_{k}(f)([g])=\binom{f(g)}{k}$.

## 6. Free binomial $\cup_{1}$-differential graded algebras

In this section, we define $\mathrm{T}(\boldsymbol{X})$, the free binomial graded cup-one algebra over the rings $R=\mathbb{Z}$ and $R=\mathbb{Z}_{p}$ generated by a set $X$. Given a map $d: \mathrm{T}(\boldsymbol{X}) \rightarrow \mathrm{T}(\boldsymbol{X})$ that satisfies the $\cup_{1}-d$ formula and the Leibniz rule, we show that $d^{2} \equiv 0$ if and only if $d^{2}(x)=0$ for all $x \in \boldsymbol{X}$. In particular, setting $d_{0}(x)=0$ for all $x \in \boldsymbol{X}$ yields the dga $\left(\mathrm{T}(\boldsymbol{X}), d_{0}\right)$, which we call the free binomial $\cup_{1}$-dga generated by $\boldsymbol{X}$.
6.1. The free binomial cup-one graded algebra. Let $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. Given a set $\boldsymbol{X}$, let $\mathfrak{m}_{X, R}$ denote the $R$-submodule of $\operatorname{Int}\left(R^{X}\right)$ consisting of polynomials with zero constant term. We let $\mathrm{T}=\mathrm{T}_{R}(\boldsymbol{X})$ denote the free graded algebra over $R$ with $\mathrm{T}^{1}$ equal to $\mathfrak{m}_{X, R}$. We define a cup-one map, $U_{1}:\left(T^{1} \otimes T^{1}\right) \otimes T^{1} \rightarrow T^{2}$, by means of the Hirsch identity (4.4), and a map $\circ: \mathrm{T}^{2} \otimes \mathrm{~T}^{2} \rightarrow \mathrm{~T}^{2}$ by means of equation (4.8). Then $\mathrm{T}_{R}(\boldsymbol{X})$ is a graded $R$-algebra with cup-one products which we call the free binomial graded algebra over $R$ with cup-one products generated by $\boldsymbol{X}$.
We now make this all more precise by starting with a definition.
 $\mathrm{T}=\mathrm{T}_{R}(\boldsymbol{X})$, is the tensor algebra on the free $R$-module $\mathfrak{m}_{X, R}$; that is,

$$
\begin{equation*}
\mathrm{T}_{R}^{*}(\boldsymbol{X})=T^{*}\left(\mathfrak{m}_{\boldsymbol{X}, R}\right) . \tag{6.1}
\end{equation*}
$$

By construction, $\mathrm{T}_{R}^{0}(\boldsymbol{X})=R$ and $\mathrm{T}_{R}^{1}(\boldsymbol{X})=\mathfrak{m}_{\boldsymbol{X}, R}$, and so $\mathrm{T}_{R}^{\leq 1}(\boldsymbol{X})=\mathrm{T}^{0} \oplus \mathrm{~T}^{1}$ is isomorphic to the free binomial algebra $\operatorname{Int}\left(R^{X}\right)$. By Theorem 5.2 and Lemma 5.8, respectively, $\mathrm{T}^{1}$ is a free $R$-module, with basis consisting of all $R$-valued polynomials of the form $\zeta_{I}$, where $I: X \rightarrow \mathbb{Z}_{\geq 0}$ has finite, non-empty support when $R=\mathbb{Z}$, and $I: X \rightarrow\{0,1, \ldots, p-1\}$ when $R=\mathbb{Z}_{p}$ excluding the constant- 0 function. Furthermore, the $R$-module $\mathrm{T}^{1}$ comes endowed with a cup-one product map, $\cup_{1}: T^{1} \otimes T^{1} \rightarrow T^{1}$, given by

$$
\begin{equation*}
a \cup_{1} b=a b \tag{6.2}
\end{equation*}
$$

By analogy with the classical Hirsch formula for cochain algebras, we use this cup-one product to define a linear map $\mathrm{T}^{2} \otimes \mathrm{~T}^{1} \rightarrow \mathrm{~T}^{2}$ by

$$
\begin{equation*}
(a \otimes b) \otimes c \mapsto a c \otimes b+a \otimes b c \tag{6.3}
\end{equation*}
$$

Recall that the $\cup_{1}-d$ formula (4.10) involves an operation (denoted by o) between degree 2 elements. For this to work, we include the linear map $\circ: T^{2} \otimes T^{2} \rightarrow T^{2}$ defined on basis elements by

$$
\begin{equation*}
\left(a_{1} \otimes a_{2}\right) \circ\left(b_{1} \otimes b_{2}\right)=\left(a_{1} b_{1}\right) \otimes\left(a_{2} b_{2}\right) \tag{6.4}
\end{equation*}
$$

With this structure, $\mathrm{T}(\boldsymbol{X})$ is a graded $R$-algebra with cup-one products.
The assignment $X \leadsto \mathrm{~T}_{R}(\boldsymbol{X})$ is functorial: a map of sets, $h: X \rightarrow \boldsymbol{X}^{\prime}$, extends to a map of polynomials from $\operatorname{Int}\left(R^{X}\right)$ to $\operatorname{Int}\left(R^{X^{\prime}}\right)$ that restricts to an $R$-linear map $\mathfrak{m}_{X, R} \rightarrow \mathfrak{m}_{X^{\prime}, R}$ which then extends to a map between tensor algebras, $\mathrm{T}(h): \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}\left(\boldsymbol{X}^{\prime}\right)$. Clearly, $\mathrm{T}(h)$ is a morphism of graded algebras that preserves $\cup_{1}$-products; moreover, $\mathrm{T}(h \circ g)=$ $\mathrm{T}(h) \circ \mathrm{T}(g)$.
In the following, the ring $R$ will be equal to either $\mathbb{Z}$ or $\mathbb{Z}_{p}$. A graded $R$-algebra $A$ with cup-one products such that the augmented algebra $R \oplus A^{1}$ is a binomial algebra is called a binomial graded $R$-algebra with cup-one products. In this category, the free binomial graded $R$-algebra $\mathrm{T}_{R}(\boldsymbol{X})$ enjoys the following universality property.

Lemma 6.2. Let A be a binomial graded $R$-algebra with cup-one products, let $\boldsymbol{X}$ be a set, and let $\phi: X \rightarrow A^{1}$ be a map of sets. There is then an extension of $\phi$ to a map $f: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A$ of binomial graded $R$-algebras with cup-one products.

Proof. From [18, Lemmas 7.4 and 8.13], it follows that there is a unique extension of $\phi$ to a degree-preserving map $f^{>0}: \mathrm{T}_{R}^{>0}(\boldsymbol{X}) \rightarrow A^{>0}$ which commutes with cup products, cup-one products, and the $\zeta$ maps. Let $\sigma_{\mathrm{T}}: R \rightarrow \mathrm{~T}_{R}^{0}(\boldsymbol{X})$ and $\sigma_{A}: R \rightarrow A^{0}$ be the structure maps for T and $A$; respectively, and define $f^{0}: \mathrm{T}_{R}^{0}(\boldsymbol{X}) \rightarrow A^{0}$ to be the composition $\sigma_{A} \circ \sigma_{\mathrm{T}}^{-1}$. Then the resulting map $f: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A$ is a morphism of binomial graded $R$-algebras and the proof is complete.
6.2. Maps from free binomial cup-one dgas. Consider now a differential $d$ : $\mathrm{T}_{R}(\boldsymbol{X}) \rightarrow$ $\mathrm{T}_{R}(\boldsymbol{X})$ making $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ into a binomial cup-one dga, and let $\left(A, d_{A}\right)$ be an arbitrary binomial cup-one dga over $R$. The next lemma gives a handy criterion for deciding whether a map $f: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A$ between the underlying binomial graded algebras with cup-one products is a morphism of binomial cup-one dgas.

Lemma 6.3. A map $f:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(A, d_{A}\right)$ of binomial graded $R$-algebras with cupone products commutes with the differentials if and only if $d_{A} f(x)=f(d x)$ for all $x \in \boldsymbol{X}$.

Proof. Recall from Section 6.1 that $\mathrm{T}_{R}^{1}(\boldsymbol{X})$ is the free $R$-module with basis consisting of all $R$-valued polynomials of the form $\zeta_{I}$, where $I: X \rightarrow \mathbb{Z}_{\geq 0}$ has finite, non-empty support when $R=\mathbb{Z}$ and $I: X \rightarrow\{0, \ldots, p-1\}$ has non-empty support when $R=\mathbb{Z}_{p}$. The claim follows from formulas (4.8) and (4.10); formula (6.14) expressing $\zeta_{n+1}(x)$ in terms of $\zeta_{n}(x)$; and induction on $n$.

Under a connectivity assumption on $H^{*}(A)$, we may improve on the conclusion of Lemma 6.2 , as follows.

Lemma 6.4. Let $\left(A, d_{A}\right)$ be a binomial cup-one $R$-dga with $H^{0}(A) \cong R$, let $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ be a free binomial cup-one dga, and let $\phi: X \rightarrow A^{1}$ be a map of sets. There is then a unique extension of $\phi$ to a map $f: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A$ of binomial graded $R$-algebras with cup-one products such that $H^{0}(f): H^{0}\left(\mathrm{~T}_{R}(\boldsymbol{X})\right) \rightarrow H^{0}(A)$ is an isomorphism.

Proof. Let $f: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A$ be the extension of $\phi$ constructed in Lemma 6.2 and let $\varepsilon$ denote the isomorphism from $H^{0}(A)$ to $R$. Since $H^{0}(A)=\operatorname{ker} d_{A}: A^{0} \rightarrow A^{1}$, it follows that the image of the structure map $\sigma_{A}$ is contained in $H^{0}(A) \subseteq A^{0}$. Thus, the composition $\varepsilon \circ \sigma_{A}$ is an isomorphism of rings from $R$ to $R$, and so equals $\operatorname{id}_{R}$. It then follows that $f^{0}$ is the unique $R$-linear map from $\mathrm{T}^{0}(\boldsymbol{X})$ to $A^{0}$ that commutes with the structure maps; that is, $f^{0} \circ \sigma_{T}=\sigma_{A}$, and the proof is complete.
6.3. Homotopies between binomial cup-one dga maps. As we saw in Theorem 3.8, homotopic dga maps induce the same homomorphism on cohomology. The next lemma provides a partial converse to this theorem, in the context of binomial cup-one algebras.

Lemma 6.5. Let $\left(A, d_{A}\right)$ be a binomial cup-one dga over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ such that $H^{1}(A)$ is a finitely generated, free $R$-module. Suppose $\varphi_{0}, \varphi_{1}:\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{0}\right) \rightarrow\left(A, d_{A}\right)$ are morphisms of binomial $\cup_{1}$-dgas such that $H^{1}\left(\varphi_{0}\right)=H^{1}\left(\varphi_{1}\right)$. Then $\varphi_{0} \simeq \varphi_{1}$.

Proof. We construct a homotopy $\Phi: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A \otimes_{R} C^{*}(I ; R)$ between $\varphi_{0}$ and $\varphi_{1}$, as follows. For each $x \in X$, set

$$
\begin{equation*}
\Phi(x)=\varphi_{0}(x) t_{0}+\varphi_{1}(x) t_{1}-c(x) u \tag{6.5}
\end{equation*}
$$

where $c(x)$ is an element in $A^{0}$ given by

$$
\begin{equation*}
d_{A}(c(x))=\varphi_{0}(x)-\varphi_{1}(x) \tag{6.6}
\end{equation*}
$$

such an element exists by our assumption that $H^{1}\left(\varphi_{0}\right)([x])=H^{1}\left(\varphi_{1}\right)([x])$. It is readily verified that $\Phi(x)$ is a 1-cocycle in $A \otimes_{R} C^{*}(I ; R)$; indeed,

$$
\begin{equation*}
d_{A \otimes C^{*}(I ; R)}(\Phi(x))=\varphi_{0}(x) u-\varphi_{1}(x) u-\left(\varphi_{0}(x)-\varphi_{1}(x)\right) u=0 . \tag{6.7}
\end{equation*}
$$

It now follows from Corollary 7.9 that the set map $\Phi: X \rightarrow Z^{1}\left(A \otimes_{R} C^{*}(I ; R)\right)$ extends to a map of binomial $\cup_{1}$-dgas, $\Phi: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A \otimes_{R} C^{*}(I ; R)$. By construction, this map is a homotopy between $\varphi_{0}$ and $\varphi_{1}$.
6.4. Differentials on $\mathrm{T}_{R}(\boldsymbol{X})$. As before, let $\mathrm{T}=\mathrm{T}_{R}(\boldsymbol{X})$ be a free binomial $\cup_{1}$-graded $R$-algebra on a set $X$. In this section we show that if a map $d: \mathrm{T} \rightarrow \mathrm{T}$ satisfies the $\cup_{1-d}$ formula and the Leibniz rule, then $d^{2}(u)=0$ for all $u \in \mathrm{~T}$ if and only if $d^{2}(x)=0$ for all $x \in X$. For that, we define additional $\cup_{1}$ and $\circ$ maps in T , as follows. First, we define a linear map $\cup_{1}: T^{3} \otimes T^{1} \rightarrow T^{3}$ by

$$
\begin{equation*}
\left(u_{1} \cup u_{2} \cup u_{3}\right) \cup_{1} v=\left(u_{1} \cup_{1} v\right) \cup u_{2} \cup u_{3}+u_{1} \cup\left(u_{2} \cup_{1} v\right) \cup u_{3}+u_{1} \cup u_{2} \cup\left(u_{3} \cup_{1} v\right) \tag{6.8}
\end{equation*}
$$

and a map $\cup_{1}: T^{2} \otimes T^{2} \rightarrow T^{3}$ by

$$
\begin{align*}
\left(a_{1} \cup a_{2}\right) \cup\left(b_{1} \cup b_{2}\right)= & -a_{1} \cup\left(b_{1} \cup_{1} a_{2}\right) \cup b_{2}-a_{1} \cup b_{1} \cup\left(b_{2} \cup_{1} a_{2}\right) \\
& +\sum_{j} a_{1} \cup\left(a_{2,1, j} \cup_{1} b_{1}\right) \cup\left(a_{2,2, j} \cup_{1} b_{2}\right) \\
& +\left(b_{1} \cup_{1} a_{1}\right) \cup b_{2} \cup a_{2}+b_{1} \cup\left(b_{2} \cup_{1} a_{1}\right) \cup a_{2}  \tag{6.9}\\
& -\sum_{i}\left(a_{1,1, i} \cup_{1} b_{1}\right) \cup\left(a_{1,2, i} \cup_{1} b_{2}\right) \cup a_{2},
\end{align*}
$$

where $d a_{1}=\sum_{i} a_{1,1, i} \cup a_{1,2, i}$ and $d a_{2}=\sum_{j} a_{2,1, j} \cup a_{2,2, j}$. Next, we define a map $\circ: \mathrm{T}^{2} \otimes \mathrm{~T}^{3} \rightarrow$ $\mathrm{T}^{3}$ by

$$
\begin{align*}
\left(a_{1} \cup a_{2}\right) \circ\left(v_{1} \cup v_{2} \cup v_{3}\right)= & \left(a_{1} \cup_{1} v_{1}\right) \cup\left(a_{2} \cup_{1} v_{2}\right) \cup v_{3} \\
& +\left(a_{1} \cup_{1} v_{1}\right) \cup v_{2} \cup\left(a_{2} \cup_{1} v_{3}\right) \\
& +v_{1} \cup\left(a_{1} \cup_{1} v_{2}\right) \cup\left(a_{2} \cup_{1} v_{3}\right)  \tag{6.10}\\
& -\sum_{i}\left(a_{1,1, i} \cup_{1} v_{1}\right) \cup\left(a_{1,2, i} \cup_{1} v_{2}\right) \cup\left(a_{2} \cup_{1} v_{3}\right),
\end{align*}
$$

where $d a_{1}=\sum_{i} a_{1,1, i} \cup a_{1,2, i}$. Finally, we define a map $\circ: \mathrm{T}^{3} \otimes \mathrm{~T}^{2} \rightarrow \mathrm{~T}^{3}$ by

$$
\begin{aligned}
\left(u_{1} \cup u_{2} \cup u_{3}\right) \circ\left(b_{1} \cup b_{2}\right)= & u_{1} \cup\left(u_{2} \cup_{1} b_{1}\right) \cup\left(u_{3} \cup_{1} b_{2}\right) \\
& +\left(u_{1} \cup_{1} b_{1}\right) \cup\left(u_{2} \cup_{1} b_{2}\right) \cup u_{3} \\
& +\left(u_{1} \cup_{1} b_{1}\right) \cup u_{2} \cup\left(u_{3} \cup_{1} b_{2}\right) \\
& -\sum_{k}\left(u_{1} \cup_{1} b_{1}\right) \cup\left(u_{2} \cup_{1} b_{2,1, k}\right) \cup\left(u_{3} \cup_{1} b_{2,2, k}\right),
\end{aligned}
$$

where $d b_{2}=\sum_{k} b_{2,1, k} \cup b_{2,2, k}$.
The proof of the following two equations is a straightforward, though computationally intensive, verification using the definitions of the $\cup_{1}-d$ formula along with the $\cup_{1}$ and $\circ$ maps in T .

$$
\begin{align*}
& d\left(d a \cup_{1} b\right)=d a \cup b-b \cup d a+d a \cup_{1} d b+d^{2} a \cup_{1} b  \tag{6.12}\\
& d(d a \circ d b)=d a \cup_{1} d b+d b \cup_{1} d a+d^{2} a \circ d b+d a \circ d^{2} b . \tag{6.13}
\end{align*}
$$

Note that these equations are analogous to equation (4.1) for $i=1,2$ and $|a|+|b|-i \leq 3$, with the $U_{2}$ map replaced by the o map.

Lemma 6.6. If $d: \mathrm{T} \rightarrow \mathrm{T}$ is a degree one map satisfying the $\cup_{1}-d$ formula and the Leibniz rule, and if $a, b$ are elements in $T^{1}$ with $d^{2} a=d^{2} b=0$, then $d^{2}\left(a \cup_{1} b\right)=0$.

Proof. By the $\cup_{1}-d$ formula (4.10), we have that

$$
d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b+d b \cup_{1} a-d a \circ d b .
$$

Then from equations (6.12) and (6.13), we have

$$
\begin{aligned}
d^{2}\left(a \cup_{1} b\right)= & -d(a \cup b)-d(b \cup a)+d\left(d a \cup_{1} b\right)+d\left(d b \cup_{1} a\right)-d(d a \circ d b) \\
= & -d a \cup b+a \cup d b-d b \cup a+b \cup d a \\
& +d a \cup b-b \cup d a+d a \cup_{1} d b+d^{2} a \cup_{1} b \\
& +d b \cup a-a \cup d b+d b \cup_{1} d a+d^{2} b \cup_{1} a \\
& -d a \cup_{1} d b-d b \cup_{1} d a-d^{2} a \circ d b-d a \circ d^{2} b \\
= & d^{2} a \cup_{1} b+d^{2} b \cup_{1} a-d^{2} a \circ d b-d a \circ d^{2} b,
\end{aligned}
$$

and the result follows since $d^{2} a=d^{2} b=0$.
Theorem 6.7. Let $d: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X})$ be a degree-one map satisfying the $\cup_{1}$-d formula and the Leibniz rule. Then $d^{2}(x)=0$ for all $x \in \boldsymbol{X}$ if and only if $d^{2}(u)=0$ for all $u \in \mathrm{~T}_{R}(\boldsymbol{X})$, in which case $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ is a binomial $\cup_{1}$-dga.

Proof. Let $x \in \boldsymbol{X}$. Recall that the binomial $\zeta$-functions satisfy the formula

$$
\begin{equation*}
\zeta_{n+1}(x)=\frac{\zeta_{n}(x) \cup_{1} x-n \zeta_{n}(x)}{n+1} . \tag{6.14}
\end{equation*}
$$

where $n+1 \leq p-1$ for $R=\mathbb{Z}_{p}$. Using this formula and Lemma 6.6, induction on $n$ shows that $d^{2} \zeta_{n}(x)=0$ for all $n \geq 1$ in the case $R=\mathbb{Z}$ and for all $n \leq p-2$ in the case $R=\mathbb{Z}_{p}$.

Making use of this fact and of Lemma 6.6 once again, induction on the number of elements in the support of $I$ shows that $d^{2}\left(\zeta_{I}(\mathbf{x})\right)=0$ for all $I$. The claim now follows by the Leibniz rule.

As a corollary, we recover a result from [18], which gives the free binomial graded algebra $\mathrm{T}_{R}(\boldsymbol{X})$ the structure of a binomial $\cup_{1}$-dga structure, with differential $d_{0}$ vanishing on all the generators $x \in X$.

Corollary 6.8 ([18]). For any set $X$, the algebra $\mathrm{T}_{R}(\boldsymbol{X})$ is a binomial $\cup_{1}-d g a$, with differential $d_{0}$ satisfying $d_{0}(x)=0$ for all $x \in \boldsymbol{X}$.

## 7. Differentials defined by admissible maps

In this section, we define embeddings of the free binomial $R$-algebra $\mathrm{T}_{R}(\boldsymbol{X})$ (or its truncation in degree 2 ) into a suitable cochain algebra. Using these embeddings, we show in Theorem 7.3 that there is a bijection between degree one linear maps $d: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X})$ that satisfy the $\cup_{1}-d$ formula and the Leibniz rule and maps of sets $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$. Theorem 7.6 gives a sufficient condition on $\tau$, called admissibility, for $d^{2}$ to be the zero map.
7.1. Embedding $\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})$ into a cochain algebra. Recall that the ring $R$ equals $\mathbb{Z}$ or $\mathbb{Z}_{p}$ with $p$ a prime. Given a set $\boldsymbol{X}$, we let $M(\boldsymbol{X}, R)$ be the set of all functions $\mathbf{a}: \boldsymbol{X} \rightarrow$ $R$. This is an abelian group under pointwise addition, with neutral element the zero function, denoted $\mathbf{0}$. Alternatively, we may view $M(\boldsymbol{X}, R)$ as a free $R$-module with basis $\boldsymbol{X}$. Furthermore, to every set map $f: \boldsymbol{X} \rightarrow \boldsymbol{Y}$ we assign (in a functorial way) the $R$-linear map $f^{\vee}: M(\boldsymbol{Y}, R) \rightarrow M(\boldsymbol{X}, R)$ given by $f^{\vee}(\mathbf{b})(x)=\mathbf{b}(f(x))$, for $\mathbf{b}: \boldsymbol{Y} \rightarrow R$ and $x \in \boldsymbol{X}$.

Now let $\mu: M \times M \rightarrow M$ be an arbitrary binary operation on $M=M(X, R)$. By the construction from Section 2.2, the magma $(M, \mu)$ determines a 2 -dimensional $\Delta$-set,
$\Delta^{(2)}(M, \mu)$. Let $C_{\mu}(\boldsymbol{X})=C_{\mu}(\boldsymbol{X} ; R)$ denote the cochain complex (over $R$ ) of this $\Delta$-set. Next, we define a degree-preserving, $R$-linear map,

$$
\begin{equation*}
\psi=\psi_{\boldsymbol{X}, \mu}: \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \longrightarrow C_{\mu}(\boldsymbol{X}) \tag{7.1}
\end{equation*}
$$

as follows. First define a map $\psi: \mathrm{T}_{R}^{0}(\boldsymbol{X}) \rightarrow C_{\mu}^{0}(\boldsymbol{X})$ by sending $1 \in \mathrm{~T}_{R}^{0}(\boldsymbol{X})=R$ to the unit cochain $1 \in C_{\mu}^{0}(\boldsymbol{X})$. For each polynomial $p \in \mathrm{~T}_{R}^{1}(\boldsymbol{X})=\mathfrak{m}_{\boldsymbol{X}}$, we set $\psi(p) \in C_{\mu}^{1}(\boldsymbol{X})$ equal to the 1-cochain whose value on a 1-simplex a is $p(\mathbf{a})$. Finally, we set $\psi(p \otimes q) \in C_{\mu}^{2}(\boldsymbol{X})$ equal to the 2 -cochain whose value on a 2 -simplex ( $\left.\mathbf{a}, \mathbf{a}^{\prime}\right)$ is $p(\mathbf{a}) \cdot q\left(\mathbf{a}^{\prime}\right)$. If the zero function $\mathbf{0}$ is a two-sided identity in $(M, \mu)$, it is readily seen that the image of $\psi$ is contained in the normalized cochains on $\Delta^{(2)}(M, \mu)$.

Lemma 7.1. With notation as above, the map $\psi=\psi_{\boldsymbol{X}, \mu}: \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \rightarrow C_{\mu}(\boldsymbol{X})$ is a monomorphism that commutes with cup products, cup-one products, and the $\circ$ map.

Proof. The proof of the lemma follows in general outline the proof of the first part of [18, Theorem 7.2]; since the context here is somewhat different, we provide full details.

It follows directly from the definitions that the map $\psi$ commutes with cup products, cupone products, and the $\circ$ map, so it suffices to show that $\psi$ is a monomorphism. To prove this, first suppose that $\psi(p)=0$, for some $p \in \mathfrak{m}_{X}$. Then for all functions $\mathbf{a}: X \rightarrow R$, we have that $p(\mathbf{a})=0$, and so $p$ is the zero polynomial. Therefore, $\psi: \mathrm{T}_{R}^{1}(\boldsymbol{X}) \rightarrow C_{\mu}^{1}(\boldsymbol{X})$ is a monomorphism.

Now suppose $\psi\left(\sum \alpha_{I, J} \zeta_{I} \otimes \zeta_{J}\right)=0$, for some $\alpha_{I, J} \in R$ and each of $I, J$ not identically 0 . Then $\sum \alpha_{I, J} \zeta_{I}(\mathbf{a}) \cdot \zeta_{J}\left(\mathbf{a}^{\prime}\right)=0$, for all functions $\mathbf{a}, \mathbf{a}^{\prime}: X \rightarrow R$. Let $X^{\prime}$ be another (disjoint) copy of $\boldsymbol{X}$, and for each $J$, let $J^{\prime}: X^{\prime} \rightarrow \mathbb{Z}_{\geq 0}$ be the corresponding indexing function. Viewing each $\zeta_{J^{\prime}}$ as a polynomial in $\operatorname{Int}\left(R^{X^{\prime}}\right)$, it follows that the polynomial $\sum \alpha_{I, J} \zeta_{I} \cdot \zeta_{J^{\prime}} \in \operatorname{Int}\left(R^{X \sqcup X^{\prime}}\right)$ is the zero polynomial. Note that, for each pair $I$ and $J$ of indexing functions, the functions $I$ and $J^{\prime}$ have disjoint supports; hence, $\zeta_{I} \cdot \zeta_{J^{\prime}}=\zeta_{K}$, where $\left.K\right|_{X}=I$ and $\left.K\right|_{X^{\prime}}=J^{\prime}$. Since these polynomials are elements in a basis for $\operatorname{Int}\left(R^{X \sqcup X^{\prime}}\right)$, it follows that each $\alpha_{I, J}$ is equal to 0 , thus showing that $\psi: \mathrm{T}_{R}^{2}(\boldsymbol{X}) \rightarrow C_{\mu}^{2}(\boldsymbol{X})$ is a monomorphism.

The map $\psi=\psi_{X, \mu}$ constructed above enjoys a naturality property that we now proceed to describe. Let $h: X \rightarrow X^{\prime}$ be a map of sets. By the discussion in section 6.1, we have an induced morphism, $\mathrm{T}(h): \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}^{\leq 2}\left(\boldsymbol{X}^{\prime}\right)$. Now let $\mu^{\prime}$ be a binary operation on $M^{\prime}=M\left(X^{\prime}, R\right)$. Composition with $h$ defines a map $h^{*}: M^{\prime} \rightarrow M$. Suppose this map is a morphism of magmas, that is, $\mu^{\prime}\left(\mathbf{a} \circ h, \mathbf{a}^{\prime} \circ h\right)=h\left(\mu\left(\mathbf{a}, \mathbf{a}^{\prime}\right)\right)$ for all $\mathbf{a}, \mathbf{a}^{\prime}: \boldsymbol{X} \rightarrow$ $R$. Then, as noted in Section 2.2, $h^{*}$ yields a simplicial map between the respective $\Delta$ complexes, $\Delta\left(h^{*}\right): \Delta^{(2)}\left(M^{\prime}, \mu^{\prime}\right) \rightarrow \Delta^{(2)}(M, \mu)$, which in turn induces a morphism between the corresponding cochain algebras, $\Delta\left(h^{*}\right)^{*}: C_{\mu}(\boldsymbol{X} ; R) \rightarrow C_{\mu^{\prime}}\left(\boldsymbol{X}^{\prime} ; R\right)$. The next lemma now follows straight from the definitions.

Lemma 7.2. Let $h: X \rightarrow X^{\prime}$ be a set map and suppose $h^{*}:\left(M^{\prime}, \mu^{\prime}\right) \rightarrow(M, \mu)$ is a magma map. Then the following diagram commutes


In particular, if $h: X \rightarrow X^{\prime}$ is injective, $\mu^{\prime}: M^{\prime} \times M^{\prime} \rightarrow M^{\prime}$ is a binary operation on $M^{\prime}$, and $\mu$ is the restriction of $\mu^{\prime}$ to $M \times M$, then clearly $h^{*}:\left(M^{\prime}, \mu^{\prime}\right) \rightarrow(M, \mu)$ is a magma map, and thus $\psi_{X^{\prime}, \mu^{\prime}} \circ \mathrm{T}(h)=\Delta\left(h^{*}\right)^{*} \circ \psi_{X, \mu}$.
7.2. From 2-tensors to simplicial complexes. We now refine the above construction, in a more specialized setting. Consider a set map $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$. Recall from (6.1) that $\mathrm{T}_{R}(\boldsymbol{X})$ is the tensor algebra on the maximal ideal $\mathfrak{m}_{\boldsymbol{X}, R} \subset \operatorname{Int}\left(R^{X}\right)$. Thus, for each $x \in \boldsymbol{X}$, the 2-tensor $\tau(x) \in \mathrm{T}_{R}^{2}(\boldsymbol{X})$ may be written as

$$
\begin{equation*}
\tau(x)=\sum_{i=1}^{s_{x}} p_{x, i} \otimes q_{x, i} \tag{7.3}
\end{equation*}
$$

for some polynomials $p_{x, i}, q_{x, i} \in \operatorname{Int}\left(R^{X}\right)$ with $p_{x, i}(\mathbf{0})=q_{x, i}(\mathbf{0})=0$.
As before, let $M=M(\boldsymbol{X}, R)$ be the set of all functions a: $\boldsymbol{X} \rightarrow R$, with $R$-module structure given by pointwise addition. We define a map $f_{\tau}: M \times M \rightarrow M$ by setting

$$
\begin{equation*}
f_{\tau}\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(x)=\sum_{i=1}^{s_{x}} p_{x, i}(\mathbf{a}) \cdot q_{x, i}\left(\mathbf{a}^{\prime}\right) \tag{7.4}
\end{equation*}
$$

The map $f_{\tau}$ then determines an operation, $\mu_{\tau}: M \times M \rightarrow M$, given by

$$
\begin{equation*}
\mu_{\tau}\left(\mathbf{a}, \mathbf{a}^{\prime}\right)=\mathbf{a}+\mathbf{a}^{\prime}-f_{\tau}\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \tag{7.5}
\end{equation*}
$$

The pair $M_{\tau}:=\left(M, \mu_{\tau}\right)$ is a unital magma, with unit the zero function $\mathbf{0}$; indeed, equations (7.4) and (7.5) imply that $\mu_{\tau}(\mathbf{a}, \mathbf{0})=\mu_{\tau}(\mathbf{0}, \mathbf{a})=\mathbf{a}$, for all $\mathbf{a}$. In the particular case when $\tau$ itself is the zero function (that is, $\tau(x)=0$ in $\mathrm{T}_{R}^{2}(\boldsymbol{X})$ for all $\left.x \in \boldsymbol{X}\right)$, the corresponding magma is just the aforementioned abelian group $M$. In general, though, the operation $\tau$ is not associative, and so $M_{\tau}$ need not be a (unital) semigroup (also known as monoid).
We denote by $\Delta^{(2)}\left(M_{\tau}\right)$ the 2 -dimensional $\Delta$-set associated to the magma $M_{\tau}$ by the constructions from Sections 2.2 and 7.1, and we let $C_{\tau}(\boldsymbol{X}):=\left(C^{\bullet}\left(\Delta^{(2)}\left(M_{\tau}\right)\right), d_{\Delta}\right)$ denote the simplicial cochain algebra associated to the $\Delta$-set $\Delta^{(2)}\left(M_{\tau}\right)$.
7.3. The endomorphism $d_{\tau}$ of $\mathrm{T}_{R}(\boldsymbol{X})$. Our next objective is to define a degree-1 endomorphism $d_{\tau}: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X})$ of the free binomial algebra on $\boldsymbol{X}$ that extends the map $\tau$ and satisfies some desirable properties. We achieve this by embedding $\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})$ into the cochain algebra $C_{\tau}(\boldsymbol{X})$ defined above.

Theorem 7.3. Given a map of sets, $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$, there is a unique degree-1 linear map, $d_{\tau}: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X})$, such that $d_{\tau}(x)=\tau(x)$ for all $x \in \boldsymbol{X}$ and both the $\cup_{1}-d$ formula and the graded Leibniz rule are satisfied.

Proof. Let $\psi=\psi_{X, \tau}: \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \hookrightarrow C_{\tau}(\boldsymbol{X})$ be the monomorphism from Lemma 7.1. First note that from the formula for the coboundary of a 1 -cochain in a $\Delta$-complex, we have

$$
\begin{align*}
d_{\Delta} \psi(x)\left(\mathbf{a}, \mathbf{a}^{\prime}\right) & =\mathbf{a}(x)+\mathbf{a}^{\prime}(x)-\left(\mathbf{a}(x)+\mathbf{a}^{\prime}(x)-\tau\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(x)\right) \\
& =\tau\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(x) \\
& =\sum_{i}\left(\varphi\left(p_{x, i}\right)(\mathbf{a}) \cdot \varphi\left(q_{x, i}\right)\left(\mathbf{a}^{\prime}\right)\right) \\
& =\sum_{i}\left(\varphi\left(p_{x, i}\right)(\mathbf{x}) \cup \varphi\left(q_{x, i}\right)(\mathbf{x})\right)\left(\mathbf{a}, \mathbf{a}^{\prime}\right)  \tag{7.6}\\
& \left.=\sum_{i} \varphi\left(p_{i, x}(\mathbf{x})\right) \otimes \varphi\left(q_{x, i}(\mathbf{x})\right)\right)\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \\
& =(\varphi \tau(x))\left(\mathbf{a}, \mathbf{a}^{\prime}\right) .
\end{align*}
$$

Since this holds for all pairs $\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in M \times M$, it follows that $d_{\Delta}(\psi(x))=\psi(\tau(x))$.
The next step is to show that the differential $d_{\Delta}: C_{\tau}^{\bullet}(\boldsymbol{X}) \rightarrow C_{\tau}^{\bullet+1}(\boldsymbol{X})$ leaves invariant the subgroup $\psi\left(\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})\right) \subset C_{\tau}(\boldsymbol{X})$. Let $p=p(\mathbf{x})$ be a polynomial in $\mathrm{T}_{R}^{1}(\boldsymbol{X})$, where $\mathbf{x} \subset \boldsymbol{X}$ denotes the subset of variables appearing in the monomials comprising $p$. Given 1-chains $\mathbf{a}, \mathbf{a}^{\prime}$, we have

$$
\begin{equation*}
d_{\Delta} p\left(\mathbf{a}, \mathbf{a}^{\prime}\right)=p(\mathbf{a})+p\left(\mathbf{a}^{\prime}\right)-p\left(\mathbf{a}+\mathbf{a}^{\prime}-\tau\left(\mathbf{a}, \mathbf{a}^{\prime}\right)\right) \tag{7.7}
\end{equation*}
$$

By Theorem 5.2, we may write

$$
\begin{equation*}
d_{\Delta} p\left(\mathbf{x}, \mathbf{x}^{\prime}\right)=\sum c_{I_{i}, J_{i}} \cdot \zeta_{I_{i}}(\mathbf{x}) \otimes \zeta_{J_{i}}\left(\mathbf{x}^{\prime}\right) \tag{7.8}
\end{equation*}
$$

for some constants $c_{I_{i}, J_{i}} \in R$. From equation (7.7) it follows that the polynomial $d_{\Delta} p\left(\mathbf{0}, \mathbf{x}^{\prime}\right)=$ $\sum c_{0, J_{i}} \cdot \zeta_{J_{i}}\left(\mathbf{x}^{\prime}\right)$ vanishes for all values of $\mathbf{x}^{\prime}$; thus all coefficients $c_{0, J_{i}}$ vanish. A similar argument shows that $c_{I_{i}, 0}=0$ for all $i$. Therefore, $d_{\Delta} p$ is a sum of products of polynomials in $\operatorname{Int}\left(R^{X}\right)$ and polynomials in $\operatorname{Int}\left(R^{X^{\prime}}\right)$ with zero constant term in each factor; that is, $d_{\Delta} p$ is in the image of $\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})$ under the map $\psi$. This completes the proof that $\psi\left(\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})\right)$ is closed under $d_{\Delta}$.
Now set $d_{\tau}: \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})$ equal to the restriction of $d_{\Delta}$ to the invariant subgroup $\psi\left(\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})\right)$. Since the differential $d_{\Delta}$ satisfies the $\cup_{1}-d$ formula (4.10), it follows that
$d_{\tau}$ also satisfies this formula. Finally, we extend $d_{\tau}$ to the whole free cup-one algebra $\mathrm{T}=\mathrm{T}_{R}(\boldsymbol{X})$ using the graded Leibniz rule.

The final step is to show that the map $d_{\tau}: \mathrm{T} \rightarrow \mathrm{T}$ defined above is the unique degree 1 linear map for which $d_{\tau}(x)=\tau(x)$ for all $x \in \boldsymbol{X}$, and the $\cup_{1}-d$ formula and the graded Leibniz rule are satisfied. Let $d: \mathrm{T}^{1} \rightarrow \mathrm{~T}^{2}$ be any map that satisfies the $\cup_{1}-d$ formula with $d(x)=d_{\tau}(x)$ for all $x \in \boldsymbol{X}$. It suffices to show that $d(p)=d_{\tau}(p)$ for all $p \in \mathrm{~T}^{1}$. Since both $d$ and $d_{\tau}$ satisfy the $\cup_{1}-d$ formula, it follows that

$$
\begin{equation*}
d\left(p \cup_{1} q\right)=d_{\tau}\left(p \cup_{1} q\right) \text { if } d(p)=d_{\tau}(p) \text { and } d(q)=d_{\tau}(q) \tag{7.9}
\end{equation*}
$$

Then from equation (7.9) and induction on $i$ using the formula $\zeta_{i+1}(x)=\left(\zeta_{i}(x) \cup_{1} x-\right.$ $\left.i \zeta_{i}(x)\right) /(i+1)$, it follows that $d\left(\zeta_{i}(x)\right)=d_{\tau}\left(\zeta_{i}(x)\right.$ for all $x \in \boldsymbol{X}$ and all $i \geq 1$ in the case $R=\mathbb{Z}$ and $1 \leq i \leq p-2$ in the case $R=\mathbb{Z}_{p}$.
It then follows using (7.9) and induction on the length of $\operatorname{supp}(I)$ that $d\left(\zeta_{I}(\mathbf{x})\right)=d_{\tau}\left(\zeta_{I}(\mathbf{x})\right)$ for all $I$ and $\mathbf{x}$. Since the polynomials $\zeta_{I}(\mathbf{x})$ form a basis for $\mathrm{T}^{1}$, the proof of uniqueness is complete.
This completes the proof of the theorem.
7.4. On the invariance of $\psi\left(\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})\right)$. Let $f: M \times M \rightarrow M$ be an arbitrary map, with $M=M(\boldsymbol{X}, R)$ equal to the set of functions from $\boldsymbol{X}$ to $R$, let $\mu: M \times M \rightarrow M$ be defined by $\mu=\mathbf{a}+\mathbf{a}^{\prime}-f\left(\mathbf{a}, \mathbf{a}^{\prime}\right)$, and let $\psi: \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \hookrightarrow C_{\mu}(\boldsymbol{X})$ be the monomorphism in Lemma 7.1. The next step is to characterize the maps $f$ for which the simplicial differential $d_{\Delta}: C_{\mu}^{\bullet}(\boldsymbol{X}) \rightarrow C_{\mu}^{\bullet+1}(\boldsymbol{X})$ leaves invariant the subgroup $\psi\left(\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})\right) \subset C_{\mu}(\boldsymbol{X})$.
Proposition 7.4. The differential $d_{\Delta}: C_{\mu}(\boldsymbol{X}) \rightarrow C_{\mu}(\boldsymbol{X})$ leaves invariant the subgroup $\psi\left(\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})\right) \subset C_{\mu}(\boldsymbol{X})$ if and only if $f=f_{\tau}$, for some function $\tau: \mathrm{T}^{1}(\boldsymbol{X}) \rightarrow \mathrm{T}^{2}(\boldsymbol{X})$.

Proof. The "if" part is included in the proof of Theorem 7.3. To prove the "only if" part, assume the map $d_{\Delta}$ leaves invariant the subgroup $\psi\left(\mathrm{T}_{R}^{\leq 2}(\boldsymbol{X})\right)$ of $C_{\tau}^{\leq 2}(\boldsymbol{X})$. Then there is a linear map $d: \mathrm{T}_{R}^{1}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}^{2}(\boldsymbol{X})$ such that the following diagram commutes.


For each $x \in \boldsymbol{X}$, let $p_{x, i}, q_{x, i}$ be polynomials in $\operatorname{Int}\left(R^{X}\right)$ with $p_{x, i}(\mathbf{0})=q_{x, i}(\mathbf{0})=0$ such that $d(x)=\sum_{i} p_{x, i} \otimes q_{x, i}$; then for each pair (a, $\left.\mathbf{a}^{\prime}\right) \in M \times M$ we have

$$
\begin{equation*}
\psi \circ d(x)\left(\mathbf{a}, \mathbf{a}^{\prime}\right)=\sum_{i} p_{x, i}(\mathbf{a}) \cup q_{x, i}\left(\mathbf{a}^{\prime}\right) \tag{7.11}
\end{equation*}
$$

Note that for $x \in \boldsymbol{X}$, the cochain $\psi(x)$ is given by $\psi(x)(\mathbf{a})=\mathbf{a}(x)$, and then from equation (7.7) we have

$$
\begin{align*}
d_{\Delta} \circ \psi(x)\left(\mathbf{a}, \mathbf{a}^{\prime}\right) & =\mathbf{a}(x)+\mathbf{a}^{\prime}(x)-\left(\mathbf{a}+\mathbf{a}^{\prime}-f\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(x)\right. \\
& =\mathbf{a}(x)+\mathbf{a}^{\prime}(x)-\mathbf{a}(x)-\mathbf{a}^{\prime}(x)+f\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(x)  \tag{7.12}\\
& =f\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(x)
\end{align*}
$$

Since $\psi \circ d(x)\left(\mathbf{a}, \mathbf{a}^{\prime}\right)=d_{\Delta} \circ \psi(x)\left(\mathbf{a}, \mathbf{a}^{\prime}\right)$, it follows that for all $x \in \boldsymbol{X}$ and all pairs $\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in$ $M \times M$, we have

$$
\begin{equation*}
\sum_{i} p_{x, i}(\mathbf{a}) \cup q_{x, i}\left(\mathbf{a}^{\prime}\right)=f\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(x) \tag{7.13}
\end{equation*}
$$

Hence, $f=f_{\tau}$ where $\tau:=\left.d\right|_{X}: \boldsymbol{X} \rightarrow \mathrm{T}_{R}^{2}(\boldsymbol{X})$, and the argument is complete.
7.5. The differential $d_{\tau}$ associated to an admissible map $\tau$. Recall that a map $\tau: X \rightarrow$ $\mathrm{T}_{R}^{2}(\boldsymbol{X})$ given by (7.3) determines a map $f_{\tau}: M \times M \rightarrow M$, given by (7.4), where $M$ is the set of all functions from $\boldsymbol{X}$ to $R$. In turn, the map $f_{\tau}$ determines an operation, $\mu_{\tau}: M \times M \rightarrow M$, given by (7.5). The case when the corresponding unital magma, $M_{\tau}=\left(M, \mu_{\tau}\right)$, is a monoid is particularly interesting.

Definition 7.5. A map of sets $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$ is said to be admissible if corresponding the binary operation, $\mu_{\tau}: M \times M \rightarrow M$, is associative, or, equivalently, the magma $M_{\tau}$ is a monoid.

Theorem 7.6. If the map $\tau: \boldsymbol{X} \rightarrow \mathrm{T}_{R}^{2}(\boldsymbol{X})$ is admissible, then the map $\psi=\psi_{\boldsymbol{X}, \tau}: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow$ $C^{*}\left(\Delta\left(M_{\tau}\right)\right)$ is a monomorphism and $d_{\tau}^{2} \equiv 0$.

Proof. Let $\Delta_{\tau}=\Delta\left(M_{\tau}\right)$ be the $\Delta$-set associated to the monoid $M_{\tau}$. Note that the 2skeleton of $\Delta_{\tau}$ is the previously defined $\Delta$-set $\Delta^{(2)}\left(M_{\tau}\right)$. The arguments in the proofs of Lemma 7.1 and Theorem 7.3 generalize as follows to show that the map $\psi_{\leq 2}: \mathrm{T}_{R}^{\leq 2}(\boldsymbol{X}) \rightarrow$ $C^{\leq 2}\left(\Delta_{\tau}^{(2)}\right)$ extends to a monomorphism $\psi: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow C^{*}\left(\Delta_{\tau}\right)$ with $d_{\Delta} \circ \psi=\psi \circ d_{\tau}$.
Set $\psi: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow C^{*}\left(\Delta_{\tau}\right)$ equal to the unique map of algebras that restricts to $\psi_{\leq 2}$ on elements of degree less than or equal to 2 .
The next step is to show that $\psi$ is a monomorphism. Let $p_{1}(\mathbf{x}) \otimes \cdots \otimes p_{n}(\mathbf{x})$ be a basis element in $\mathrm{T}_{R}^{n}(\boldsymbol{X})$ and let $\boldsymbol{X}_{1}, \ldots, \boldsymbol{X}_{n}$ be disjoint copies of $\boldsymbol{X}$. Set $e: \mathrm{T}_{R}^{n}(\boldsymbol{X}) \rightarrow$ $\operatorname{Int}\left(R^{X_{1} \cup \cdots \cup X_{n}}\right)$ equal to the map that sends $p_{1}(\mathbf{x}) \otimes \cdots \otimes p_{n}(\mathbf{x})$ to the product of polynomials $p_{1}\left(\mathbf{x}_{1}\right) \cdots p_{n}\left(\mathbf{x}_{n}\right)$, where $p_{i}\left(\mathbf{x}_{i}\right)$ denotes the polynomial $p_{i}(\mathbf{x})$ with the variables $x \in \boldsymbol{X}$ replaced by the corresponding variables $x_{i} \in \boldsymbol{X}_{i}$. Since $e$ is a bijection on basis elements, it follows that $e$ is a bijection. If $\psi\left(\sum_{i} p_{1, i}(\mathbf{x}) \otimes \cdots \otimes p_{n, i}(\mathbf{x})\right)$ is the zero element in $C^{n}\left(\Delta_{\tau}\right)$, then $e\left(\sum_{i} p_{1, i}\left(\mathbf{a}_{1}\right) \otimes \cdots \otimes p_{n, i}\left(\mathbf{a}_{n}\right)\right)$ is zero for all maps $\mathbf{a}_{i}: \boldsymbol{X}_{i} \rightarrow R$, and hence, is the zero polynomial. The result that $\psi$ is a monomorphism now follows since $e$ is a monomorphism.

Since $d_{\Delta} \circ \psi=\psi \circ d_{\tau}: \mathrm{T}_{R}^{1}(\boldsymbol{X}) \rightarrow C^{2}\left(\Delta_{\tau}\right)$ by Theorem 7.3, and since $\mathrm{T}_{R}(\boldsymbol{X})$ is generated by products of elements in $\mathrm{T}_{R}^{1}(\boldsymbol{X})$, it follows that $d_{\Delta} \circ \psi=\psi \circ d_{\tau}: \mathrm{T}_{R}^{i}(\boldsymbol{X}) \rightarrow C^{i+1}\left(\Delta_{\tau}\right)$ for all $i \geq 1$. Then since $d_{\Delta}^{2} \equiv 0$ and $\psi$ is a monomorphism, it follows that $d_{\tau}^{2} \equiv 0$, and the proof is complete.
Remark 7.7. It can be shown that the homomorphism $\psi: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow C^{\bullet}\left(\Delta_{\tau}\right)$ sends the $\cup_{1}$ and $\circ$ products in $T_{R}(\boldsymbol{X})$ given in Section 6.4; respectively, to the $\cup_{1}$ and $\cup_{2}$ product maps in $C^{\bullet}\left(\Delta_{\tau}\right)$.
7.6. The differentials of the $\zeta$-maps. We consider now in more detail the simplest possible case of Theorem 7.3; namely, the case when $\tau=0$. To begin, we recall the following result, which is proved in [18, Theorems 7.5 and 8.14].

Theorem 7.8. Let $\left(A, d_{A}\right)$ be a $\cup_{1}$-dga over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, let $X$ be a set, and let $f: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A$ be a morphism of graded $R$-algebras with cup-one products. Then $f:\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{\mathbf{0}}\right) \rightarrow\left(A, d_{A}\right)$ is a map of $\cup_{1}-d g a s$ if and only if $d_{A} \circ f(x)=f \circ d_{\mathbf{0}}(x)$ for all $x \in \boldsymbol{X}$.

Lemma 6.4 and the above theorem have the following immediate corollary.
Corollary 7.9. If $\left(A, d_{A}\right)$ is a binomial cup-one $R$-dga with $H^{0}(A)=R$, then there is a bijection between binomial $\cup_{1}-d g$ maps from $\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{0}\right)$ to $\left(A, d_{A}\right)$ inducing an isomorphism on $H^{0}$ and maps of sets from $\boldsymbol{X}$ to $Z^{1}(A)$.

The following theorem gives an explicit formula for the differential $d_{\mathbf{0}}: \mathrm{T}_{R}^{1}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}^{2}(\boldsymbol{X})$ associated to this map. This result recovers Theorem 6.11 from [18], proved there by other methods.

Theorem 7.10. Let $X$ be a set, let $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$ be the zero map, and let $d_{0}$ be the corresponding differential on $\mathrm{T}_{R}(\boldsymbol{X})$, given by $d_{0}(x)=0$ for all $x \in \boldsymbol{X}$. Then we have

$$
\begin{equation*}
d_{0}\left(\zeta_{k}(x)\right)=-\sum_{\ell=1}^{k-1} \zeta_{\ell}(x) \otimes \zeta_{k-\ell}(x) \tag{7.14}
\end{equation*}
$$

for all $x \in X$ and for $k \geq 1$ in the case $R=\mathbb{Z}$ and for $1 \leq k \leq p-1$ in the case $R=\mathbb{Z}_{p}$. More generally,

$$
\begin{equation*}
d_{0}\left(\zeta_{I}(\mathbf{x})\right)=-\sum_{\substack{I_{1}+I_{2}=I \\ I_{j} \neq 0}} \zeta_{I_{1}}(\mathbf{x}) \otimes \zeta_{I_{2}}(\mathbf{x}) \tag{7.15}
\end{equation*}
$$

where in the case $R=\mathbb{Z}_{p}$ we have $k \leq p-1$ and each of the indices in $I$ is less than or equal to $p-1$.

Proof. Since $\tau$ is the zero map, the magma $M=M_{\tau}$ is a monoid (in fact, an abelian group), and hence, $\tau$ is admissible. By Theorem 7.6, we have that $d_{0}=d_{\tau}$ is a differential on $\mathrm{T}_{R}(\boldsymbol{X})$, and hence, $\mathrm{T}=\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{\mathbf{0}}\right)$ is a binomial $\cup_{1}$-dga.

To prove equations (7.14) and (7.15) recall that in a binomial algebra with elements $a, b$, we have

$$
\begin{equation*}
\zeta_{k}(a+b)=\sum_{i+j=k} \zeta_{i}(a) \zeta_{j}(b) \tag{7.16}
\end{equation*}
$$

for $k \geq 1$ in the case $R=\mathbb{Z}$ and for $1 \leq k \leq p-1$ in the case $R=\mathbb{Z}_{p}$. Set $C_{\tau}(\boldsymbol{X})=$ $\left(C^{\bullet}\left(\Delta_{\tau}(\boldsymbol{X})\right), d_{\Delta}\right)$, and let $\psi: \mathrm{T}^{\leq 2} \hookrightarrow C_{\tau}^{\leq 2}(\boldsymbol{X})$ be the injective map of binomial $\cup_{1}$-dgas defined in the proof of Theorem 7.3, so that $\psi\left(\zeta_{k}(x)\right)=\zeta_{k}(\mathbf{a}(x))$ for all $\mathbf{a} \in C_{\tau}^{1}(\boldsymbol{X})$ and all $k \geq 1$. Then,

$$
\begin{align*}
\psi\left(d_{\tau} \zeta_{k}(x)\right)\left(\mathbf{a}, \mathbf{a}^{\prime}\right) & =d_{\Delta} \psi\left(\zeta_{k}(x)\right)\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \\
& =\mathbf{a}(x)+\mathbf{a}^{\prime}(x)-\zeta_{k}\left(\mathbf{a}(x)+\mathbf{a}^{\prime}(x)\right)  \tag{7.7}\\
& =\mathbf{a}(x)+\mathbf{a}^{\prime}(x)-\mathbf{a}(x)-\mathbf{a}^{\prime}(x)-\sum_{\ell=1}^{k-1} \zeta_{\ell}\left(\mathbf{a}(x) \cdot \zeta_{k-\ell}\left(\mathbf{a}^{\prime}(x)\right) \quad \text { by }(7.7)\right.  \tag{7.16}\\
& =-\sum_{\ell=1}^{k-1} \zeta_{\ell}\left(\mathbf{a}(x) \cdot \zeta_{k-\ell}\left(\mathbf{a}^{\prime}(x)\right)\right. \\
& =-\sum_{\ell=1}^{k-1}\left[\psi\left(\zeta_{\ell}(x)\right) \cup \psi\left(\zeta_{k-\ell}(x)\right)\right]\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \\
& =-\sum_{\ell=1}^{k-1} \psi\left[\zeta_{\ell}(x) \otimes \psi\left(\zeta_{k-\ell}(x)\right)\right]\left(\mathbf{a}, \mathbf{a}^{\prime}\right)
\end{align*}
$$

Since this equality holds for all 2 -simplices ( $\mathbf{a}, \mathbf{a}^{\prime}$ ) in $\Delta_{\tau}(\boldsymbol{X})$, equation (7.14) now follows. Equation (7.15) follows by a similar argument, by applying equation (7.16) to products of the form $\zeta_{i_{1}}\left(a_{1}+b_{1}\right) \zeta_{i_{2}}\left(a_{2}+b_{2}\right) \cdots \zeta_{i_{n}}\left(a_{n}+b_{n}\right)$.

Corollary 7.11. Let A be a binomial $\cup_{1}$-dga over $R$. Then for $a \in Z^{1}(A)$ we have

$$
\begin{equation*}
d_{A}\left(\zeta_{k}(a)\right)=-\sum_{\ell=1}^{k-1} \zeta_{\ell}(a) \otimes \zeta_{k-\ell}(a), \tag{7.17}
\end{equation*}
$$

for all $k \geq 1$ in the case $R=\mathbb{Z}$ and for $1 \leq k \leq p-1$ in the case $R=\mathbb{Z}_{p}$.
Proof. By Corollary 7.9, an element $a \in Z^{1}(A)$ corresponds to a map of binomial $\cup_{1}$-dgas from $\left(\mathrm{T}_{R}(\{x\}), d_{0}\right)$ to $\left(A, d_{A}\right)$ which sends $x$ to $a$, and the result follows from Theorem 7.10.

## 8. Hirsch extensions

In this section, we consider Hirsch extensions of $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$, the free binomial graded algebra with cup-one products on a set $\boldsymbol{X}$ equipped with a differential $d: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow$
$\mathrm{T}_{R}(\boldsymbol{X})$ making it into a $\cup_{1}$-dga. Furthermore, for $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, we show that the map $\psi_{X, R}:\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{\mathbf{0}}\right) \rightarrow C^{*}\left(B\left(R^{n}\right) ; R\right)$ induces an isomorphism of cohomology rings.
8.1. Hirsch extensions of $\mathrm{T}_{R}(\boldsymbol{X})$. The following definition is motivated by the notion of Hirsch extension in the context of commutative dgas over fields of characteristic 0. Let ( $\mathrm{T}_{R}(\boldsymbol{X}), d$ ) be as above.

Definition 8.1. Let $\boldsymbol{Y}$ be a set. An inclusion $i:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ of binomial $\cup_{1}$-dgas is called a Hirsch extension if $\bar{d}(y)$ is a cocycle in $\left(\mathrm{T}_{R}^{2}(\boldsymbol{X}), d\right)$ for all $y \in \boldsymbol{Y}$. If $Y=\{y\}$ is a singleton, we call such an extension an elementary Hirsch extension.

Theorem 8.2. Let $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ be a free binomial $\cup_{1}-d g$ a on a set $\boldsymbol{X}$. Then,
(1) For every set $\boldsymbol{Y}$, there is a bijection between Hirsch extensions $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow$ $\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ and maps of sets $\tau_{\boldsymbol{Y}}: \boldsymbol{Y} \rightarrow Z\left(\mathrm{~T}_{R}^{2}(\boldsymbol{X})\right)$.
(2) If $\tau=\left.d\right|_{X}$ is admissible, then $\bar{\tau}=\left.\bar{d}\right|_{X \cup Y}$ is admissible.

Proof. Given a Hirsch extension $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \hookrightarrow\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$, the restriction of $\bar{d}$ to $\boldsymbol{Y}$ gives a map $\tau_{\boldsymbol{Y}}=\left.\bar{d}\right|_{Y}: \boldsymbol{Y} \rightarrow Z\left(\mathrm{~T}_{R}^{2}(\boldsymbol{X})\right)$.
In the opposite direction, assume that the map $\tau_{Y}: \boldsymbol{Y} \rightarrow Z\left(\mathrm{~T}_{R}^{2}(\boldsymbol{X})\right)$ is given. Set $\bar{\tau}: X \cup$ $\boldsymbol{Y} \rightarrow \mathrm{T}_{R}^{2}(\boldsymbol{X} \cup \boldsymbol{Y})$ equal to the map given by $\left.\bar{\tau}\right|_{X}=\left.d\right|_{X}$ and $\left.\bar{\tau}\right|_{\boldsymbol{Y}}=\left.\tau\right|_{\boldsymbol{Y}}$. By Theorem 7.3, the map $\bar{\tau}$ determines an extension of $d$ to a map $\bar{d}=d_{\bar{\tau}}: \mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y})$ satisfying the $\cup_{1}-d$ formula and the Leibniz rule with $\left.d_{\bar{\tau}}\right|_{X \cup Y}=\bar{\tau}$. Since $\tau_{Y}(y)$ is a cocycle for all $y \in \boldsymbol{Y}$, it follows from Theorem 6.7 that $\bar{d}^{2}(u)=0$ for all $u \in \mathrm{~T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y})$, and the proof of claim (1) is complete.

Recall from Definition 7.5 that $\tau$ is admissible precisely when the corresponding magma, $M_{\tau}$, is a monoid. It follows from the above proof that $M_{\bar{\tau}}$ is the extension of $M_{\tau}$ by the $R$-module $M(\boldsymbol{Y}, R)$ of functions from $\boldsymbol{Y}$ to $R$ given by the normalized cocycle $v \in$ $Z^{2}\left(\Delta\left(M_{\tau}\right) ; M(\boldsymbol{Y}, R)\right.$, where for $y \in \boldsymbol{Y}$, we have $\tau(y)=\sum_{i=1}^{s_{y}} p_{y, i} \otimes q_{y, i}$ and $v\left(\mathbf{a}, \mathbf{a}^{\prime}\right)(y)=$ $\sum_{i=1}^{s_{y}} p_{\mathbf{y}, i}(\mathbf{a}) \cdot q_{y, i}\left(\mathbf{a}^{\prime}\right)$.
Claim (2) now follows from Lemma 2.4, part (3), and the proof is complete.
Given a map $\tau: \boldsymbol{Y} \rightarrow Z\left(\mathrm{~T}_{R}^{2}(\boldsymbol{X}), d\right)$, denote by $[\tau]$ the map from $\boldsymbol{Y}$ to $H^{2}\left(\mathrm{~T}_{R}(\boldsymbol{X}), d\right)$ that sends each element $y \in \boldsymbol{Y}$ to the cohomology class of $\tau(y)$.

Definition 8.3. Given maps $\tau$ and $\tau^{\prime}$ from a set $\boldsymbol{Y}$ to $Z\left(\mathrm{~T}_{R}^{2}(\boldsymbol{X}), d\right)$, the corresponding Hirsch extensions, $\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ and $\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}^{\prime}\right)$, are called equivalent Hirsch extensions if $[\tau]=\left[\tau^{\prime}\right]$.

Lemma 8.4. If $\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ and $\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}^{\prime}\right)$ are equivalent Hirsch extensions, then the cohomology algebras $H^{*}\left(\mathrm{~T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ and $H^{*}\left(\mathrm{~T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}^{\prime}\right)$ are isomorphic.

Proof. First recall from Lemmas 6.3 and 6.4 that a map $f:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(A, d_{A}\right)$ of binomial graded $R$-algebras with cup-one products commutes with the differentials if and only if $d_{A} f(x)=f(d x)$ for all $x \in \boldsymbol{X}$; moreover, if $H^{0}(A)=R$ and $f$ induces an isomorphism on $H^{0}$, then $f$ is determined by its restriction to the set $\boldsymbol{X}$.
From the definition of equivalent Hirsch extensions it follows that for each $y \in \boldsymbol{Y}$ there is an element $c_{1}(y) \in \mathrm{T}_{R}^{1}(\boldsymbol{X})$ with $\tau^{\prime}(y)=\tau(y)+d c_{1}(y)$. Define a linear map $f:\left(\mathrm{T}_{R}(\boldsymbol{X} \cup\right.$ $\boldsymbol{Y}), \bar{d}) \rightarrow\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}^{\prime}\right)$ by setting $f(u)=u$ for $u \in \mathrm{~T}_{R}(\boldsymbol{X})$ and $f(y)=y-c_{1}(y)$ for $y \in \boldsymbol{Y}$. Then

$$
\begin{aligned}
\bar{d}^{\prime}(f(y)) & =\bar{d}^{\prime}\left(y-c_{1}(y)\right) \\
& =\bar{d}^{\prime}(y)-d c_{1}(y) \\
& =\tau^{\prime}(y)-d c_{1}(y) \\
& =\left(\tau(y)+d c_{1}(y)\right)-d c_{1}(y) \\
& =\tau(y) \\
& =f(\bar{d}(y))
\end{aligned}
$$

Thus, $f$ commutes with the differentials. Similarly, a linear map $g:\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}^{\prime}\right) \rightarrow$ $\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ is defined by setting $g(u)=u$ for $u \in \mathrm{~T}_{R}(\boldsymbol{X})$ and $g(y)=y+c_{1}(y)$ for $y \in \boldsymbol{Y}$. Then $g$ commutes with the differentials, and the result follows since $f$ and $g$ are inverses of each other.
8.2. A spectral sequence. We now set up a cohomological spectral sequence that will prove useful for our purposes.

Lemm 8.5. Let $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ be a free binomial $\cup_{1}-d g$ a on a set $X$. Given an elementary Hirsch extension $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(\mathrm{T}_{R}(\boldsymbol{X} \cup\{y\}), \bar{d}\right)$, there is a spectral sequence, $\left(E_{r}^{p, q}, d_{r}\right)$, with $d_{r}: E_{r}^{p, q} \rightarrow E_{r}^{p+r, q-r+1}$ and $E_{2}^{p, q} \cong H^{p}\left(\mathrm{~T}_{R}(\boldsymbol{X}), d\right) \otimes H^{q}\left(\mathrm{~T}_{R}(\{y\}), d_{0}\right)$, where $d_{0}(y)=0$.

Proof. Denote $\mathrm{T}_{R}(\boldsymbol{X} \cup\{y\})$ by T . A basis for $\mathrm{T}^{1}$ is given by elements of the form

$$
\begin{equation*}
\zeta_{I}\left(x_{1}, \ldots, x_{\ell}\right) \zeta_{k}(y) \tag{8.1}
\end{equation*}
$$

where $I=\left(i_{1}, \ldots, i_{\ell}\right)$ and $\zeta_{I}\left(x_{1}, \ldots, x_{\ell}\right)=\zeta_{i_{1}}\left(x_{1}\right) \cdots \zeta_{i_{\ell}}\left(x_{\ell}\right) \in \mathrm{T}_{R}(\boldsymbol{X})$, with the $x_{j}$ distinct elements in $\boldsymbol{X}$. If $k=0$, then $\zeta_{I}\left(x_{1}, \ldots, x_{\ell}\right) \zeta_{k}(y)$ denotes $\zeta_{I}\left(x_{1}, \ldots, x_{\ell}\right)$; and if $I=(0, \ldots, 0)$, then $\zeta_{I}\left(x_{1}, \ldots, x_{\ell}\right) \zeta_{k}(y)$ denotes $\zeta_{k}(y)$, where in the case $R=\mathbb{Z}_{p}$, we have that $1 \leq i_{j} \leq p-1$.
Define a bigrading on T by setting $D^{p, q}$ equal to the summand of $\mathrm{T}^{p+q}$ generated by the tensor products $u_{1} \otimes \cdots \otimes u_{p+q}$ of basis elements in $\mathrm{T}^{1}$ for which exactly $p$ of the factors have $I \neq(0, \ldots, 0)$. We claim that the differential $d$ restricts to maps

$$
\begin{equation*}
d: D^{0,1} \rightarrow D^{2,0} \oplus D^{0,2} \quad \text { and } \quad d: D^{1,0} \rightarrow D^{2,0} \oplus D^{1,1} . \tag{8.2}
\end{equation*}
$$

The claim follows by induction using the $\cup_{1}-d$ formula, the Hirsch identity, and the formula $\zeta_{n+1}(y)=\left[\zeta_{n}(y) y-n \zeta_{n}(y)\right] /(n+1)$. The group $D^{0,1}$ is free abelian, with basis $\left\{\zeta_{i}(y)\right\}$, and the induction is on $i$ with base case $i=1$. The group $D^{1,0}$ has basis $\left\{\zeta_{I}\left(x_{1}, \ldots x_{\ell}\right) \zeta_{i}(y)\right\}$, where $I \neq\{0\}$, and the induction is on $i$ with base case $i=0$.
From equation (8.2), it follows that $F^{\ell}(\mathrm{T}):=\bigoplus_{p \geq \ell, q \geq 0} D^{p, q}$ defines a decreasing filtration, $\mathrm{T}=F^{0} \supseteq F^{1} \supseteq F^{2} \supseteq \cdots$, of subcomplexes. A direct computation shows that in the resulting spectral sequence the $E_{2}$ term is given by

$$
\begin{equation*}
E_{2}^{p, q}=H^{p}\left(\mathrm{~T}_{R}(\boldsymbol{X}), d\right) \otimes H^{q}\left(\mathrm{~T}_{R}(y), d_{0}\right), \tag{8.3}
\end{equation*}
$$

where $d_{\mathbf{0}}(y)=0$, and the proof is complete.
8.3. The cohomology of $\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{0}\right)$. We are now in a position to compute the cohomology algebra of the free binomial graded cup-one algebra $\mathrm{T}_{R}(\boldsymbol{X})$, endowed with the differential $d_{\tau}=d_{0}$ corresponding to the admissible function $\tau: X \rightarrow \mathrm{~T}_{R}^{2}(\boldsymbol{X})$ given by $\tau(x)=0$ for all $x \in \boldsymbol{X}$. We first assume $R=\mathbb{Z}$, in which case we write $\mathrm{T}(\boldsymbol{X}):=\mathrm{T}_{R}(\boldsymbol{X})$.
Proposition 8.6. Given a finite set $\boldsymbol{X}$, there is a natural isomorphism

$$
\begin{equation*}
\kappa_{X}: H^{*}(\mathrm{~T}(\boldsymbol{X})) \xrightarrow{\simeq} \bigwedge^{*}(\boldsymbol{X}) \tag{8.4}
\end{equation*}
$$

between the cohomology algebra of the dga $\left(\mathrm{T}(\boldsymbol{X}), d_{0}\right)$ and the exterior algebra on the free abelian group $\mathbb{Z}^{X}$.

Proof. We establish the existence of the isomorphism $\kappa_{X}$ is by induction on $k$, the size of $\boldsymbol{X}$. For the base case $k=1$, write $\mathrm{T}=\mathrm{T}(\boldsymbol{X})$, and define two subcomplexes, $\mathrm{T}_{0}$ and $\mathrm{T}_{1}$, as follows. Set $\mathrm{T}_{0}^{0}=\mathbb{Z}, \mathrm{T}_{0}^{1}=\mathbb{Z}$ with generator $x$, and $\mathrm{T}_{0}^{i}=0$ for $i \geq 2$. Furthermore, set $\mathrm{T}_{1}^{1}$ equal to the submodule of $\mathrm{T}^{1}(\{x\})$ generated by the elements of the form $\zeta_{k}(x)$ with $k \geq 2$, and set $\mathrm{T}_{1}^{j}=\mathrm{T}^{j}(\{x\})$ for $j \geq 2$. It is now readily verified that $\mathrm{T}=\mathrm{T}_{0} \oplus \mathrm{~T}_{1}$.
Clearly, $\mathrm{T}_{0}=\Lambda(x)$, with zero differential; thus, $H^{*}\left(\mathrm{~T}_{0}\right)=\bigwedge(x)$. Denote $\zeta_{k}(x)$ by $\zeta_{k}$, and define homomorphisms $h_{\ell}: \mathrm{T}_{1}^{\ell} \rightarrow \mathrm{T}_{1}^{\ell-1}$ by

$$
h_{\ell}\left(\zeta_{i_{1}} \otimes \zeta_{i_{2}} \otimes \cdots \otimes \zeta_{i_{\ell}}\right)= \begin{cases}-\zeta_{i_{2}+1} \otimes \cdots \otimes \zeta_{i_{\ell}} & \text { if } i_{1}=1  \tag{8.5}\\ 0 & \text { if } i_{1}>1\end{cases}
$$

By direct computation using equation (7.15), it follows that

$$
\begin{equation*}
d_{\mathbf{0}} \circ h_{\ell}+h_{\ell+1} \circ d_{\mathbf{0}}=\mathrm{id}_{\mathrm{T}_{1}} \tag{8.6}
\end{equation*}
$$

Hence, the cohomology of $\mathrm{T}_{1}$ is zero, and we conclude that the cohomology of $\left(\mathrm{T}(\{x\}), d_{\mathbf{0}}\right)$ is the exterior algebra with generator $x$.
For the inductive step, we assume the result holds for $\boldsymbol{X}_{k}=\left\{x_{1}, \ldots, x_{k}\right\}$ and show the result then holds for $\boldsymbol{X}_{k+1}=\left\{x_{1}, \ldots, x_{k}, x_{k+1}\right\}$. We use the spectral sequence in Lemma 8.5 with $\boldsymbol{X}=\boldsymbol{X}_{k}$ and $y=x_{k+1}$; applying the base case with $\mathrm{T}=\mathrm{T}\left(x_{k+1}\right)$, we have by induction that the $E_{2}$ term in the spectral sequence of the form $E_{2}=\bigwedge\left(x_{1}, \ldots, x_{k}\right) \otimes \bigwedge\left(x_{k+1}\right)$. Since
$d_{0} x_{k+1}=0$ in T , it follows that the spectral sequence collapses, from which we obtain an isomorphism of graded $\mathbb{Z}$-modules,

$$
\begin{equation*}
H^{*}\left(\mathrm{~T}(\boldsymbol{X}), d_{\mathbf{0}}\right) \cong \wedge\left(x_{1}, \ldots, x_{k+1}\right) . \tag{8.7}
\end{equation*}
$$

For $x \in \boldsymbol{X}$, set $[x]$ equal to the cohomology class of $x$ in $H^{1}\left(\mathbf{T}(\boldsymbol{X}), d_{\mathbf{0}}\right)$; note that $d_{0} \zeta_{2}(x)=$ $-x \cup x$, and so $[x] \cup[x]=0$ in $H^{*}\left(\mathrm{~T}(\boldsymbol{X})\right.$ ). Moreover, for $x_{1}, x_{2}$ distinct elements in $X$, we have $d_{0}\left(x_{1} \cup_{1} x_{2}\right)=-x_{1} \cup x_{2}-x_{2} \cup x_{1}$, so $\left[x_{1}\right] \cup\left[x_{2}\right]=-\left[x_{2}\right] \cup\left[x_{1}\right]$. It follows that the isomorphism $\kappa_{X}$ of graded $\mathbb{Z}$-modules between $H^{*}\left(\mathrm{~T}(\boldsymbol{X}), d_{\mathbf{0}}\right)$ and the exterior algebra is a map of graded algebras.
To prove the naturality of the isomorphism $\kappa_{X}$, let $h: \boldsymbol{X} \rightarrow \boldsymbol{Y}$ be a map of sets, let $\bigwedge(h): \bigwedge(\boldsymbol{X}) \rightarrow \bigwedge(\boldsymbol{Y})$ be its extension to exterior algebras, and let $\mathrm{T}(h): \mathrm{T}(\boldsymbol{X}) \rightarrow \mathrm{T}(\boldsymbol{Y})$ be its extension to free binomial graded algebras with cup-one products constructed in Section 6.1. It is readily verified that $\kappa_{Y} \circ \mathrm{~T}(h)=\bigwedge(h) \circ \kappa_{X}$, and this completes the proof.

Corollary 8.7. If $\boldsymbol{X}$ is a set with $n$ elements, then the map $\psi_{X}:\left(\mathrm{T}(\boldsymbol{X}), d_{\mathbf{0}}\right) \rightarrow C^{*}\left(B\left(\mathbb{Z}^{n}\right)\right)$ induces an isomorphism of cohomology rings.

Proof. The proof is by induction. To prove the result in the case $n=1$, let $\boldsymbol{X}=\{x\}$. The morphism $\psi$ in Theorem 7.6 maps $\mathrm{T}(\{x\})$ to $C^{*}\left(\Delta(M(\{x\}) ; \mathbb{Z})=C^{*}(B(\mathbb{Z}) ; \mathbb{Z})\right.$. Note that $C^{1}(B(\mathbb{Z}), \mathbb{Z})$ is the free abelian group of maps of sets from $\mathbb{Z}$ to $\mathbb{Z}$ and $\psi(x)$ is the identity map from $\mathbb{Z}$ to $\mathbb{Z}$. The identity map of $\mathbb{Z}$ is a generator for $H^{1}(\mathbb{Z})$ and it now follows from Proposition 8.6 that the map $H^{*}(\psi)$ is an isomorphism.

For the inductive step, write $\boldsymbol{X}^{n}=\left\{x_{1}, \ldots, x_{n}\right\}$ and $\boldsymbol{X}=\boldsymbol{X}^{n} \cup\left\{x_{n+1}\right\}$, and consider the morphism $\psi=\psi_{X}:\left(\mathrm{T}(\boldsymbol{X}), d_{\mathbf{0}}\right) \rightarrow C^{*}\left(B\left(\mathbb{Z}^{n} \oplus \mathbb{Z}\right) ; \mathbb{Z}\right)$ from Theorem 7.6. Assume by induction that the restriction of $\psi$ to a map $\mathrm{T}\left(\boldsymbol{X}^{n}\right) \rightarrow C^{*}\left(B\left(\mathbb{Z}^{n}\right) ; \mathbb{Z}\right)$ induces an isomorphism on cohomology. By the case $n=1$ above, the restriction of $\psi$ to the map from ( $\left.\mathrm{T}\left(\left\{x_{n+1}\right\}\right), d_{0}\right)$ to $C^{*}(B(\mathbb{Z}) ; \mathbb{Z})$ induces an isomorphism on cohomology. Thus the map of $E_{2}$ terms from the spectral sequence of the Hirsch extension $\mathrm{T}\left(\boldsymbol{X}^{n}\right) \rightarrow \mathrm{T}(\boldsymbol{X})$ to the spectral sequence of the central extension $\mathbb{Z} \rightarrow \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}^{n}$ is an isomorphism. Since both spectral sequences collapse, it follows that $H^{*}(\psi)$ is an isomorphism and the proof is complete.

The next step is to consider the case $R=\mathbb{Z}_{p}$. Recall that for $R=\mathbb{Z}_{2}$, the cohomology ring $H^{*}(B(R), R)=R[x]$ is the polynomial algebra over $R$ on a single generator $x$ in degree 1 , and for $R=\mathbb{Z}_{p}$ with $p$ odd, $H^{*}(B(R) ; R)=\bigwedge(x) \otimes_{R} R[y]$ is the free commutative algebra over $R$ with one generator $x$ in degree 1 and one generator $y$ in degree 2 , with the relation $x^{2}=0$.

Proposition 8.8. For $R=\mathbb{Z}_{p}$, the map $\psi:\left(\mathrm{T}_{R}(\{x\}), d_{0}\right) \rightarrow C^{*}(B(R) ; R)$ induces an isomorphism $\psi^{*}: H^{*}\left(\mathrm{~T}_{R}(\{x\}), d_{\mathbf{0}}\right) \xrightarrow{\simeq} H^{*}(B(R) ; R)$.

Proof. Consider first the case $R=\mathbb{Z}_{2}$. Note that in this case $\mathrm{T}_{R}(\{x\})$ is equal to $\mathbb{Z}_{2}[x]$, the polynomial algebra on a single generator $x$. Hence, the differential $d_{0}$ is identically zero and we have that $H^{*}\left(\mathrm{~T}_{R}(\{x\}), d_{0}\right)=\mathbb{Z}_{2}[x]$. To see that $\psi$ induces an isomorphism on cohomology, note that the 1-chain, [1], in the chain complex of the bar construction on $\mathbb{Z}_{2}$ is a generator of $H_{1}\left(B\left(\mathbb{Z}_{2}\right) ; \mathbb{Z}_{2}\right)$ and $\psi(x)([1])=1 \in \mathbb{Z}_{2}$ so $\psi(x)$ is a generator of $H^{1}\left(B\left(\mathbb{Z}_{2}\right) ; \mathbb{Z}_{2}\right)$. The result follows, since $\psi^{*}$ is an isomorphism in degree 1 and both its source and target are polynomial algebras on a single generator in degree 1 .

Now consider the case $R=\mathbb{Z}_{p}$ with $p \geq 3$. Denoting $\zeta_{i}(x)$ by $\zeta_{i}$, the $R$-vector space $\mathrm{T}=\mathrm{T}_{R}(\{x\})$ has basis consisting of all elements of the form $\zeta_{i_{1}} \otimes \zeta_{i_{2}} \otimes \cdots \otimes \zeta_{i \iota}$, where each $1 \leq i_{j} \leq p-1$. Set $D$ equal to the graded $R$-submodule of $T$ generated by the basis elements $\zeta_{i}$ for $2 \leq i \leq p-1$ and $\zeta_{i_{1}} \otimes \zeta_{i_{2}} \otimes \cdots \otimes \zeta i_{\ell}$ for $\left(i_{1}, i_{2}\right) \neq(1, p-1)$. Clearly, $D$ is closed under the differential $d_{0}$; moreover, the cochain homotopy used over $\mathbb{Z}$ restricts to $D$. Hence, $H^{*}\left(D, d_{0}\right)=0$, and it follows that $H^{*}\left(\mathrm{~T}, d_{0}\right) \cong H^{*}\left(\mathrm{~T} / D, d_{0}\right)$.
Write $x$ and $\zeta_{i}$ for the images of those elements from T in $\mathrm{T} / D$, and note that both $x$ and $\zeta_{1} \otimes \zeta_{p-1}$ are cocycles in $\mathrm{T} / D$. The $\mathbb{Z}_{p}$-algebra $\mathrm{T} / D$ is generated in degree 1 by $[x]$ and in degree 2 by $\left[\zeta_{1} \otimes \zeta_{p-1}\right]$. It follows that $H^{1}\left(\mathrm{~T} / D ; \mathbb{Z}_{p}\right) \cong \mathbb{Z}_{p}$, with generator [ $x$ ], and $H^{2}\left(\mathrm{~T} / D ; \mathbb{Z}_{p}\right) \cong \mathbb{Z}_{p}$, with generator $\left[\zeta_{1} \otimes \zeta_{p-1}\right]$.
Now note that $(\mathrm{T} / D)^{i+2}$ is isomorphic to $\zeta_{1} \otimes \zeta_{p-1} \otimes \mathrm{~T}^{i}$ for $i \geq 1$. The degree-2 map $\mathrm{T}^{*} \rightarrow(\mathrm{~T} / D)^{*+2}$ given by $\alpha \mapsto \zeta_{1} \otimes \zeta_{p-1} \otimes \alpha$ for $\alpha \in \mathrm{T}^{i}$ is an isomorphism of graded $R$-algebras. Since $\zeta_{1} \otimes \zeta_{p-1}$ is a cocycle in T/D, it follows from the graded Leibniz rule that this map commutes with the differentials, and hence induces an isomorphism on cohomology. This gives $H^{i}(\mathrm{~T}) \cong H^{i}(\mathrm{~T} / D) \cong H^{i+2}(\mathrm{~T})$ for $i \geq 1$, and it follows that $H^{i}(\mathrm{~T} / D) \cong \mathbb{Z}_{p}$ for $i \geq 1$. The generators of these groups are

$$
\begin{array}{ll}
\zeta_{1} \otimes \zeta_{p-1} \otimes \zeta_{1} \otimes \zeta_{p-1} \otimes \cdots \otimes \zeta_{1} \otimes \zeta_{p-1} & \text { if } i \text { is even } \\
\zeta_{1} \otimes \zeta_{p-1} \otimes \zeta_{1} \otimes \zeta_{p-1} \otimes \cdots \otimes \zeta_{1} \otimes \zeta_{p-1} \otimes \zeta_{1} & \text { if } i \text { is odd } \tag{8.8}
\end{array}
$$

The next step is to see that $\psi$ induces isomorphisms $\psi^{i}: H^{i}\left(\mathrm{~T}(\{x\}), d_{\mathbf{0}}\right) \rightarrow H^{i}\left(B\left(\mathbb{Z}_{p}\right) ; \mathbb{Z}_{p}\right)$ in degrees $i=1$ and 2 . The 1 -chain [1] is a generator of $H_{1}\left(B\left(\mathbb{Z}_{p}\right) ; \mathbb{Z}_{p}\right)=\mathbb{Z}_{p}$ and $\psi(x)([1])=1$, so $\psi^{1}$ is an isomorphism. Now note that the cocycle $c=\sum_{i=1}^{p} \zeta_{i}(x) \cup \zeta_{k-i}(x)$ in T projects to the cocycle $\zeta_{1} \otimes \zeta_{p-1}$ in T $/ D$. Moreover, the homology class of the 2 -cycle $g=\sum_{i=1}^{p-1}[i \mid 1]$ is a generator of $H_{2}\left(B\left(\mathbb{Z}_{p}\right) ; \mathbb{Z}_{p}\right)$. Since $\psi(c)(g)=1$, we conclude that $\psi^{2}$ is also an isomorphism.

Finally, since $H^{*}\left(B\left(\mathbb{Z}_{p}\right) ; \mathbb{Z}_{p}\right)$ is generated in degrees 1 and 2 and $\psi^{*}$ is an isomorphism in those degrees, it follows that $\psi^{*}$ is an epimorphism. Since $H^{i}\left(\mathrm{~T}(\{x\}), d_{0}\right)$ and $H^{i}\left(B\left(\mathbb{Z}_{p}\right) ; \mathbb{Z}_{p}\right)$ are both isomorphic to $\mathbb{Z}_{p}$ for each $i \geq 0$, it follows that $\psi^{*}$ is an isomorphism, and the proof is complete.

The next theorem is a synthesis of the preceding results.

Theorem 8.9. If $\boldsymbol{X}$ is a finite set with $n$ elements and if $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, then the dga morphism $\psi_{X, R}:\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{\mathbf{0}}\right) \rightarrow C^{*}\left(B\left(R^{n}\right) ; R\right)$ induces an isomorphism of cohomology rings.

Proof. For $R=\mathbb{Z}$ the result is Corollary 8.7. For $R=\mathbb{Z}_{p}$ and $n=1$, the result is Proposition 8.8. For $n>1$, the result follows by induction on $n$, using the property that the spectral sequence of the Hirsch extension $\left(\mathrm{T}_{R}(\boldsymbol{X}), d_{\mathbf{0}}\right) \rightarrow\left(\mathrm{T}_{R}(\boldsymbol{X} \cup\{y\}), d_{\mathbf{0}}\right)$ collapses with $E_{2}=E_{\infty}$.
8.4. Colimits of Hirsch extensions. We now consider a type of free binomial $\cup_{1}$-dgas that arise as unions (or, colimits) of certain sequences of Hirsch extensions. These objects will play an important role for the rest of this paper.
Definition 8.10. A free binomial $\cup_{1}$-dga $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ is called a colimit of Hirsch extensions if the following conditions hold.
(1) $X=\bigcup_{i \geq 1} X_{i}$ with each set $X_{i}$ finite.
(2) For $\boldsymbol{X}^{n}=\bigcup_{i=1}^{n} \boldsymbol{X}_{i}$ and $n \geq 1$, the differential $d$ on $\mathrm{T}_{R}(\boldsymbol{X})$ restricts to a differential $d_{n}$ on $\mathrm{T}_{R}\left(X^{n}\right)$.
(3) Each map $i_{n}:\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right), d_{n}\right) \rightarrow\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n+1}\right), d_{n+1}\right)$ is a Hirsch extension.
(4) $\boldsymbol{X}_{1} \neq \emptyset$, and $d_{1}(x)=0$ for all $x \in \boldsymbol{X}_{1}$.

A morphism of colimits of Hirsch extensions is a map of binomial $\cup_{1}$-dgas as above, $f:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{\prime}\right), d^{\prime}\right)$, with the property that for each $n \geq 1$, the map restricts to a morphism $f_{n}: \mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right) \rightarrow \mathrm{T}_{R}\left(\boldsymbol{X}^{\prime n}\right)$. Note that these morphisms are compatible with the respective colimits; that is, the diagram below commutes for each $n \geq 1$.

8.5. The group associated to a colimit of Hirsch extensions. We now associate in a functorial way to each colimit of Hirsch extensions a pronilpotent group.

Lemma 8.11. Let $\mathrm{T}=\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ be a colimit of Hirsch extensions.
(1) There is a pronilpotent group $G_{\top}$ and a $\cup_{1}$-dga map $\psi_{\top}: \mathrm{T} \rightarrow C^{*}\left(B\left(G_{\top}\right) ; R\right)$.
(2) If $\boldsymbol{X}$ is finite, then $G_{\mathrm{T}}$ is a nilpotent group and $\psi_{\mathrm{T}}$ is a quasi-isomorphism. Moreover, if $R=\mathbb{Z}$, then $G_{\top}$ is torsion-free.
(3) Every morphism of colimits of Hirsch extensions, $f: \mathrm{T} \rightarrow \mathrm{T}$, induces (in a functorial way) a group homomorphism, $\tilde{f}: G_{\top^{\prime}} \rightarrow G_{\top}$.

Proof. (1) We start by defining for each $n \geq 1$ a finitely generated nilpotent group, $G_{n}=G_{\mathrm{T}_{n}}$, corresponding to the free $\cup_{1}$-dgas $\mathrm{T}_{n}=\left(\mathrm{T}_{R}\left(X^{n}\right), d_{n}\right)$, as well as a $\cup_{1}$-dga map $\psi_{n}: \mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right) \rightarrow C^{*}\left(B\left(G_{n}\right) ; R\right)$ inducing an isomorphism on cohomology. This is done inductively, as follows.
First let $G_{1}=M\left(\boldsymbol{X}_{1}, R\right)$. As noted in Section 7.1, this is a free $R$-module with basis $\boldsymbol{X}_{1}$; we view it now as a finitely generated abelian group. By Theorem 8.9, there is a quasi-isomorphism $\psi_{1}:\left(\mathrm{T}_{R}\left(\boldsymbol{X}_{1}\right), d_{\mathbf{0}}\right) \rightarrow C^{*}\left(B\left(G_{1}\right) ; R\right)$. Assume now that a finitely generated nilpotent group $G_{n}$ has been constructed, together with a $\cup_{1}$-dga quasi-isomorphism $\psi_{n}:\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right), d_{n}\right) \rightarrow C^{*}\left(B\left(G_{n}\right) ; R\right)$ inducing an isomorphism on $H^{1}$. By Theorem 8.2, the differential $d_{n+1}$ on $\mathrm{T}\left(\boldsymbol{X}_{n+1}\right)$ restricts to an admissible map $\tau_{n}: \boldsymbol{X}_{n+1} \rightarrow Z\left(\mathrm{~T}_{R}^{2}\left(\boldsymbol{X}^{n}\right)\right)$. The composition $\psi_{n} \circ \tau_{n}$, then, defines a cocycle in $Z^{2}\left(B\left(G_{n}\right) ; M\left(\boldsymbol{X}_{n+1}, R\right)\right)$; let

$$
\begin{equation*}
0 \longrightarrow M\left(\boldsymbol{X}_{n+1}, R\right) \longrightarrow G_{n+1} \xrightarrow{q_{n}} G_{n} \longrightarrow 1 \tag{8.10}
\end{equation*}
$$

be the corresponding central extension. Since $G_{n}$ is a group, Lemma 2.4 insures that $G_{n+1}$ is also a group; by construction, this is again a finitely generated nilpotent group (torsion-free if $R=\mathbb{Z}$ ). Since the map $\tau_{n}$ is admissible, Theorem 8.2 insures that the map $\tau_{n+1}=\left.d_{n+1}\right|_{n+1}$ is also admissible. Since $\boldsymbol{X}_{n+1}$ is finite, the Hirsch extension $\mathrm{T}_{R}\left(\boldsymbol{X}_{n}\right) \hookrightarrow$ $\mathrm{T}_{R}\left(\boldsymbol{X}_{n+1}\right)$ can be realized as a sequence of elementary Hirsch extensions. The inductive assumption together with Lemma 8.5 and Theorem 8.9 then show that $\psi_{n+1}$ is a quasiisomorphism. This completes the construction of the $G_{n}$ and the argument that $\psi_{n}$ is a quasi-isomorphism.
We now let $G_{\top}=\lim _{\leftrightarrows} G_{n}$ be the limit of the inverse system of groups $\left\{G_{n}, q_{n}\right\}_{n \geq 1}$ and $\psi_{\mathrm{T}}: \mathrm{T} \rightarrow C^{*}\left(B\left(G_{\mathrm{T}}\right) ; R\right)$ be the colimit of the directed system of maps $\psi_{n}:\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right), d_{n}\right) \rightarrow$ $C^{*}\left(B\left(G_{n}\right) ; R\right)$. By construction, both $G_{\top}$ and $\psi_{\top}$ satisfy the claimed properties. Note that the underlying magma of $G_{\top}$ is $\left(M(\boldsymbol{X}, R), \mu_{\tau}\right)$, where $\tau: X \rightarrow Z\left(\mathrm{~T}_{R}^{2}(\boldsymbol{X})\right)$ is the colimit of the maps $\tau_{n}$, while $\psi_{\mathrm{T}}$ coincides with the map $\psi_{\boldsymbol{X}}: \mathrm{T}(\boldsymbol{X}) \rightarrow C^{*}\left(\Delta\left(M_{\tau}\right) ; R\right)$.
(2) If $\boldsymbol{X}$ is finite, then $\boldsymbol{X}=\boldsymbol{X}^{n}$ for some $n \geq 1$, and the claimed properties for $G_{\top}$ and $\psi_{\top}$ follow from the above proof.
(3) Let $f: \mathbf{T} \rightarrow \mathbf{T}^{\prime}$ be a morphism of colimits Hirsch extensions, so that, for each $n \geq 1$, the diagram (8.9) commutes and $d_{n}^{\prime} \circ f_{n}=f_{n+1} \circ d_{n}$. Setting $\tilde{i}_{n}:=q_{n}$, we define inductively homomorphisms $\tilde{f}_{n}: G_{n}^{\prime} \rightarrow G_{n}$ which satisfy $\tilde{f}_{n} \circ \tilde{i}_{n}=\tilde{i}_{n} \circ \tilde{f}_{n+1}$, as follows. We first let $\tilde{f}_{1}: G_{1}^{\prime} \rightarrow G_{1}$ be equal to the $R$-linear map $f_{1}^{\vee}: M\left(\boldsymbol{X}_{1}^{\prime}, R\right) \rightarrow M\left(\boldsymbol{X}_{1}, R\right)$ from Section 7.1. Assuming that $\tilde{f}_{n}$ has been defined, we let $\left(f_{n+1} \mid X_{n+1}\right)^{\vee}: M\left(X_{n+1}^{\prime} ; R\right) \rightarrow M\left(\boldsymbol{X}_{n+1} ; R\right)$ be equal to the $\operatorname{Hom}(-, R)$-dual of the set map $\left.f_{n+1}\right|_{X_{n+1}}: \boldsymbol{X}_{n+1} \rightarrow \boldsymbol{X}_{n+1}^{\prime}$. The fact that $f_{n}$ and $f_{n+1}$ are compatible dga-maps implies that the homomorphisms $\tilde{f}_{n}$ and $\left(\left.f_{n+1}\right|_{X_{n+1}}\right)^{\vee}$ are compatible with the $k$-invariants of the extensions (8.10), and thus define a homomorphism $\tilde{f}_{n+1}: G_{n+1}^{\prime} \rightarrow G_{n+1}$ with the claimed property. Passing to the limit yields a homomorphism $\tilde{f}: G_{\mathrm{T}^{\prime}} \rightarrow G_{\mathrm{T}}$. Finally, if $g: \mathrm{T}^{\prime} \rightarrow \mathrm{T}^{\prime \prime}$ is another morphism of colimits of Hirsch extensions, it is readily verified that $\tilde{f} \circ \tilde{g}=\widetilde{g \circ f}$.

The next lemma describes another type of functoriality property of the above construction, related to maps between classifying spaces of pronilpotent groups.

Theorem 8.12. Let $\mathrm{T}=\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ be a colimit of Hirsch extensions, let $G=G_{\mathrm{T}}$ be the corresponding pronilpotent group, and assume the map $\psi_{\top}: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow C^{*}(B G ; R)$ is a quasi-isomorphism. Moreover, let $\pi: B \bar{G} \rightarrow B G$ be the fibration corresponding to $a$ central extension of groups, $0 \rightarrow F \rightarrow \bar{G} \rightarrow G \rightarrow 1$, with $F$ a finitely generated, free $R$-module. Then
(1) There is a Hirsch extension $i: \mathrm{T} \hookrightarrow \overline{\mathrm{T}}=\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$ such that $\bar{G}=G_{\bar{T}}$.
(2) The diagram below commutes

(3) The map $\psi_{\bar{\top}}$ is a quasi-isomorphism.

Proof. It suffices to prove the case where $F=R$. Let $c \in \mathrm{~T}_{R}^{2}(\boldsymbol{X})$ be a cocycle such that the cohomology class of $\psi_{X}(c)$ in $H^{2}(B G ; R)$ corresponds to the central extension. Set $\overline{\mathrm{T}}=\left(\mathrm{T}_{R}(\boldsymbol{X} \cup\{y\}), \bar{d}\right)$ equal to the Hirsch extension with $\bar{d}(y)=c$ and set $\bar{G}=G_{\overline{\mathrm{T}}}$. This gives the commutative diagram (8.11).

The map $\psi_{\overline{\mathrm{T}}}$ induces a map of spectral sequences from the spectral sequence of the Hirsch extension to the spectral sequence of the fibration. The map of the terms $E_{2}^{p, q}$ is

$$
\begin{equation*}
H^{p}\left(\mathrm{~T}_{R}(\boldsymbol{X}) \otimes H^{q}\left(\mathrm{~T}_{R}(\{y\})\right) \xrightarrow{\psi_{T} \otimes f} H^{p}(B G ; R) \otimes H^{q}(B R ; R),\right. \tag{8.12}
\end{equation*}
$$

where $H^{q}(\mathrm{~T}(\{y\})$ denotes the cohomology computed with $d y=0$ and $f$ denotes the map on cohomology induced by the map $\psi_{\mathrm{T}(\mid y)}: \mathrm{T}_{R}(\{y\}) \rightarrow C^{*}(B R ; R)$. The map $\psi_{\mathrm{T}}$ is an isomorphism by assumption, while $f$ has been shown to be an isomorphism in Theorem 8.9. Hence, the map of $E_{2}$ terms is an isomorphism and it follows that $\psi_{\overline{\bar{T}}}$ is a quasiisomorphism.

## 9. The existence of 1-minimal models

In this section, we define 1-minimal models and show that every binomial $\cup_{1}$-dga $A$ over the ring $R=\mathbb{Z}$ or $R=\mathbb{Z}_{p}$ admits a 1-minimal model, provided that $H^{0}(A)=R$ and $H^{1}(A)$ is a finitely generated, free $R$-module.
9.1. Solving an extension problem. We start by setting up an extension problem in the category of binomial $\cup_{1}$-dgas and give an if and only if criterion to solve it.

Definition 9.1. Let $f:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(A, d_{A}\right)$ be a morphism of binomial $\cup_{1}$-dgas, and let $i: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y})$ be a Hirsch extension of $\mathrm{T}_{R}(\boldsymbol{X})$. A morphism $\bar{f}: \mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}) \rightarrow A$ is an extension of $f$ if the following diagram commutes.


Theorem 9.2. Given a morphism of binomial $\cup_{1}$-dgas, $f:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(A, d_{A}\right)$, and a Hirsch extension, $i:\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \rightarrow\left(\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}), \bar{d}\right)$, there is an extension $\bar{f}$ of $f$ if and only if $[f(d y)]=0$ for all $y \in \boldsymbol{Y}$. Moreover, if there is an extension of $f$, then there is a bijection between the set of extensions and functions $\sigma: Y \rightarrow A^{1}$ with $d_{A}(\sigma(y))=f(d y)$.

Proof. Given an extension $\bar{f}: \mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}) \rightarrow A$, the corresponding map $\sigma: \boldsymbol{Y} \rightarrow A^{1}$ is given by $\sigma(y)=\bar{f}(y)$ for $y \in \boldsymbol{Y}$. The condition that $d_{A}(\sigma(y))=f(d y)$ follows from the assumption that $\bar{f}$ is a map of dgas.
In the opposite direction, assume $\sigma: \boldsymbol{Y} \rightarrow A^{1}$ is a map of sets with $d_{A}(\sigma(y))=f(d y)$. Then by Lemma 6.2, the function $y \mapsto \sigma(y)$ extends $f$ uniquely to a map $\bar{f}$ of binomial cup-one algebras from $\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y})$ to $A$. Since $d_{A}(\sigma(y))=f(d y)$, it then follows from Theorem 7.8 that this extension commutes with the differentials on $\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y})$ and on $A$, and the proof is complete.
Lemma 9.3. With notation as above, let $\bar{f}: \mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y}) \rightarrow$ A be an extension of $f: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow$ A with $Y=\{y\}$, where $\bar{d}$ denotes the differential on $\mathrm{T}_{R}(\boldsymbol{X} \cup \boldsymbol{Y})$ and d denotes its restriction to $\mathrm{T}_{R}(\boldsymbol{X})$. Assume both $H^{1}(A)$ and $\operatorname{ker}\left(H^{2}(f)\right)$ are finitely generated, free $R$-modules, and that the cohomology classes of the cocycles $d y, c_{1}, c_{2}, \ldots, c_{\ell}$ in $\mathrm{T}_{R}(\boldsymbol{X})$ form a basis for $\operatorname{ker}\left(H^{2}(f)\right)$. Then,
(1) The inclusion $i: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow \mathrm{T}_{R}(\boldsymbol{X} \cup\{y\})$ induces an isomorphism on $H^{1}$.
(2) The set $\left\{\left[i\left(c_{1}\right)\right], \ldots,\left[i\left(c_{\ell}\right)\right]\right\}$ is a basis for $\mathrm{im}\left(H^{2}(i)\right) \cap \operatorname{ker}\left(H^{2}(\bar{f})\right)$.
(3) The kernel of $H^{2}(\bar{f})$ is a finitely generated, free $R$-module.

Proof. Consider the spectral sequence from Lemma 8.5 associated with the elementary Hirsch extension $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right) \hookrightarrow\left(\mathrm{T}_{R}(\boldsymbol{X} \cup\{y\}), \bar{d}\right)$. We then have $E_{2}^{0,1}=R$ with generator $y$, and $\bar{d} y \in \operatorname{ker}\left(H^{2}(f)\right)$. Hence, by assumption, $n d y \neq 0$ for all $n \in R, n \neq 0$. It follows that $E_{3}^{0,1}=0$. Therefore, the induced homomorphism $H^{1}(f): H^{1}\left(\mathrm{~T}_{R}(\boldsymbol{X}) \rightarrow H^{1}\left(\mathrm{~T}_{R}(\boldsymbol{X} \cup\{y\})\right)\right.$ is an isomorphism, and the proof of claim (1) is complete.
To prove claim (2) in the case $R=\mathbb{Z}_{p}$, note that since $\mathbb{Z}_{p}$ is a field, we can write $H^{2}\left(\mathrm{~T}_{R}(\boldsymbol{X})\right)$ as a direct sum $\operatorname{span}\left([d y],\left[c_{1}\right], \ldots,\left[c_{\ell}\right]\right) \oplus \bar{B}$, with $H^{2}(f)$ restricted to $\bar{B}$ a
monomorphism. Then $\operatorname{im}\left(H^{2}(i)\right)=E_{\infty}^{2,0}=\operatorname{span}\left(\left[c_{1}\right], \ldots,\left[c_{\ell}\right]\right) \oplus \bar{B}$, and the result follows since $H^{2}(\bar{f}) \circ H^{2}(i)=H^{2}(f)$.
To prove claim (2) in the case $R=\mathbb{Z}$, note that since $E_{2}^{0,2}=0$, the terms $E_{\infty}^{2,0}$ and $E_{\infty}^{1,1}$ give an exact sequence,

$$
\begin{equation*}
0 \longrightarrow E_{\infty}^{2,0}=H^{2}\left(\mathrm{~T}_{R}(\boldsymbol{X})\right) /[\bar{d} y] \longrightarrow H^{2}\left(\mathrm{~T}_{R}(\boldsymbol{X} \cup\{y\})\right) \longrightarrow E_{\infty}^{1.1} \longrightarrow 0, \tag{9.2}
\end{equation*}
$$

and claim (2) follows at once.
For $R=\mathbb{Z}_{p}$, claim (3) follows since in this case every submodule of a finitely generated, free $R$-module is a finitely generated, free $R$-module.
To prove claim (3) in the case $R=\mathbb{Z}$, let $\left\{[d y],\left[c_{1}\right], \ldots,\left[c_{\ell}\right]\right\}$ be a basis for $\operatorname{ker}\left(H^{2}(f)\right)$, and let $\left[c_{2}, \ldots, c_{\ell}\right]$ denote the submodule of $H^{2}\left(\mathrm{~T}_{R}(X \cup\{y\})\right)$ generated by the elements $\left[i\left(c_{2}\right)\right], \ldots,\left[i\left(c_{\ell}\right)\right]$. Then since $E_{\infty}^{1,1}$ is finitely generated and torsion-free, the sequence (9.2) is split exact and the kernel of the map from $H^{2}\left(\mathrm{~T}_{R}(\boldsymbol{X} \cup\{y\})\right) /\left[c_{2}, \ldots, c_{\ell}\right]$ to $H^{2}(A)$ is the submodule $K$ of $E_{\infty}^{1,1}$ consisting of all elements $k$ for which there is an element $\alpha \in$ $\operatorname{im}\left(H^{2}(i)\right) /\left[c_{2}, \ldots, c_{\ell}\right]$ with $H^{2}(\bar{f})(k+\alpha)=0$. It follows that $\operatorname{ker}\left(H^{2}(\bar{f})\right) \cong\left[c_{2}, \ldots, c_{\ell}\right] \oplus K$. This establishes claim (3) in the case $R=\mathbb{Z}$, and the proof is complete.
9.2. A lifting criterion. The next theorem corresponds to an analogous rational homotopy result from [6] (Lemma 12.4).

Theorem 9.4. Let $\left(A, d_{A}\right)$ and $\left(A^{\prime}, d_{A^{\prime}}\right)$ be binomial cup-one $R$-dgas over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, let $f: A \rightarrow A^{\prime}$ be a surjective 1-quasi-isomorphism, and let $\varphi$ be a morphism from a colimit of Hirsch extensions, $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$, to $\left(A^{\prime}, d_{A^{\prime}}\right)$. There is then a lift of $\varphi$ through $f$; that is, a morphism $\widehat{\varphi}: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A$ such that the following diagram commutes


Proof. As in Definition 8.10, let $\left\{\left(\mathrm{T}_{R}\left(X^{n}\right), d_{n}\right)\right\}_{n \geq 1}$ be the sequence of binomial $\cup_{1}$-dgas whose colimit is $\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$. Set $\varphi_{n}: \mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right) \rightarrow A^{\prime}$ equal to the restriction of $\varphi$ to $\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right)$. It suffices to show that for each $n \geq 1$, there is a lift $\widehat{\varphi}_{n}$ of $\varphi_{n}$ through $f$.

The argument proceeds by induction. For $n=1$, we have that $d_{1}(x)=0$ for all $x \in \boldsymbol{X}_{1}$. Thus, by Lemmas 6.2 and 6.3 it suffices to show that for each $x \in X_{1}$ there is a cocycle $a_{x}$ in $A$ with $f\left(a_{x}\right)=\varphi(x)$. To do this, given $x \in X_{1}$, let $b_{x}$ be an element in $A^{1}$ with $f\left(b_{x}\right)=$ $\varphi(x)$. Then $d_{A}\left(b_{x}\right)$ is a cocycle in $\operatorname{ker}(f)$. By Remark 3.3, we have that $H^{2}(\operatorname{ker}(f))=0$. Hence, there is an element $c_{x} \in \operatorname{ker}(f)$ with $d_{A}\left(c_{x}\right)=d_{A}\left(b_{x}\right)$. Then $a_{x}=b_{x}-c_{x}$ is a cocycle in $A$ with $f\left(a_{x}\right)=\varphi(x)$. This completes the argument for the case $n=1$.

Now assume there is a lifting $\widehat{\varphi}_{n}$ of $\varphi_{n}$ through $f$. In order to show that $\widehat{\varphi}_{n}$ extends to a lifting $\widehat{\varphi}_{n+1}$ through $f$, it suffices by Lemmas 6.2 and 6.3 to show that for each $x \in \boldsymbol{X}_{n+1}$ there is an element $a_{x}$ in $A$ with $f\left(a_{x}\right)=\varphi(x)$ and $\widehat{\varphi}_{n}(d x)=d_{A}\left(a_{x}\right)$. Given $x \in \boldsymbol{X}_{n+1}$, let $b_{x}$ be an element in $A$ with $f\left(b_{x}\right)=\varphi(x)$. Then

$$
\begin{aligned}
f\left(\widehat{\varphi}_{n}(d x)-d_{A}\left(b_{x}\right)\right) & =\varphi(d x)-f\left(d_{A}\left(b_{x}\right)\right) \\
& =\varphi(d x)-d_{A^{\prime}}\left(f\left(b_{x}\right)\right) \\
& =\varphi(d x)-d_{A^{\prime}}(\varphi(x)) \\
& =0,
\end{aligned}
$$

and so $\widehat{\varphi}_{n}(d x)-d_{A}\left(b_{x}\right) \in \operatorname{ker}(f)$. We have that $\widehat{\varphi}_{n}(d x)-d_{A}\left(b_{x}\right)$ is a cocycle in $\operatorname{ker}(f)$ and $H^{2}(\operatorname{ker}(f))=0$; therefore, there is an element $c_{x} \in \operatorname{ker}(f)$ with $d_{A}\left(c_{x}\right)=\widehat{\varphi}_{n}(d x)-d_{A}\left(b_{x}\right)$. Setting $a_{x}=b_{x}-c_{x}$, we have that $f\left(a_{x}\right)=\varphi(x)$ and $\widehat{\varphi}_{n}(d x)=d_{A}\left(a_{x}\right)$, and the argument is complete.
9.3. 1-minimal models. Colimits of Hirsch extensions lead to the notion of 1-minimal model, which is central to the study done in this paper. Let $\left(A, d_{A}\right)$ be a binomial $\cup_{1}$-dga over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ such that $H^{0}(A)=R$ and $H^{1}(A)$ is a finitely generated, free $R$-module.

Definition 9.5. A 1-minimal model for $A$ is a free binomial $\cup_{1}$-dga $\mathcal{M}=\left(\mathrm{T}_{R}(\boldsymbol{X}), d\right)$ which arises as the colimit of a sequence of Hirsch extensions, $\mathcal{M}_{n}=\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right), d_{n}\right)$, together with morphisms $\rho_{n}: \mathcal{M}_{n} \rightarrow A$ such that, for each $n \geq 1$, the diagram below,

is a commutative diagram of binomial $\cup_{1}$-dgas and the following conditions are satisfied:
(1) The maps $H^{i}\left(\rho_{1}\right): H^{i}\left(\mathcal{M}_{1}\right) \rightarrow H^{i}(A)$ are isomorphisms for $i=0$ and $i=1$.
(2) The submodule $\operatorname{ker}\left(H^{2}\left(\rho_{n}\right)\right) \subset H^{2}\left(\mathcal{M}_{n}\right)$ is a free $R$-module with basis given by the cohomology classes of the cocycles $\left\{d_{n+1}(x) \mid x \in \boldsymbol{X}_{n+1}\right\} \subset Z^{2}\left(\mathrm{~T}_{R}\left(\boldsymbol{X}^{n}\right)\right)$.

Set $\mathcal{M}:=\bigcup_{n} \mathcal{M}_{n}$. Since all diagrams of type (9.3) commute, there is a morphism of $\cup_{1}$-dgas, $\rho:(\mathcal{M}, d) \rightarrow\left(A, d_{A}\right)$, whose restriction to $\mathcal{M}_{n}$ coincides with $\rho_{n}$ for all $n \geq 1$. We will oftentimes refer to $\rho: \mathcal{M} \rightarrow A$, or simply to $\mathcal{M}$ as being a 1-minimal model for $A$; when needed, we will refer to the map $\rho: \mathcal{M} \rightarrow A$ as the structural morphism for $\mathcal{M}$.
From the definition of colimit of Hirsch extensions we have that $\mathcal{M}_{1}=\left(\mathrm{T}_{R}\left(\boldsymbol{X}_{1}\right), d_{\mathbf{0}}\right)$. Also note that from part (2) of Theorem 8.2 it follows that the map $\left.d\right|_{X^{n}}$ is admissible for each $n \geq 1$.

Lemma 9.6. Let $\mathcal{M}=\bigcup_{n \geq 1} \mathcal{M}_{n}$ be a 1-minimal model for $A$. Then, for all $n \geq 1$,
(1) $Z^{1}\left(\mathcal{M}_{n}\right)=H^{1}\left(\mathcal{M}_{n}\right)$.
(2) The inclusion $\mathcal{M}_{1} \hookrightarrow \mathcal{M}_{n}$ induces an isomorphism, $H^{1}\left(\mathcal{M}_{1}\right) \xrightarrow{\simeq} H^{1}\left(\mathcal{M}_{n}\right)$.

Proof. Part (1) follows since $d:\left(\mathcal{M}_{n}\right)^{0} \rightarrow\left(\mathcal{M}_{n}\right)^{1}$ is the zero map for all $n$. Part (2) follows from Lemma 9.3, part (1).

The next corollary follows at once from the lemma.
Corollary 9.7. If $\mathcal{M}$ is a 1 -minimal model for $A$, then, for all $n \geq 1$,
(1) $H^{1}\left(\mathcal{M}_{n}\right) \cong M\left(X_{1}, R\right)$, the free $R$-module with basis given by the elements in $\boldsymbol{X}_{1}$.
(2) $Z^{1}\left(\mathcal{M}_{n+1}\right)=Z^{1}\left(\mathcal{M}_{n}\right)$.
9.4. Existence of 1 -minimal models. We are now in a position to state and prove the main result of this section.

Theorem 9.8. Let $A$ be a binomial $\cup_{1}$-dga over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, with $H^{0}(A)=R$ and $H^{1}(A)$ a finitely-generated, free $R$-module. There is then a 1-minimal model, $\mathcal{M}$, and a structural morphism, $\rho: \mathcal{M} \rightarrow A$, which is a 1-quasi-isomorphism.

Proof. Since $\mathcal{M}$ is connected, we can define $\rho^{0}: \mathcal{M}^{0} \rightarrow A^{0}$ to be the composition of the inverse of the structure map from $R$ to $\mathcal{M}^{0}$ followed by the structure map for $A$.
Now let $u_{1}, \ldots, u_{k}$ be cocycles in $A^{1}$ whose cohomology classes give a basis for $H^{1}(A)$. Let $\boldsymbol{X}_{1}=\left\{x_{1}, \ldots, x_{k}\right\}$ and set $\mathcal{M}_{1}=\left(\mathrm{T}_{R}\left(\boldsymbol{X}_{1}\right), d_{\tau_{1}}\right)$, where $\tau_{1}=0$, that is, $d_{\tau_{1}}\left(x_{i}\right)=0$ for all $i$. In view of Corollary 7.9, we may define a morphism $\rho_{1}: \mathcal{M}_{1} \rightarrow A$ by setting $\rho_{1}\left(x_{i}\right)=u_{i}$ for $1 \leq i \leq k$ such that the induced map on $H^{0}$ is an isomorphism. By construction, the map $H^{1}\left(\rho_{1}\right)$ is also an isomorphism.
Assume by induction that an extension $\left(\mathcal{M}_{n}, d_{\tau_{n}}\right)=\left(\mathrm{T}_{R}\left(\boldsymbol{X}_{1} \cup \cdots \cup \boldsymbol{X}_{n}\right), d_{\tau_{n}}\right)$ has been constructed, along with a map $\rho_{n}: \mathcal{M}_{n} \rightarrow A$ inducing isomorphisms on $H^{0}$ and $H^{1}$ and such that the kernel of $\rho_{n}$ is a finitely generated, free $R$-module. Then by repeated applications of Corollary 7.9, Theorem 8.2, and Lemma 9.3, it follows that there is a finite set $\boldsymbol{X}_{n+1}$, an extension $\mathcal{M}_{n+1}=\mathrm{T}_{R}\left(\boldsymbol{X}_{1} \cup \cdots \cup \boldsymbol{X}_{n+1}\right)$ with differential $d_{\tau_{n+1}}$, and an extension $\rho_{n+1}$ of $\rho_{n}$ such that $\rho_{n+1}$ induces isomorphisms on $H^{0}$ and $H^{1}$, the kernel of $H^{2}\left(\rho_{n+1}\right)$ is a finitely generated, free $R$-module, and the restriction of $H^{2}\left(\rho_{n+1}\right)$ to the image of $\mathcal{M}_{n}$ in $\mathcal{M}_{n+1}$ is a monomorphism.
If for some $n$ the map $H^{2}\left(\rho_{n}\right)$ is a monomorphism, then set $\mathcal{M}=\mathcal{M}_{n}$; if not, then set $\mathcal{M}=$ $\bigcup_{n \geq 1} \mathcal{M}_{n}$. It then follows that $\mathcal{M}$ is a 1-minimal model for $A$. Its structural morphism, $\rho: \mathcal{M} \rightarrow A$, is defined to be the direct limit of the morphisms $\rho_{n}: \mathcal{M}_{n} \rightarrow A$; that is, $\left.\rho\right|_{\mathcal{M}_{n}}=\rho_{n}$, for all $n \geq 1$. By construction, the map $H^{i}(\rho): H^{i}(\mathcal{M}) \rightarrow H^{i}(A)$ is an isomorphism for $i=0$ and 1 and a monomorphism for $i=2$. Therefore, $\rho$ is a 1 -quasiisomorphism, and the proof is complete.

The theorem has an immediate corollary in the case when $A$ is a cochain algebra of a space.

Corollary 9.9. Let $X$ be a connected $\Delta$-complex, and assume $H^{1}(X ; R)$ is a finitely generated module over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. There is then a 1-minimal model, $\rho: \mathcal{M} \rightarrow A$, for the cochain algebra $A=C^{*}(X ; R)$.
9.5. Augmented 1-minimal models. When the binomial $\cup_{1}$-dga $A$ admits an augmentation, the above theorem can be enhanced, accordingly.
Theorem 9.10. Let $A$ be binomial $\cup_{1}$-dga such that there is an augmentation $\varepsilon_{A}: A \rightarrow R$ which induces an isomorphism from $H^{0}(A)$ to $R$ and such that $H^{1}(A)$ is a finitely generated, free $R$-module. There is then an augmented 1-minimal model, $\mathcal{M}$, such that the structural morphism, $\rho: \mathcal{M} \rightarrow A$, is an augmentation-preserving 1-quasi-isomorphism.

Proof. By Theorem 9.8, the binomial $\cup_{1}$-dga $A$ has a 1-minimal model, $\rho: \mathcal{M} \rightarrow A$, which is a 1-quasi-isomorphism. Since the tensor algebra $\mathcal{M}=\mathrm{T}_{R}(\boldsymbol{X})$ is connected, it admits a canonical augmentation, $\varepsilon_{\mathcal{M}}: \mathcal{M} \rightarrow R$, which sends $\mathcal{M}^{>0}$ to 0 and identifies $\mathcal{M}^{0}$ with $R$.

Since both $\varepsilon_{A}$ and $\rho$ are dga maps, their composite, $\varepsilon_{A} \circ \rho: \mathcal{M} \rightarrow R$ is again a dga map. Owing to our hypothesis on $\varepsilon_{A}$, the map from $R=H^{0}(\mathcal{M})$ to $R$ induced by the composition is an isomorphism of rings from $R$ to $R$ and so equals the identity of $R$. It follows that $\varepsilon_{A} \circ \rho$ is an augmentation for $\mathcal{M}$. By the uniqueness of augmentation maps for connected dgas, we have that $\varepsilon_{A} \circ \rho=\varepsilon_{\mathcal{M}}$, and the proof is complete.

Recall from Section 3.4 that a choice of basepoint $x_{0}$ for a space $X$ yields an augmentation map, $\varepsilon_{0}: C^{*}(X ; R) \rightarrow R$.
Corollary 9.11. Let $\left(X, x_{0}\right)$ be a connected, pointed $\Delta$-complex, and assume $H^{1}(X ; R)$ is a finitely generated module over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. There is then an augmented 1-minimal model, $\mathcal{M}$, for the cochain algebra $C^{*}(X ; R)$ and a structural morphism, $\rho: \mathcal{M} \rightarrow A$, which is a 1-quasi-isomorphism preserving augmentations, that is, $\varepsilon_{0} \circ \rho=\varepsilon_{\mathcal{M}}$.
Example 9.12. In this example we find the (integral) 1-minimal model for the cochain algebra $A=C^{*}(B(G(k)) ; \mathbb{Z})$, where $G(k)$ is the Heisenberg group of upper triangular matrices of the form

$$
\left(\begin{array}{ccc}
1 & a_{1} & a_{1,2} / k  \tag{9.4}\\
0 & 1 & a_{2} \\
0 & 0 & 1
\end{array}\right)
$$

with $a_{1}, a_{2}, a_{1,2} \in \mathbb{Z}$ and $k$ is a fixed positive integer. The multiplication in $G(k)$ has the form

$$
\begin{equation*}
\left(a_{1}, a_{2}, a_{1,2}\right) \cdot\left(a_{1}^{\prime}, a_{2}^{\prime}, a_{1,2}^{\prime}\right)=\left(a_{1}+a_{1}^{\prime}, a_{2}+a_{2}^{\prime}, a_{1,2}+a_{1,2}^{\prime}+k a_{1} a_{2}\right) . \tag{9.5}
\end{equation*}
$$

Hence, $G(k)$ has presentation with generators $g_{1}, g_{2}, g_{1,2}$ and relators $\left[g_{1}, g_{2}\right] g_{1,2}^{-k},\left[g_{1}, g_{1,2}\right]$, and $\left[g_{2}, g_{1,2}\right.$ ]. Let $X(k)$ be the corresponding CW-complex, with 1 -skeleton equal to a wedge of circles, one circle for each generator, and with 2-cells given by the relators. Since a classifying space $B(G(k))$ can be constructed from $X(k)$ by adding cells of dimension 3 or more, it follows that the corresponding inclusion of $X(k)$ into $B(G(k))$ induces a 1-quasi-isomorphism of cochain complexes, and hence, it suffices to find the 1-minimal model for $C^{*}(X(k) ; \mathbb{Z})$.

Let $c_{1}, c_{2}, c_{1,2}$ be 1 -cochains in $X(k)$ whose restrictions to the cochains on the 1 -skeleton are dual to the corresponding generators $g_{1}, g_{2}$, and $g_{1,2}$, with $c_{1}$ and $c_{2}$ cocycles, $d c_{1,2}=$ $-k c_{1} \cup c_{2}$, the restriction of $c_{2}$ to the 2-cell [ $g_{1}, g_{1,2}$ ] is zero, and the restriction of $c_{1}$ to the 2 -cell $\left[g_{2}, g_{1,2}\right]$ is zero. Set $\mathcal{M}(k)=\left(\mathrm{T}\left(\left\{x_{1}, x_{2}, x_{1,2}\right\}\right), d\right)$, where $d x_{1}=d x_{2}=0$ and $d x_{1,2}=-k x_{1} \otimes x_{2}$, and define $\rho: \mathcal{M}(k) \rightarrow C^{*}(X(k) ; \mathbb{Z})$ by $\rho\left(x_{i}\right)=c_{i}$ and $\rho\left(x_{1,2}\right)=c_{1,2}$.
We can see that $(\mathcal{M}(k), \rho)$ is a 1-minimal model for $C^{*}(X(k) ; \mathbb{Z})$ as follows. Note that $\mathcal{M}(k)_{1}=\mathrm{T}\left(\left\{x_{1}, x_{2}\right\}\right.$ and the kernel of $H^{2}\left(\rho_{1}\right)$ is the submodule generated by the cohomology class of $k x_{1} \otimes x_{2}$. Hence, it suffices to show that $H^{2}(\rho): H^{2}(\mathcal{M}(k)) \rightarrow H^{2}(X(k) ; \mathbb{Z})$ is a monomorphism.

By using the spectral sequence of a Hirsch extension of dgas and then finding cocycle representatives of elements in $E_{\infty}$, it follows that $H^{1}(\mathcal{M}(k))=\mathbb{Z} \oplus \mathbb{Z}$ with basis given by the cohomology classes of $x_{1}$ and $x_{2}$. Moreover, $H^{2}(\mathcal{M}(k))=\mathbb{Z}_{k} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where the cohomology class of $x_{1} \otimes x_{2}$ generates the $\mathbb{Z}_{k}$-summand and the cohomology classes of the cocycles $x_{1} \otimes x_{1,2}-k \zeta_{2}\left(x_{1}\right) \otimes x_{2}$ and $x_{1,2} \otimes x_{2}-k x_{1} \otimes \zeta_{2}\left(x_{2}\right)$ generate the $\mathbb{Z} \oplus \mathbb{Z}$-summand.
Since $\rho^{*}: H^{*}(\mathcal{M}(k)) \rightarrow H^{*}(X(k) ; \mathbb{Z})$ commutes with cup products, it follows that $H^{2}(\rho)$ maps the $\mathbb{Z}_{k}$-summand in $H^{2}(\mathcal{M}(k))$ isomorphically to the $\mathbb{Z}_{k}$-summand in $H^{2}(X(k) ; \mathbb{Z})$. To get information about $H^{2}(\rho)$ restricted to the $\mathbb{Z} \oplus \mathbb{Z}$-summand, note that the cohomology class $u=\left[x_{1} \otimes x_{1,2}-k \zeta_{2}\left(x_{1}\right) \otimes x_{2}\right]$ is an element in the Massey product $\left\langle\left[x_{1}\right], k\left[x_{1}\right],\left[x_{2}\right]\right\rangle$, which has indeterminacy equal to the $\mathbb{Z}_{k}$-summand. Moreover, $H^{2}(\rho)(u)=$ $\left[c_{1} \otimes c_{1,2}-k \zeta_{2}\left(c_{1}\right) \otimes c_{2}\right] \in\left\langle\left[c_{1}\right], k\left[c_{1}\right],\left[c_{2}\right]\right\rangle$, which has indeterminacy equal to the $\mathbb{Z}_{k^{-}}$ summand of $H^{2}(X(k) ; \mathbb{Z})$. From the conditions on the cochains $c_{1}, c_{2}, c_{1,2}$, it follows that $H^{2}(\rho)(u)$ evaluated on the torus $\left[g_{1}, g_{1,2}\right.$ ] is equal to $\pm 1$ and $H^{2}(\rho)(u)$ evaluated on the torus [ $\left.g_{2}, g_{1,2}\right]$ is zero. Similarly, for $v=\left[x_{1,2} \otimes x_{2}-k x_{1} \otimes \zeta_{2}\left(x_{2}\right)\right] \in\left\langle\left[x_{1}\right], k\left[x_{2}\right],\left[x_{2}\right]\right\rangle$, it follows that $H^{2}(\rho)(v)$ evaluated on the torus [ $\left.g_{2}, g_{1,2}\right]$ is equal to $\pm 1$, and $H^{2}(\rho)(v)$ evaluated on the torus [ $g_{1}, g_{1,2}$ ] is zero. It follows that $H^{2}(\rho)$ is an isomorphism, and the argument that $(\mathcal{M}(k), \rho)$ is a 1-minimal model for $X(k)$ is complete.

## 10. Uniqueness and functoriality of 1-minimal models

10.1. Maps between 1 -minimal models. Let $(A, d)$ and $\left(A^{\prime}, d^{\prime}\right)$ be binomial $\cup_{1}$-dgas over the ring $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$, with $H^{0}=R$ and $H^{1}$ a finitely generated, free $R$-module. By Theorem 9.8, there are 1 -minimal models $\rho: \mathcal{M} \rightarrow A$ and $\rho^{\prime}: \mathcal{M}^{\prime} \rightarrow A^{\prime}$, in the sense
of Definition 9.5. We then have $\mathcal{M}=\bigcup_{n \geq 1} \mathcal{M}_{n}$, with inclusion maps $i_{n}: \mathcal{M}_{n} \hookrightarrow \mathcal{M}_{n+1}$ which are Hirsch extensions, and morphisms $\rho_{n}: \mathcal{M}_{n} \rightarrow A$ such that $\rho_{n}=\left.\rho\right|_{\mathcal{M}_{n}}$ and $\rho_{n+1} \circ i_{n}=\rho_{n}$ for all $n \geq 1$, and similarly for $A^{\prime}$ and $\mathcal{M}^{\prime}$.
Let $\varphi: A \rightarrow A^{\prime}$ be a map of binomial $\cup_{1}$-dgas. A map $f: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ between the corresponding 1 -minimal models is said to be a morphism compatible with $\varphi$ if $f$ is a map of binomial $\cup_{1}$-dgas such that $\rho^{\prime} \circ f=\varphi \circ \rho$. That is, we have a sequence of morphisms, $f_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$, such the following diagrams commute, for all $n \geq 1$,



The commutativity of the diagrams (10.1) means that the morphisms $f_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$ between the $n$-th stages of the respective colimits of Hirsch extensions are compatible, in the sense delineated in Section 8.4.

A weaker notion is that of a morphism compatible up to homotopy; that is, a morphism $f: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ preserving dga and binomial structures and such that $\rho^{\prime} \circ f \simeq \varphi \circ \rho$. In this case, the diagram (10.1) commutes for all $n \geq 1$ and the diagram (10.2) commutes only up to homotopy, in the sense that there are homotopies $\Phi_{n}: \mathcal{M}_{n} \rightarrow A^{\prime} \otimes_{R} C^{*}(I ; R)$ between $\varphi \circ \rho_{n}$ and $\rho_{n}^{\prime} \circ f_{n}$ for all $n \geq 1$, with the restriction of $\Phi_{n+1}$ to $\mathcal{M}_{n}$ equal to $\Phi_{n}$ for all $n \geq 1$. Since by Theorem 3.8 homotopic maps induce the same map on cohomology, we still get commuting diagrams in cohomology,

for all $i \geq 0$ and all $n \geq 1$.
10.2. Extending dga maps to 1 -minimal models. Our next goal is to show that, given a morphism $\varphi: A \rightarrow A^{\prime}$, there is a morphism $f: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ compatible with $\varphi$ up to homotopy. We start with a lemma.

Lemma 10.1. Let $\varphi: A \rightarrow A^{\prime}$ be a morphism of binomial $\cup_{1}$-dgas as above. Let $\rho: \mathcal{M} \rightarrow$ $A$ and $\rho^{\prime}: \mathcal{M}^{\prime} \rightarrow A$ be 1-minimal models for $(A, d)$ and $\left(A^{\prime}, d^{\prime}\right)$, respectively, and let $n$ be a positive integer. Assume there is a morphism $f_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$ such that the diagram
below commutes.


Then
(1) There is a morphism $f_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{M}_{n+1}^{\prime}$ such that diagram (10.1) commutes.
(2) There is a bijection between morphisms $f_{n+1}$ as above and maps of sets from $\boldsymbol{X}_{n+1}$ to $H^{1}\left(\mathcal{M}_{n}^{\prime}\right)$.
(3) If $f_{n}$ is an isomorphism and $H^{2}(\varphi)$ is a monomorphism, then $f_{n+1}$ is also an isomorphism.

Proof. Set $K_{n}=\operatorname{ker} H^{2}\left(\rho_{n}\right)$ and $K_{n}^{\prime}=\operatorname{ker} H^{2}\left(\rho_{n}^{\prime}\right)$. To prove parts (1) and (2), there are three cases to consider: (a) $K_{n}=0$, (b) $K_{n} \neq 0, K_{n}^{\prime}=0$, and (c) $K_{n} \neq 0, K_{n}^{\prime} \neq 0$.
(a) If $K_{n}=0$, then $\mathcal{M}_{n}=\mathcal{M}_{n+1}$ and the results in both parts follow, since $f_{n+1}=i_{n}^{\prime} \circ f_{n}$ is the unique map for which the diagram (10.1) commutes.
(b) Next consider the case where $K_{n} \neq 0$ and $K_{n}^{\prime}=0$. In this case, $\mathcal{M}_{n+1}=\mathrm{T}\left(\boldsymbol{X}_{n} \cup \boldsymbol{X}_{n+1}\right)$ with $X_{n+1} \neq \emptyset$, and $\mathcal{M}_{n+1}^{\prime}=\mathcal{M}_{n}^{\prime}$. For $x \in X_{n+1}$, it follows from the commutativity of the diagram (10.4) that the cocycle $f_{n}\left(d_{\mathcal{M}_{n}}(x)\right)$ is cohomologous to 0 in $H^{2}\left(\mathcal{M}_{n}^{\prime}\right)$. Thus, for each $x \in \boldsymbol{X}_{n+1}$, we can choose an element, $f_{n+1}(x) \in \mathcal{M}_{n}^{\prime}$ with $d_{\mathcal{M}_{n}^{\prime}}\left(f_{n+1}(x)\right)=f_{n}\left(d_{\mathcal{M}_{n}}(x)\right)$. Then by Theorem 9.2 the map of sets $\boldsymbol{X}_{n+1} \rightarrow \mathcal{M}_{n}^{\prime}$ given by $x \mapsto f_{n+1}(x)$ extends uniquely to a morphism $f_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{M}_{n+1}^{\prime}$ of binomial $\cup_{1}$-dgas. This completes the proof of part (1) in this case.
Now note that if $\hat{f}_{n+1}$ is a morphism from $\mathcal{M}_{n+1}$ to $\mathcal{M}_{n+1}^{\prime}$ with $\hat{f}_{n+1} \circ i_{n}=f_{n}$, then $d_{\mathcal{M}_{n+1}^{\prime}}\left(f_{n+1}(x)\right)-d_{\mathcal{M}_{n+1}^{\prime}}\left(\hat{f}_{n+1}(x)\right)=f_{n}\left(d_{\mathcal{M}_{n}}(x)\right)-f_{n}\left(d_{\mathcal{M}_{n}}(x)\right)=0$. Hence, the map of sets $X_{n+1} \rightarrow Z^{1}\left(\mathcal{M}_{n}^{\prime}\right)$ given by $x \mapsto c_{x}:=f_{n+1}(x)-\hat{f}_{n+1}(x)$ is a bijection, in this case, between morphisms $\hat{f}_{n+1}$ with $\hat{f}_{n+1} \circ i_{n}=f_{n}$ and maps of sets from $X_{n+1}$ to $Z^{1}\left(\mathcal{M}_{n}^{\prime}\right)$. From part (1) of Lemma 9.6, we have that $Z^{1}\left(\mathcal{M}_{n}\right)=H^{1}\left(\mathcal{M}_{n}\right)$. This completes the proof of part (2) in case (b).
(c) Finally, consider the case where $K_{n} \neq 0$ and $K_{n}^{\prime} \neq 0$. Since the diagram (10.4) commutes, the map $H^{2}\left(f_{n}\right)$ restricts to a homomorphism $k_{n}: K_{n} \rightarrow K_{n}^{\prime}$ which fits into the commuting diagram below,


Note that by assumption both $K_{n}$ and $K_{n}^{\prime}$ are non-zero, finitely generated, free $R$-modules. Now let $\mathcal{M}_{n+1}=\mathrm{T}\left(\boldsymbol{X}^{n} \cup \boldsymbol{X}_{n+1}\right)$ and $\mathcal{M}_{n+1}^{\prime}=\mathrm{T}\left(\boldsymbol{X}^{\prime n} \cup \boldsymbol{X}_{n+1}^{\prime}\right)$. From hypothesis (2) in the definition of a 1-minimal for $A$, it follows that the composition of $d: \mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}\right) \rightarrow Z^{2}\left(\mathcal{M}_{n}\right)$ followed by the projection of a cocycle to its cohomology class gives an isomorphism $\mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}\right) \xrightarrow{\leftrightharpoons} K_{n}$. Similarly, the differential from $\mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}^{\prime}\right)$ to $\mathcal{M}_{n+1}^{\prime}$ gives an isomorphism $\mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}^{\prime}\right) \xrightarrow{\simeq} K_{n}^{\prime}$. Therefore, $k_{n}$ gives a homomorphism $\mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}\right) \rightarrow \mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}^{\prime}\right)$. By Theorem 7.8, this homomorphism extends uniquely to a map $f_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{M}_{n+1}^{\prime}$ of binomial $\cup_{1}$-dgas such that the diagram (10.1) commutes. This completes the proof of part (1) in this case.
Now fix a choice for $f_{n+1}$ and let $\hat{f}_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{M}_{n+1}^{\prime}$ be any morphism such that the diagram (10.1) commutes. Let $x$ be any element in $X_{n+1}$. Since $\hat{f}_{n+1}$ commutes with the differentials and $d x \in \mathcal{M}_{n}$, we have $f_{n} \circ d(x)=d^{\prime} \circ f_{n+1}(x)$ and similarly $f_{n} \circ d(x)=$ $d^{\prime} \circ \hat{f}_{n+1}(x)$. Hence,

$$
\begin{equation*}
d^{\prime} f_{n+1}(x)-d^{\prime} \hat{f}_{n+1}(x)=f_{n}(d x)-f_{n}(d x)=0, \tag{10.6}
\end{equation*}
$$

and it follows that $f_{n+1}(x)-\hat{f}_{n+1}(x)$ is a cocycle in $\left(\mathcal{M}_{n}^{\prime}\right)^{1}$. Therefore, for each $x \in \boldsymbol{X}_{n+1}$ there is a cocycle $c(x) \in\left(\mathcal{M}_{n}^{\prime}\right)^{1}$ such that

$$
\begin{equation*}
\hat{f}_{n+1}(x)=f_{n+1}(x)+c(x) \tag{10.7}
\end{equation*}
$$

By Lemma 9.6, part (1) we have that $Z^{1}\left(\mathcal{M}_{n}^{\prime}\right)=H^{1}\left(\mathcal{M}_{n}^{\prime}\right)$. This shows that a choice for $f_{n+1}$ gives a map that sends isomorphisms $\hat{f}_{n+1}$ such that the diagram (10.1) commutes to maps from $\boldsymbol{X}_{n+1}$ to $H^{1}\left(\mathcal{M}_{n}^{\prime}\right)$. This completes the proof of part (2) in the last case.
We now turn to part (3). Suppose that $f_{n}$ is an isomorphism and $H^{2}(\varphi)$ is a monomorphism. Then $H^{2}\left(f_{n}\right)$ is also an isomorphism, and so chasing diagram (10.5) shows that $H^{2}\left(f_{n}\right)$ restricts to an isomorphism $k_{n}: K_{n} \xrightarrow{\leftrightharpoons} K_{n}^{\prime}$. Since the differentials $d: \mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}\right) \rightarrow$ $Z^{2}\left(\mathcal{M}_{n}\right)$ and $d^{\prime}: \mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}^{\prime}\right) \rightarrow Z^{2}\left(\mathcal{M}_{n}^{\prime}\right)$ are monomorphisms, it follows that $k_{n}$ gives an isomorphism $\mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}\right) \xrightarrow{\Rightarrow} \mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}^{\prime}\right)$. By Theorem 7.8 , this isomorphism extends uniquely to a morphism $f_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{M}_{n+1}^{\prime}$ of binomial $\cup_{1}$-dgas such that the diagram (10.1) commutes.

We claim that if $\hat{f}_{n+1}$ is any morphism from $\mathcal{M}_{n+1}$ to $\mathcal{M}_{n+1}^{\prime}$ such that the diagram (10.1) commutes, then $\hat{f}_{n+1}$ is in fact, an isomorphism. To prove the claim, first note that since $f_{n+1}$ restricts to an isomorphism from $\mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}\right)$ to $\mathrm{T}^{1}\left(\boldsymbol{X}_{n+1}^{\prime}\right)$, it follows that $f_{n+1}: \mathcal{M}_{n+1} \rightarrow$ $\mathcal{M}_{n+1}^{\prime}$ is an isomorphism. By equation (10.7) it follows that $f_{n+1}$ and $\hat{f}_{n+1}$ induce the same map of $R$-modules from $\mathcal{M}_{n+1} / \mathcal{M}_{n}$ to $\mathcal{M}_{n+1}^{\prime} / \mathcal{M}_{n}^{\prime}$. Consider the following commutative
diagram of exact sequences of $R$-modules.


By assumption, $f_{n}$ is an isomorphism; moreover, $\left[\hat{f}_{n+1}\right]$ is an isomorphism, since $\left[f_{n+1}\right]$ is an isomorphism. The claim now follows from the Five Lemma, and this completes the proof.
10.3. Lifting homotopies to 1 -minimal models. The next step is to show that homotopies between maps of binomial $\cup_{1}$-dgas lift to the respective 1-minimal models.

Lemma 10.2 (Homotopy Lifting Lemma). Let $\left(A, d_{A}\right)$ and $\left(A^{\prime}, d_{A^{\prime}}\right)$ be binomial cupone dgas over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ with $H^{0}=R$ and $H^{1}$ finitely generated, free $R$-modules. Let $\rho: \mathcal{M}(A) \rightarrow A$ and $\rho^{\prime}: \mathcal{M}\left(A^{\prime}\right) \rightarrow A^{\prime}$ be 1-minimal models, and let $\varphi: A \rightarrow A^{\prime}$ be a morphism. Suppose for a given $n \geq 1$ there is a morphism $f_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$ and a homotopy $\Phi_{n}: \mathcal{M}_{n} \rightarrow A^{\prime} \otimes_{R} C^{*}(I ; R)$ between $\varphi \circ \rho_{n}$ and $\rho_{n}^{\prime} \circ f_{n}$. Then,
(1) There is a unique morphism $f_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{M}_{n+1}^{\prime}$ such that $f_{n+1} \circ i_{n}=i_{n}^{\prime} \circ f_{n}$ and such that there is a homotopy $\Phi_{n+1}: \mathcal{M}_{n+1} \rightarrow A^{\prime} \otimes_{R} C^{*}(I ; R)$ between $\varphi \circ \rho_{n+1}$ and $\rho_{n+1}^{\prime} \circ f_{n+1}$ with $\left.\Phi_{n+1}\right|_{\mathcal{M}_{n}}=\Phi_{n}$.
(2) If in addition $f_{n}$ is an isomorphism and $H^{2}(\varphi)$ is a monomorphism, then $f_{n+1}$ is also an isomorphism.

Proof. As before, let $\mathcal{M}_{n+1}=\mathrm{T}\left(\boldsymbol{X}^{n} \cup \boldsymbol{X}_{n+1}\right)$ and let $\mathcal{M}_{n+1}^{\prime}=\mathrm{T}\left(\boldsymbol{X}^{\prime n} \cup \boldsymbol{X}_{n+1}^{\prime}\right)$, with corresponding maps $\rho$ and $\rho^{\prime}$ as pictured in the diagram below.


Let $d_{n+1}$ denote the differential on $\mathcal{M}_{n+1}$, and note that for $x \in \boldsymbol{X}_{n+1}$, we have $d_{n+1}(x) \in$ $\mathcal{M}_{n}$. By Theorem 7.8, it suffices to define for each $x \in \boldsymbol{X}_{n+1}$ an element $\Phi_{n+1}(x) \in$ $A^{\prime} \otimes_{R} C^{*}(I ; R)$ such that $\Phi_{n+1} \circ d_{n+1}(x)=d_{A^{\prime} \otimes_{R} C^{*}(I ; R)} \circ \Phi_{n+1}(x)$. Let $x \in X_{n+1}$. Then $d_{n+1}(x)$
is a cocycle in $\mathcal{M}_{n}$, and hence, $\Phi_{n}\left(d_{n+1}(x)\right)$ is a cocycle in $A^{\prime} \otimes_{R} C^{*}(I ; R)$. We can assume this cocycle has the form

$$
\begin{equation*}
\Phi_{n}\left(d_{n+1}(x)\right)=\varphi\left(\rho_{n}\left(d_{n+1} x\right)\right) t_{0}+\rho_{n}^{\prime}\left(f_{n}\left(d_{n+1} x\right)\right) t_{1}+c_{1}(x) u \tag{10.10}
\end{equation*}
$$

with $c_{1}(x) \in\left(A^{\prime}\right)^{1}$.
The condition that $\Phi_{n}\left(d_{n+1}(x)\right)$ is a cocycle then leads to an equation for $d_{A^{\prime}}\left(c_{1}(x)\right)$, as follows. Recall that if $u_{0}, u_{1}, c$ are homogeneous elements in $A^{\prime}$ with $\left|u_{0}\right|=\left|u_{1}\right|=|c|+1$, then $u=u_{0} t_{0}+u_{1} t_{1}+c u$ is a homogeneous element in $A^{\prime} \otimes_{R} C^{*}(I ; R)$ with

$$
\begin{align*}
d_{A^{\prime} \otimes C^{*}(I ; R)}\left(u_{0} t_{0}+u_{1} t_{1}+c u\right)= & d_{A^{\prime}}\left(u_{0}\right) t_{0}+d_{A^{\prime}}\left(u_{1}\right) t_{1} \\
& +\left((-1)^{\left|u_{0}\right|+1} u_{0}+(-1)^{\left|u_{1}\right|} u_{1}+d_{A^{\prime}}(c)\right) u . \tag{10.11}
\end{align*}
$$

In particular, if $u_{0} t_{0}+u_{1} t_{1}+c u$ is a cocycle in $A^{\prime} \otimes_{R} C^{*}(I ; R)$, then $u_{0}$ and $u_{1}$ are cocycles in $A^{\prime}$ and $d_{A^{\prime}}(c)=(-1)^{\left|u_{0}\right|} u_{0}+(-1)^{\left|u_{1}\right|+1} u_{1}$. Since $\Phi_{n}\left(d_{n+1} x\right)$ is a cocycle in $A^{\prime} \otimes_{R} C^{*}(I ; R)$, we have that

$$
\begin{equation*}
d_{A^{\prime}}\left(c_{1}(x)\right)=\varphi\left(\rho_{n}\left(d_{n+1} x\right)\right)-\rho_{n}^{\prime}\left(f_{n}\left(d_{n+1} x\right)\right) \tag{10.12}
\end{equation*}
$$

Now by Lemma 10.1 there are morphisms $f_{n+1}: \mathcal{M}_{n+1} \rightarrow \mathcal{M}_{n+1}^{\prime}$ such that the diagram (10.1) commutes. Choose such a map $f_{n+1}$. The map $f_{n+1}$ then determines a map from $X_{n+1}$ to $H^{1}\left(\mathcal{M}_{n}^{\prime}\right)$, as follows. For each $x \in X_{n+1}$, we have

$$
\begin{equation*}
d_{A^{\prime}}\left(\varphi\left(\rho_{n+1}(x)\right)\right)=\varphi\left(\rho_{n}\left(d_{n+1} x\right)\right) \quad \text { and } \quad d_{A^{\prime}}\left(\rho_{n+1}^{\prime}\left(f_{n+1}(x)\right)\right)=\rho_{n}^{\prime}\left(f_{n}\left(d_{n+1} x\right)\right) \tag{10.13}
\end{equation*}
$$

and it follows from equations (10.12) and (10.13) that the element

$$
\begin{equation*}
z(x):=c_{1}(x)-\varphi\left(\rho_{n+1}(x)\right)+\rho_{n+1}^{\prime}\left(f_{n+1}(x)\right) \tag{10.14}
\end{equation*}
$$

is a cocycle in $A^{\prime}$. Thus, we have a map of sets from $\boldsymbol{X}_{n+1}$ to $H^{1}\left(A^{\prime}\right)$ given by $x \mapsto[z(x)]$, where $[w]$ denotes the cohomology class of a cocycle $w$. By Definition 9.5 and Lemma 9.6, this map corresponds uniquely to a map of sets from $X_{n+1}$ to $H^{1}\left(\mathcal{M}_{n}^{\prime}\right)$.

By Lemma 10.1 we can assume that $f_{n+1}$ has been chosen so that for each $x \in \boldsymbol{X}_{n+1}$, we have that $[z(x)]=0$. It then follows that for each $x$ there is an element $c_{0}(x) \in\left(A^{\prime}\right)^{0}$ with

$$
\begin{equation*}
d_{A^{\prime}}\left(c_{0}(x)\right)=z(x)=c_{1}(x)-\varphi\left(\rho_{n+1}(x)\right)+\rho_{n+1}^{\prime}\left(f_{n+1}(x)\right) . \tag{10.15}
\end{equation*}
$$

Now set

$$
\begin{equation*}
\Phi_{n+1}(x):=\varphi\left(\rho_{n+1}(x)\right) t_{0}+\rho_{n+1}^{\prime}\left(f_{n+1}(x)\right) t_{1}+c_{0}(x) u \tag{10.16}
\end{equation*}
$$

The final step is to show that with this choice of $\Phi_{n+1}$, it follows that $\Phi_{n}\left(d_{n+1}(x)\right)=$ $d_{A^{\prime} \otimes C^{*}(I)}\left(\Phi_{n+1}(x)\right)$ for all $x \in X_{n+1}$. Using equations (10.16), (10.15), and (10.10), we
have that

$$
\begin{align*}
d_{A^{\prime} \otimes C^{*}(I)}\left(\Phi_{n+1}(x)\right)= & d_{A^{\prime} \otimes C^{*}(I)}\left(\varphi\left(\rho_{n+1}(x)\right) t_{0}+\rho_{n+1}^{\prime}\left(f_{n+1}(x)\right) t_{1}+c_{0}(x) u\right) \\
= & \varphi\left(\rho_{n}\left(d_{n+1} x\right)\right) t_{0}+\rho_{n}^{\prime}\left(f_{n}\left(d_{n+1} x\right)\right) t_{1} \\
& +\left[\varphi\left(\rho_{n+1}(x)\right)-\rho_{n+1}^{\prime}\left(f_{n+1}(x)\right)+d_{A^{\prime}}\left(c_{0}(x)\right)\right] u  \tag{10.17}\\
= & \varphi\left(\rho_{n}\left(d_{n+1} x\right)\right) t_{0}+\rho_{n}^{\prime}\left(f_{n}\left(d_{n+1} x\right)\right) t_{1}+c_{1}(x) u \\
= & \Phi_{n}\left(d_{n+1} x\right),
\end{align*}
$$

and the proof is complete.
10.4. Homotopy functoriality of 1-minimal models. We are now in a position to show that 1-minimal models are unique up to homotopy.
Theorem 10.3. Let $(A, d)$ and $\left(A^{\prime}, d^{\prime}\right)$ be binomial cup-one dgas over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ such that $H^{0}(A)$ and $H^{0}\left(A^{\prime}\right)$ are isomorphic to $R$ and $H^{1}(A)$ and $H^{1}\left(A^{\prime}\right)$ are finitely generated, free $R$-modules. Let $\rho: \mathcal{M}(A) \rightarrow A$ and $\rho^{\prime}: \mathcal{M}\left(A^{\prime}\right) \rightarrow A^{\prime}$ be 1-minimal models, and let $\varphi: A \rightarrow A^{\prime}$ be a morphism. Then
(1) There is a morphism $\widehat{\varphi}: \mathcal{M}(A) \rightarrow \mathcal{M}\left(A^{\prime}\right)$ compatible with the respective colimit structures (that is, $\widehat{\varphi}_{n+1} \circ i_{n}=i_{n}^{\prime} \circ \widehat{\varphi}_{n}$ for all $n$ ), and there is a homotopy $\Phi: \mathcal{M}(A) \rightarrow A^{\prime} \otimes_{R} C^{*}(I ; R)$ between $\varphi \circ \rho$ and $\rho^{\prime} \circ \widehat{\varphi}$ (also preserving colimit structures), so that the diagram below commutes up to homotopy.

(2) If $\varphi$ is a 1-quasi-isomorphism, then $\widehat{\varphi}$ is an isomorphism.

Proof. To prove part (1), first set $\mathcal{M}_{n}=\mathcal{M}_{n}(A)$ and $\mathcal{M}_{n}^{\prime}=\mathcal{M}_{n}\left(A^{\prime}\right)$. We need to construct isomorphisms $\widehat{\varphi}_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$ and homotopies $\Phi_{n}: \mathcal{M}_{n} \rightarrow A \otimes_{R} C^{*}(I ; R)$ between $\rho_{n}$ and $\rho_{n}^{\prime} \circ \widehat{\varphi}_{n}$ such that $\widehat{\varphi}_{n+1} \circ i_{n}=i_{n}^{\prime} \circ \widehat{\varphi}_{n}$ and $\left.\Phi_{n+1}\right|_{\mathcal{M}_{n}}=\Phi_{n}$. The proof is by induction on $n$.
The base case is to show that there is an isomorphism $\widehat{\varphi}_{1}: \mathcal{M}_{1} \rightarrow \mathcal{M}_{1}^{\prime}$ and a homotopy $\Phi_{1}: \mathcal{M}_{1} \rightarrow A \otimes_{R} C^{*}(I ; R)$ between $\rho_{1}$ and $\rho_{1}^{\prime} \circ \widehat{\varphi}_{1}$. Since $H^{1}\left(\rho_{1}\right)=H^{1}\left(\rho_{1}^{\prime} \circ \widehat{\varphi}_{1}\right)$, the claim follows from Lemma 6.5. The induction step now follows from the homotopy lifting lemma (Lemma 10.2).

To prove part (2), assume that $\varphi$ is a 1-quasi-isomorphism; that is, $H^{1}(\varphi): H^{1}(A) \rightarrow$ $H^{1}\left(A^{\prime}\right)$ is an isomorphism and $H^{2}(\varphi): H^{2}(A) \rightarrow H^{2}\left(A^{\prime}\right)$ is a monomorphism. Now, the map $H^{1}(\varphi)$ determines a map $\widehat{\varphi}_{1}: \mathcal{M}_{1} \rightarrow \mathcal{M}_{1}^{\prime}$, which must also be an isomorphism. Since $H^{2}(\varphi)$ is a monomorphism, Lemma 10.1, part (3) insures that $\widehat{\varphi}_{1}$ lifts to compatible isomorphisms, $\widehat{\varphi}_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$, for all $n \geq 1$. It follows that the family of maps $\left\{\widehat{\varphi}_{n}\right\}_{n \geq 1}$ defines the desired isomorphism $\widehat{\varphi}: \mathcal{M}(A) \rightarrow \mathcal{M}\left(A^{\prime}\right)$, and this completes the proof.

Taking $A=A^{\prime}$ in the above theorem, we obtain the following corollary.
Corollary 10.4. Let A be a binomial cup-one $R$-dga as above, and let $(\mathcal{M}, \rho)$ and $\left(\mathcal{M}^{\prime}, \rho^{\prime}\right)$ be any two 1-minimal models for $A$. Then there is an isomorphism $f: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ such that $\rho^{\prime} \circ f$ is homotopic to $\rho$.
10.5. 1-minimal models and homotopies. The next lemma corresponds to an analogous result in [9] (Corollary 11.4); see also [6, Proposition 12.7].

Lemma 10.5. Homotopy is an equivalence relation on the set of morphisms from a colimit of Hirsch extensions, $\mathcal{M}$, to a binomial cup-one dga, $A$.

Proof. Clearly, $\simeq$ is reflexive. To show symmetry, let $\Phi: \mathcal{M} \rightarrow A \otimes_{R} \mathbb{C}^{*}(I ; R)$ is a homotopy from $\varphi_{0}$ to $\varphi_{1}$, given on elements $a \in \mathcal{M}^{i}$ by $\Phi(a)=\varphi_{0}(a) t_{0}+\varphi_{1}(a) t_{1}+c(a) u$, for some $c(a) \in A^{i-1}$; then the map $\bar{\Phi}$ given by $\bar{\Phi}(a)=\varphi_{1}(a) t_{0}+\varphi_{0}(a) t_{1}+c(a) u$ is a homotopy from $\varphi_{1}$ to $\varphi_{0}$.
It remains to show $\simeq$ is transitive. With $\Phi$ as above, let $I^{\prime}$ be another copy of the interval, let $t_{0}^{\prime}, t_{1}^{\prime}, u^{\prime}$ be the corresponding generators of $C^{*}\left(I^{\prime} ; R\right)$, and let $\Phi^{\prime}: \mathcal{M} \rightarrow A \otimes_{R} \mathbb{C}^{*}\left(I^{\prime} ; R\right)$ be a homotopy from $\varphi_{1}$ to $\varphi_{2}$. Finally, let $C^{*}(I ; R) \vee C^{*}\left(I^{\prime} ; R\right)$ be the fiber product corresponding to the augmentations $\varepsilon: C^{*}(I ; R) \rightarrow R$ and $\varepsilon^{\prime}: C^{*}\left(I^{\prime} ; R\right) \rightarrow R$ given by $\varepsilon\left(t_{1}\right)=\varepsilon^{\prime}\left(t_{0}^{\prime}\right)=1$ and $\varepsilon\left(t_{0}\right)=\varepsilon^{\prime}\left(t_{1}^{\prime}\right)=0$. With this setup, we define a map

$$
\begin{equation*}
\Psi: \mathcal{M} \longrightarrow A \otimes_{R}\left(C^{*}(I ; R) \vee C^{*}\left(I^{\prime} ; R\right)\right) \tag{10.19}
\end{equation*}
$$

by setting $\Psi(a)=\left(\Phi(a), \Phi^{\prime}(a)\right)$. Now let $\Delta$ be a triangle with oriented edges $e_{1}=I$, $e_{2}=I^{\prime}$, and $e_{3}=I^{\prime \prime}$. The inclusions of the edges in the triangle induce epimorphisms $q_{j}: C^{*}(\Delta ; R) \rightarrow C^{*}\left(e_{j} ; R\right)$, which give a surjection $f: C^{*}(\Delta ; R) \rightarrow C^{*}(I ; R) \vee C^{*}\left(I^{\prime} ; R\right)$. By Theorem 9.4, the morphism $\Psi$ lifts through $f$ to a morphism $\widehat{\Psi}: \mathcal{M} \rightarrow A \otimes_{R} C^{*}(\Delta ; R)$. The map $q_{3} \circ \widehat{\Psi}: \mathcal{M} \rightarrow A \otimes_{R} \mathbb{C}^{*}\left(I^{\prime \prime} ; R\right)$, then, is the desired homotopy from $\varphi_{0}$ to $\varphi_{2}$.

We will write $[\mathcal{M}, A]$ for the set of homotopy classes of morphisms $\varphi: \mathcal{M} \rightarrow A$. Given a morphism $\xi: A \rightarrow A^{\prime}$, composition with $\xi$ defines a function, $\xi_{*}:[\mathcal{M}, A] \rightarrow\left[\mathcal{M}, A^{\prime}\right]$. Similar notions hold for augmented dgas and augmentation-preserving morphisms between them. The next lemma corresponds to an analogous result in [9] (Theorem 11.5); see also [6, Proposition 12.9].

Lemma 10.6. Let $\left(T_{R}(\boldsymbol{X}), d\right)$ be a colimit of Hirsch extensions and assume $A$ and $A^{\prime}$ are binomial $\cup_{1}$-dgas with augmentations, $\varepsilon_{A}, \varepsilon_{A^{\prime}}$ that induce isomorphisms from $H^{0}$ to $R$. Assume further that there is an augmentation-preserving 1-quasi-isomorphism, $\xi: A \rightarrow A^{\prime}$. Then the induced map of equivalence classes of augmentation-preserving homotopies of augmentation preserving maps, $\xi_{*}:\left[\mathrm{T}_{R}(\boldsymbol{X}), A\right] \rightarrow\left[\mathrm{T}_{R}(\boldsymbol{X}), A^{\prime}\right]$, is injective.

Proof. Let $f$ and $g$ be augmentation-preserving morphisms from $\mathrm{T}_{R}(\boldsymbol{X})$ to $A$, and let $H: \mathrm{T}_{R}(\boldsymbol{X}) \rightarrow A^{\prime} \otimes_{R} C^{*}(I ; R)$ be an augmentation-preserving homotopy between $\xi \circ f$ and
$\xi \circ g$. We will show that $H$ lifts to an augmentation-preserving homotopy $\widehat{H}$ between $f$ and $g$.
Write $\boldsymbol{X}=\left\{x_{1}, x_{2}, \ldots\right\}$ with $\boldsymbol{X}^{n}=\left\{x_{1}, \ldots, x_{n}\right\}$. Set $\mathcal{M}_{n}=\left(\mathrm{T}_{R}\left(\boldsymbol{X}^{n}\right), d_{n}\right)$ and let $f_{n}, g_{n}$ and $H_{n}$ denote the restrictions of $f, g$, and $H$ to $\mathcal{M}_{n}$. Then $x_{1}$ is a cocycle in $\mathcal{M}_{1}$, and we have

$$
\begin{equation*}
H\left(x_{1}\right)=\xi \circ f\left(x_{1}\right) t_{0}+\xi \circ g\left(x_{1}\right) t_{1}+c\left(x_{1}\right) u \tag{10.20}
\end{equation*}
$$

with

$$
\begin{equation*}
d c\left(x_{1}\right)=\xi \circ f\left(x_{1}\right)-\xi \circ g\left(x_{1}\right) \tag{10.21}
\end{equation*}
$$

Since $H^{1}(\xi): H^{1}(A) \rightarrow H^{1}\left(A^{\prime}\right)$ is a monomorphism, it follows that there is an element $\hat{c}\left(x_{1}\right) \in \operatorname{ker}\left(\varepsilon_{A}\right)$ with

$$
\begin{equation*}
d \hat{c}\left(x_{1}\right)=f\left(x_{1}\right)-g\left(x_{1}\right)=f_{1}\left(x_{1}\right)-g_{1}\left(x_{1}\right) . \tag{10.22}
\end{equation*}
$$

Since $f_{1}\left(x_{1}\right) t_{0}+g_{1}\left(x_{1}\right) t_{1}+\hat{c}\left(x_{1}\right) u$ is a cocycle in $A \otimes_{R} C^{1}(I ; R)$ and $\hat{c}\left(x_{1}\right) \in \operatorname{ker}\left(\varepsilon_{A}\right)$ with $\varepsilon_{A}$ a binomial subalgebra of $A^{0}$, it follows that the map $x_{1} \mapsto f_{1}\left(x_{1}\right) t_{0}+g_{1}\left(x_{1}\right) t_{1}+\hat{c}\left(x_{1}\right) u$ extends uniquely to an augmentation-preserving homotopy $\widehat{H}_{1}$ between $f_{1}$ and $g_{1}$.
The next step is to show that $\widehat{H}_{1}$ is a lifting of $H_{1}$. Since $\hat{c}\left(x_{1}\right) \in \operatorname{ker}\left(\varepsilon_{A}\right)$ and $\xi$ is augmentation-preserving, it follows that $\xi \circ \hat{c}\left(x_{1}\right) \in \operatorname{ker}\left(\varepsilon_{A^{\prime}}\right)$. From equations (10.21) and (10.22), we have that the elements $\xi \circ \hat{c}\left(x_{1}\right)$ and $c\left(x_{1}\right)$ have the same coboundary. Since $\varepsilon_{A^{\prime}}$ induces an isomorphism from $H^{0}\left(A^{\prime}\right)$ to $R$, it follows that two elements in $\operatorname{ker}\left(\varepsilon_{A^{\prime}}\right)$ with the same coboundary are equal to each other; hence $\xi \circ \hat{c}\left(x_{1}\right)=c\left(x_{1}\right)$ and $\hat{H}_{1}$ is a lifting of $H_{1}$.
Now assume by induction that the homotopy $H_{n}$ lifts to a homotopy $\widehat{H}_{n}$ between $f_{n}$ and $g_{n}$ and show that $\widehat{H_{n}}$ then extends to a lifting $\widehat{H}_{n+1}$ of $H_{n+1}$. Note that $d x_{n+1}$ is a cocycle in $\mathrm{T}\left(X^{n}\right)$. Hence

$$
\widehat{H}_{n}\left(d x_{n+1}\right)=f_{n}\left(d x_{n+1}\right) t_{0}+g_{n}\left(d x_{n+1}\right) t_{1}+\hat{c}\left(d x_{n+1}\right) u
$$

is a cocycle in $A \otimes_{R} C^{*}(I ; R)$, and it follows that $d \hat{c}\left(d x_{n+1}\right)=f_{n}\left(d x_{n+1}\right)-g_{n}\left(d x_{n+1}\right)$.
The obstruction to extending $\widehat{H}_{n}$ to a homotopy $\widehat{H}_{n+1}$ is finding an element $\hat{c}\left(x_{n+1}\right) \in A^{0}$ such that the map

$$
x_{n+1} \mapsto \widehat{H}_{n+1}\left(x_{n+1}\right)=f\left(x_{n+1}\right) t_{0}+g\left(x_{n+1}\right) t_{1}+\hat{c}\left(x_{n+1}\right) u
$$

commutes with the coboundary map; that is,

$$
d \hat{c}\left(x_{n+1}\right)=g\left(x_{n+1}\right)-f\left(x_{n+1}\right)+\hat{c}\left(d x_{n+1}\right) .
$$

Since $\widehat{H}_{n}$ is a lifting of $H_{n}$, we have that $H_{n+1}\left(x_{n+1}\right)$ has the form $\xi \circ f\left(x_{n+1}\right) t_{0}+\xi \circ$ $g\left(x_{n+1}\right) t_{1}+c\left(x_{n+1}\right) u$, where

$$
d c\left(x_{n+1}\right)=\xi \circ g\left(x_{n+1}\right)-\xi \circ f\left(x_{n+1}\right)+\xi \circ \hat{c}\left(d x_{n+1}\right) .
$$

In particular, the cocycle $\xi \circ g\left(x_{n+1}\right)-\xi \circ f\left(x_{n+1}\right)+\xi \circ \hat{c}\left(d x_{n+1}\right)$ in $A^{\prime}$ is cohomologous to zero, and then since $\xi: H^{1}(A) \rightarrow H^{1}\left(A^{\prime}\right)$ is a monomorphism, it follows that the cocycle
$g\left(x_{n+1}\right)-f\left(x_{n+1}\right)+\hat{c}\left(d x_{n+1}\right)$ is cohomologous to zero in $A$. Thus, there is an element $\hat{c}\left(x_{n+1}\right)$ in $\operatorname{ker}\left(\varepsilon_{A}\right)$ with $d \hat{c}\left(x_{n+1}\right)=g\left(x_{n+1}\right)-f\left(x_{n+1}\right)+\hat{c}\left(d x_{n+1}\right)$, and hence, $\widehat{H}_{n}$ extends to a homotopy $\widehat{H}_{n+1}$.
The final step is to see that $\widehat{H}_{n+1}$ is a lifting of $H_{n+1}$. Since $\hat{c}\left(x_{n+1}\right) \in \operatorname{ker}\left(\varepsilon_{A}\right)$ and $\xi$ is augmentation-preserving, it follows that $\xi \circ \hat{c}\left(x_{n+1}\right) \in \operatorname{ker}\left(\varepsilon_{A^{\prime}}\right)$. The elements $\xi \circ \hat{c}\left(x_{n+1}\right)$ and $c\left(x_{n+1}\right)$ have the same coboundary and are both elements in $\operatorname{ker}\left(\varepsilon_{A^{\prime}}\right)$; hence they are equal to each other given that the augmentation $\varepsilon_{A^{\prime}}$ induces an isomorphism from $H^{0}\left(A^{\prime}\right)$ to $R$. This completes the argument that $\widehat{H}_{n+1}$ is a lifting of $H_{n+1}$, and hence the proof is complete.

Remark 10.7. Note that if $A=C^{*}(X ; R)$ with $X$ a path-connected $\Delta$-complex, then there is an augmentation $\varepsilon_{A}: A \rightarrow R$ inducing an isomorphism from $H^{0}(A)$ to $R$.
10.6. 1-minimal models of augmented binomial dgas. Let $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$. For binomial cup-one $R$-dgas $\left(A, d_{A}\right)$ that come equipped with an augmentation, $\varepsilon_{A}: A \rightarrow R$, that induces an isomorphism from $H^{0}(A)$ to $R$, and for which $H^{1}(A)$ is a finitely generated, free $R$-module, Theorem 10.3 may be refined. Recall from Theorem 9.10 that any such dga admits an augmented 1-minimal model, $\rho: \mathcal{M} \rightarrow A$.

Theorem 10.8. Let $(A, d)$ and $\left(A^{\prime}, d^{\prime}\right)$ be augmented binomial cup-one dgas as above. Let $\rho: \mathcal{M} \rightarrow A$ and $\rho^{\prime}: \mathcal{M}^{\prime} \rightarrow A^{\prime}$ be augmented 1-minimal models, and let $\varphi: A \rightarrow A^{\prime}$ be an augmentation-preserving morphism. There is then a unique augmentation-preserving morphism $\widehat{\varphi}: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ such that $\varphi \circ \rho$ is augmentation-preserving homotopic to $\rho^{\prime} \circ \widehat{\varphi}$.

Proof. By Theorem 10.3, there is an isomorphism $\widehat{\varphi}: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ and a homotopy $\Phi: \mathcal{M} \rightarrow$ $A \otimes_{R} C^{*}(I ; R)$ between $\varphi \circ \rho$ and $\rho^{\prime} \circ \widehat{\varphi}$. Let $\widetilde{\varphi}: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ be another such isomorphism. Since $\rho^{\prime} \circ \widehat{\varphi}$ and $\rho^{\prime} \circ \widetilde{\varphi}$ are both homotopic to $\varphi \circ \rho$, it follows from Lemma 10.5 that $\rho^{\prime} \circ \widehat{\varphi}$ and $\rho^{\prime} \circ \widetilde{\varphi}$ are homotopic to each other. Then from Lemma 10.6, it follows that $\widehat{\varphi}$ and $\widetilde{\varphi}$ are homotopic morphisms from $\mathcal{M}$ to $\mathcal{M}^{\prime}$.

Since the proofs of Theorem 10.3 and Lemma 10.5 apply as well to augmentation preserving homotopies, it follows that $\widehat{\varphi}$ and $\widetilde{\varphi}$ are homotopic by an augmentation-preserving homotopy. It then follows from Lemma 3.10 that $\widehat{\varphi}=\widetilde{\varphi}$, and the proof is complete.

The following uniqueness result follows from Theorem 10.8 by talking $A=A^{\prime}$.
Corollary 10.9. Let $A$ be an augmented binomial cup-one $R$-dga as above, and let $(\mathcal{M}, \rho)$ and $\left(\mathcal{M}^{\prime}, \rho^{\prime}\right)$ be any two augmented 1-minimal models for $A$. Then there is a unique augmentation-preserving isomorphism $f: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ such that $\rho^{\prime} \circ f$ is augmentationpreserving homotopic to $\rho$.

## 11. Compatibility of integer and rational 1-minimal models

In this section we define the Postnikov tower of a connected space $Y$ with $H^{1}(Y ; \mathbb{Z})$ finitely generated and show that the corresponding sequence of Hirsch extensions is a 1minimal model for $Y$. We then use this result to show that the integral 1-minimal model of $Y$ tensored with the rationals is weakly equivalent as a dga to the 1-minimal model of $Y$ defined in rational homotopy theory.
11.1. Postnikov towers and 1-minimal models. A connected space $Y$ with $H^{1}(Y ; R)$ finitely generated determines a Postnikov tower with compatible maps from $Y$ to the tower, as pictured in display (11.1), as follows.


Set $Y_{1}$ equal to the Eilenberg-MacLane space $K\left(H^{1}(Y ; R), 1\right)$ and let $h_{1}$ be a map inducing an isomorphism from $H^{1}\left(Y_{1} ; R\right)$ to $H^{1}(Y ; R)$.
Assume $h_{n}: Y \rightarrow Y_{n}$ has been defined. Let $\pi_{n}: Y_{n+1} \rightarrow Y_{n}$ be the fibration with $k$-invariant corresponding to the kernel of $H^{2}\left(h_{n}\right)$, and let $h_{n+1}$ be a lifting of $h_{n}$. The resulting tower is called the Postikov 1-tower of the space $Y$.

Lemma 11.1. Let $Y$ be a connected space with $H^{1}(Y ; R)$ finitely generated. Let $\left\{Y_{n}\right\}_{n \geq 1}$ be a Postnikov 1-tower for $Y$, as in diagram (11.1). Then there is a colimit of Hirsch extensions $\mathcal{M}$ and quasi-isomorphisms $\psi_{n}: \mathcal{M}_{n} \rightarrow C^{*}\left(Y_{n} ; R\right)$ such that $\mathcal{M}$ with structure maps $\rho_{n}: \mathcal{M}_{n} \rightarrow C^{*}(Y ; R)$ given by $\rho_{n}=h_{n}^{*} \circ \psi_{n}$ is a 1-minimal model for $C^{*}(Y ; R)$.

Proof. Note that $H^{2}\left(K\left(H^{1}(Y ; R), 1\right) ; R\right)$ is a finitely generated free $R$-module. Then by induction, using the argument in the proof of property (3) of Lemma 9.3, it follows that the kernel of the map $H^{2}\left(h_{n}\right): H^{2}\left(Y_{n} ; R\right) \rightarrow H^{2}(Y ; R)$ is also a finitely generated, free $R$-module. The existence of the colimit of Hirsch extensions $\mathcal{M}$ and quasi-isomorphisms $\psi_{n}: \mathcal{M}_{n} \rightarrow C^{*}\left(Y_{n} ; R\right)$ such that $\mathcal{M}$ with structure maps $\rho_{n}=h_{n}^{*} \circ \psi_{n}$ is a 1-minimal model for $C^{*}(Y ; R)$ now follows from Theorem 8.12.
11.2. Polynomial differential forms. We now briefly review a construction in rational homotopy theory due to Sullivan [25]. For each integer $n \geq 0$, set

$$
\begin{equation*}
\left(A_{\mathrm{PL}}\right)_{n}=\frac{\Lambda\left(t_{0}, \ldots, t_{n}, y_{0}, \ldots, y_{n}\right)}{\left(\sum t_{i}-1, \sum y_{j}\right)} \tag{11.2}
\end{equation*}
$$

where $\Lambda\left(t_{0}, \ldots, t_{n}, y_{0}, \ldots, y_{n}\right)$ denotes the free commutative algebra over $\mathbb{Q}$ generated by elements $t_{i}$ of degree zero and elements $y_{j}$ of degree one, and define a differential $d$ on $\left(A_{\mathrm{PL}}\right)_{n}$ by setting $d t_{i}=y_{i}$ and $d y_{j}=0$.
Given a topological space $Y$, an element $u \in A_{\mathrm{pL}}^{p}(Y)$ is a rule that associates to each singular $n$-simplex $\sigma$ of $Y$ an element $u(\sigma) \in\left(A_{\mathrm{PL}}\right)_{n}^{p}$ compatible with the face and degeneracy maps $\partial_{i}:\left(A_{\mathrm{PL}}\right)_{n+1} \rightarrow\left(A_{\mathrm{PL}}\right)_{n}$ and $s_{j}:\left(A_{\mathrm{PL}}\right)_{n} \rightarrow\left(A_{\mathrm{PL}}\right)_{n+1}$ given by

$$
\partial_{i}: t_{k} \mapsto\left\{\begin{array}{ll}
t_{k} & k<i  \tag{11.3}\\
0 & k=i \\
t_{k-1} & k>i
\end{array} \quad \text { and } \quad s_{j}: t_{k} \mapsto \begin{cases}t_{k} & k<j \\
t_{k}+t_{k+1} & k=j \\
t_{k+1} & k>j\end{cases}\right.
$$

Then $A_{\mathrm{PL}}(Y)$ is a commutative differential graded algebra over the rationals and the assignment $Y \leadsto A_{\mathrm{PL}}(Y)$ is functorial.

In addition to the Sullivan algebra $A_{\mathrm{PL}}(Y)$ and the cochain algebra $C^{*}(Y ; \mathbb{Q})$, there is a differential graded algebra over the rationals, $C A(Y)$, with the following property.

Theorem 11.2 (Corollary 10.10 in [6]). For topological spaces $Y$ there are natural quasiisomorphisms

$$
C^{*}(Y ; \mathbb{Q}) \longrightarrow C A(Y) \longleftarrow A_{\mathrm{PL}}(Y)
$$

Consequently, $A_{\mathrm{PL}}(Y)$ is weakly equivalent (as a dga) to $C^{*}(Y ; \mathbb{Q})$.
11.3. Rational 1-minimal models. Let $(\Lambda(X), d)$ be the free commutative algebra over $\mathbb{Q}$ generated by the elements in a set $\boldsymbol{X}$, equipped with a differential $d$. We say that a dga $\left(\Lambda(\boldsymbol{X}) \otimes_{\mathbb{Q}} \Lambda(\boldsymbol{Y}), \bar{d}\right)$ is a Hirsch extension of $(\Lambda(\boldsymbol{X}), d)$ if $\bar{d} x=d x$ for all $x \in \boldsymbol{X}$ and $\bar{d} y$ is a cocycle in $\Lambda(\boldsymbol{X})$ for all $y \in \boldsymbol{Y}$.

Now let $Y$ be a topological space. A sequence $\mathcal{M}_{n, \mathbb{Q}}(Y)=\left(\Lambda\left(\boldsymbol{X}_{1} \cup \cdots \cup \boldsymbol{X}_{n}\right), d_{n}\right)$ of Hirsch extensions with $n \geq 1$ together with maps

is a rational 1-minimal model for $Y$ if the following conditions are satisfied:
(1) $d_{1}(x)=0$ for all $x \in \boldsymbol{X}_{1}$.
(2) The map $\rho_{1}^{1}: H^{1}\left(\mathcal{M}_{1, \mathrm{Q}}\right) \rightarrow H^{1}\left(A_{\mathrm{PL}}(Y)\right)$ is an isomorphism.
(3) Under the assignment $x \mapsto d_{n+1}(x)$, the set $\boldsymbol{X}_{n+1}$ corresponds to a basis for $\operatorname{ker}\left(\rho_{n}^{2}\right) \subset H^{2}\left(\mathcal{M}_{n, \mathbb{Q}}\right)$ given by the cohomology classes of the cocycles $\left\{d_{n+1}(x) \mid\right.$ $\left.x \in X_{n+1}\right\} \subset Z^{2}\left(\mathcal{M}_{n, \mathbb{Q}}\right)$.

As shown by Sullivan [25] (see also [9, 6, 24, 23]), every connected space $Y$ admits a rational 1-minimal model, unique up to an isomorphism of cdgas.

The following gives the definition of $n$-step equivalence in rational homotopy theory.
Definition 11.3. Given commutative dgas $\left(A, d_{A}\right)$ and $\left(A^{\prime}, d_{A^{\prime}}\right)$ over $\mathbb{Q}$ with 1-minimal models $\left(\mathcal{M}_{\mathbb{Q}}, \rho\right)$ and $\left(\mathcal{M}_{\mathbb{Q}}^{\prime}, \rho^{\prime}\right)$; respectively, and an integer $n \geq 1$, we say that $A$ and $A^{\prime}$ are $n$-step equivalent over $\mathbb{Q}$ if there are isomorphisms $f_{n}: \mathcal{M}_{n, \mathbb{Q}} \rightarrow \mathcal{M}_{n, \mathbb{Q}}^{\prime}$ and $e_{n}: H^{2}(A) \rightarrow$ $H^{2}\left(A^{\prime}\right)$ such that the diagram below commutes.

11.4. Relating the integer and rational 1-minimal models. We are now in a position to state and prove the main result of this section.

Theorem 11.4. Let $Y$ be a connected topological space with $H^{1}(Y ; \mathbb{Z})$ finitely generated. Then the 1-minimal model for $C^{*}(Y ; \mathbb{Z})$ tensored with the rationals is weakly equivalent as a differential graded algebra to the 1-minimal model in rational homotopy theory for $A_{\text {PL }}(Y)$.

Proof. Let $Y_{n}$ be a Postnikov 1-tower for $Y$, and let $\mathcal{M}=\left\{\mathcal{M}_{n, \mathbb{Z}}(Y), \rho_{n, \mathbb{Z}}\right\}_{n \geq 1}$ be an integral 1-minimal model for $Y$, with quasi-isomorphisms $\psi_{n}: \mathcal{M}_{n, \mathbb{Z}}(Y) \rightarrow C^{*}\left(Y_{n} ; \mathbb{Z}\right)$ as in Lemma 11.1. The proof is to show by induction that for the rational 1-minimal model $\left\{\mathcal{M}_{n, \mathbb{Q}}(Y), \rho_{n, \mathrm{Q}}\right\}_{n \geq 1}$ for $A_{\mathrm{PL}}(Y)$, there are 1-quasi-isomorphisms $e_{n}: \mathcal{M}_{n, \mathrm{Q}}(Y) \rightarrow A_{\mathrm{PL}}\left(Y_{n}\right)$ with $\rho_{n, \mathbb{Q}}=f_{n}^{*} \circ e_{n}$. The result then follows from the natural equivalences between $A_{\mathrm{PL}}\left(Y_{n}\right)$ and $C^{*}\left(Y_{n} ; \mathbb{Q}\right)$ and between $A_{\mathrm{PL}}(Y)$ and $C^{*}(Y ; \mathbb{Q})$.
Assume that for the sets $\boldsymbol{X}_{i}(i \geq 1)$, we have $\mathcal{M}_{n, \mathbb{Z}}(Y)=\mathrm{T}\left(\boldsymbol{X}_{1} \cup \cdots \cup \boldsymbol{X}_{n}\right)$. Then to prove the base case, set $\mathcal{M}_{1, \mathrm{Q}}=\Lambda\left(\boldsymbol{X}_{1}\right)$ and define $e_{1}: \Lambda\left(\boldsymbol{X}_{1}\right) \rightarrow A_{\mathrm{PL}}\left(Y_{1}\right)$ by setting $e_{1}(x)$ equal to a cocyle in $A_{\mathrm{PL}}\left(Y_{1}\right)$ whose cohomology class corresponds to $\psi_{1}(x)$ under the weak equivalence between $C^{*}\left(Y_{1} ; \mathbb{Q}\right)$ and $A_{\mathrm{PL}}\left(Y_{1}\right)$. Then $e_{1}$ is a 1-quasi-isomorphism and $\rho_{1, \mathrm{Q}}=f_{1}^{*} \circ e_{1}$ induces an isomorphism from $H^{1}\left(\mathcal{M}_{1, \mathrm{Q}}(Y)\right)$ to $H^{1}\left(A_{\mathrm{PL}}(Y)\right)$.
To prove the inductive step, assume $\mathcal{M}_{n, \mathbb{Q}}(Y)$ and a 1-quasi-isomorphism $e_{n}$ have been constructed. Consider the diagram (11.6). Note that the diagram of solid arrows commutes and each horizontal arrow represents a 1-quasi-isomorphism. Define $\mathcal{M}_{n+1, \mathbb{Q}}(Y)$ and $e_{n+1}$ as follows. Set $\mathcal{M}_{n+1, \mathrm{Q}}(Y)$ equal to the Hirsch extension $\left(\mathcal{M}_{n, \mathrm{Q}}(Y) \otimes \Lambda\left(\boldsymbol{X}_{n+1}\right), d_{n, \mathbb{Q}}\right)$,

where for each $x \in X_{n+1}$, its differential $d_{n, \mathbb{Q}}(x)$ is equal to a cocycle in $\mathcal{M}_{n, \mathbb{Q}}(Y)$ whose cohomology class corresponds to the cohomology class of $d_{n, \mathbb{Z}}(x) \in \mathcal{M}_{n, \mathbb{Z}}(Y) \otimes \mathbb{Q}$ under the 1-equivalence between $\mathcal{M}_{n, \mathbb{Z}}(Y) \otimes \mathbb{Q}$ and $\mathcal{M}_{n, \mathbb{Q}}(Y)$.
Given $x \in X_{n+1}$, set $e_{n+1}(x)$ equal to a cochain in $A_{\mathrm{PL}}\left(Y_{n+1}\right)$ whose coboundary equals $e_{n}\left(d_{n, \mathrm{Q}}(x)\right)$. Then $p_{n} \circ e_{n}=e_{n+1} \circ i_{n}$. As in the proof of Theorem 8.12, the map $e_{n+1}$ gives a map of the spectral sequence of the Hirsch extension $i_{n}$ to the spectral sequence of the fibration $p_{n}$ inducing an isomorphism of $E_{2}$ terms. Thus, $e_{n+1}$ is a quasi-isomorphism and the proof is complete.

## 12. Distinguishing homotopy types via 1-minimal models

In this section we use the 1 -minimal model of a binomial $\cup_{1}$-dga over $R=\mathbb{Z}$ or $\mathbb{Z}_{p}$ to define $n$-step equivalence for $n \geq 1$. We show in Theorem 12.4 that if $X$ and $X^{\prime}$ have isomorphic fundamental groups, then $C^{*}(X ; \mathbb{Z})$ and $C^{*}\left(X^{\prime} ; \mathbb{Z}\right)$ are $n$-step equivalent for all $n \geq 1$, and in Proposition 12.5 we give an example of a family of spaces that can be distinguished using $n$-step equivalence with $R=\mathbb{Z}$, where the corresponding approach in rational homotopy theory fails to distinguish the spaces.
12.1. $n$-step equivalence. Let $\left(A, d_{A}\right)$ be a binomial $\cup_{1}$-dga over the ring $R=\mathbb{Z}$ or $R=\mathbb{Z}_{p}$. We will assume throughout this section that $H^{0}(A)=R$ and $H^{1}(A)$ is a finitely generated, free $R$-module. By Theorem 9.8, there is a 1-minimal model, $(\mathcal{M}, d)$, which comes equipped with a structure map, $\rho: \mathcal{M} \rightarrow A$, that induces an isomorphism on $H^{1}$ and a monomorphism on $H^{2}$. Furthermore, by Theorem 10.3, any morphism $\varphi: A \rightarrow A^{\prime}$ between two such binomial $\cup_{1}$-dgas lifts to a morphism $\widehat{\varphi}: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ which is compatible with the respective colimit structures and with the structure maps (up to homotopy). The next result extracts further information from these data.

Proposition 12.1. Let $A$ and $A^{\prime}$ be two binomial $\cup_{1}$-dgas as above, with 1-minimal models $\rho: \mathcal{M} \rightarrow A$ and $\rho^{\prime}: \mathcal{M}^{\prime} \rightarrow A^{\prime}$. Let $\varphi: A \rightarrow A^{\prime}$ be a morphism, and let $\widehat{\varphi}: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ be a lift of $\varphi$. Then,
(1) The map $H^{2}(\varphi)$ induces homomorphisms $\bar{\varphi}_{n}: \operatorname{coker}\left(H^{2}\left(\rho_{n}\right)\right) \rightarrow \operatorname{coker}\left(H^{2}\left(\rho_{n}^{\prime}\right)\right)$, for all $n \geq 1$.
(2) If $\varphi$ is a 1-quasi-isomorphism, then all the homomorphisms $\bar{\varphi}_{n}$ are injective.

Proof. Since the morphism $\widehat{\varphi}: \mathcal{M} \rightarrow \mathcal{M}^{\prime}$ is compatible with the colimit structures, it restricts to $R$-linear maps $\widehat{\varphi}_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$. Since $\rho^{\prime} \circ \widehat{\varphi}$ is homotopic to $\varphi \circ \rho$ via a homotopy which is also compatible with the colimit structures, we have that $H^{2}\left(\rho^{\prime}\right) \circ$ $H^{2}(\varphi)=H^{2}(\varphi) \circ H^{2}(\rho)$ and similarly for the maps at level $n \geq 1$. We thus obtain for each $n \geq 1$ a commuting diagram in the category of $R$-modules,

where, by definition, $\bar{\varphi}_{n}$ is the $R$-linear map induced by $H^{2}(\varphi)$ on cokernels.
Now suppose $\varphi: A \rightarrow A^{\prime}$ is a 1-quasi-isomorphism. Then the map $H^{2}(\varphi): H^{2}(A) \rightarrow$ $H^{2}\left(A^{\prime}\right)$ is injective; moreover, by Theorem 10.3, the map $H^{2}(\widehat{\varphi})$ is an isomorphism. A straightforward diagram chase with (12.1) then shows that the map $\bar{\varphi}_{n}$ is injective, and the proof is complete.

Next we define an equivalence relation such that coker $H^{2}\left(\rho_{n}\right)$ is an invariant for each $n \geq 1$.

Definition 12.2. Given binomial cup-one dgas $A, A^{\prime}$ with 1-minimal models ( $\mathcal{M}, \rho$ ) and ( $\mathcal{M}^{\prime}, \rho^{\prime}$ ); respectively, and an integer $n \geq 1$ we say that $A$ and $A^{\prime}$ are $n$-step equivalent if there are isomorphsms $f_{n}: \mathcal{M}_{n} \rightarrow \mathcal{M}_{n}^{\prime}$ and $e_{n}: H^{2}(A) \rightarrow H^{2}\left(A^{\prime}\right)$ such that the diagram below commutes.


Note that if $A$ and $A^{\prime}$ are $q$-equivalent for some $q \geq 2$, then $A$ and $A^{\prime}$ are $n$-step equivalent for all $n \geq 1$.

For the rest of this paper, we will assume $R=\mathbb{Z}$ and that $H^{2}(A)$ and $H^{2}\left(A^{\prime}\right)$ are finitely generated. In this case, the cokernels of the maps $H^{2}\left(\rho_{n}\right)$ and $H^{2}\left(\rho_{n}^{\prime}\right)$ are also finitely
generated. Given a 1-minimal model $\rho: \mathcal{M} \rightarrow A$ over $\mathbb{Z}$, we define for each $n \geq 1$ a finite abelian group $\kappa_{n}(A)$ by

$$
\begin{equation*}
\kappa_{n}(A):=\operatorname{Tors}\left(\operatorname{coker}\left(H^{2}\left(\rho_{n}\right): H^{2}\left(\mathcal{M}_{n}\right) \rightarrow H^{2}(A)\right)\right) \tag{12.3}
\end{equation*}
$$

That is, $\kappa_{n}(A)$ is the torsion subgroup of the finitely generated abelian group coker $\left(H^{2}\left(\rho_{n}\right)\right)$.
From Proposition 12.1 it follows that the groups coker $H^{2}\left(\rho_{n}\right)$ and hence $\kappa_{n}(A)$ are independent of the choice of 1 -minimal model for $A$ used in constructing them. The considerations above yield the following proposition.
Proposition 12.3. Let $A$ and $A^{\prime}$ be binomial $\cup_{1}$-dgas over $\mathbb{Z}$, with $H^{0}=\mathbb{Z}$, $H^{1}$ finitely generated and torsion-free, and $H^{2}$ finitely generated. Let $(\mathcal{M}, \rho)$ and $\left(\mathcal{M}^{\prime}, \rho^{\prime}\right)$ be 1minimal models over $\mathbb{Z}$ for $A$ and $A^{\prime}$, respectively. Then, if $A$ and $A^{\prime}$ are $n$-step equivalent, then coker $H^{2}\left(\rho_{n}\right)$ is isomorphic to coker $H^{2}\left(\rho_{n}^{\prime}\right)$ and $\kappa_{n}(A)$ is isomorphic to $\kappa_{n}\left(A^{\prime}\right)$

Proof. The result follows from the definition of $n$-step equivalence by a diagram chase using the diagram (12.2) expanded, as in diagram (12.1), to include the cokernels.
12.2. Distinguishing homotopy 2-types. Making use of the above setup, we obtain in this section an invariant of 2-type for spaces based on a cohomological comparison between their integral cochain algebra and the corresponding 1-minimal model.
In this section, all our spaces are assumed to be connected $\Delta$-complexes $X$ with finitely generated cohomology groups $H^{1}(X ; \mathbb{Z})$ and $H^{2}(X ; \mathbb{Z})$ (for instance, presentation 2-complexes for finitely presented groups). Let $C^{*}(X ; \mathbb{Z})$ be the simplicial cochain algebra of such a space, viewed as a $\cup_{1}$-algebra over $\mathbb{Z}$. We consider the sequence of finite abelian groups $\kappa_{n}(X):=\kappa_{n}\left(C^{*}(X ; \mathbb{Z})\right)$, for $n \geq 1$, The next result shows that these groups depend only on the homotopy 2-type of $X$, or, equivalently, on the isomorphism type of its fundamental group.
Theorem 12.4. Let $X$ and $X^{\prime}$ be two $\Delta$-complexes as above.
(1) If $\pi_{1}(X) \cong \pi_{1}\left(X^{\prime}\right)$, then $\kappa_{n}(X) \cong \kappa_{n}^{\prime}(X)$ for all $n \geq 1$.
(2) If $\kappa_{n}(X) \not \not \kappa_{n}^{\prime}(X)$ for some $n \geq 1$, then the cochain algebras $C^{*}(X ; \mathbb{Z})$ and $C^{*}\left(X^{\prime} ; \mathbb{Z}\right)$ are not $n$-step equivalent.

Proof. Let $\rho: \mathcal{M} \rightarrow C^{*}(X ; \mathbb{Z})$ and $\rho^{\prime}: \mathcal{M}^{\prime} \rightarrow C^{*}\left(X^{\prime} ; \mathbb{Z}\right)$ be 1-minimal models for the respective cochain algebras. We begin by proving property (1) in the case where $X$ and $X^{\prime}$ are 2 -dimensional. By assumption, $\pi_{1}(X) \cong \pi_{1}\left(X^{\prime}\right)$, that is, $X$ and $X^{\prime}$ have the same 2-type. As in the proof of Theorem 3.6, it follows that there is a homotopy equivalence, $f: \bar{X} \rightarrow \overline{X^{\prime}}$, where $\bar{X}=X \vee \bigvee_{i \in I} S_{i}^{2}$ and $\overline{X^{\prime}}=X^{\prime} \vee \bigvee_{j \in J} S_{j}^{2}$, for some indexing sets $I$ and $J$. We let $q: \bar{X} \rightarrow X$ and $q^{\prime}: \overline{X^{\prime}} \rightarrow X^{\prime}$ be the maps that collapse the wedges of 2 -spheres to the basepoint, and we let $\bar{\rho}: \overline{\mathcal{M}} \rightarrow C^{*}(\bar{X} ; \mathbb{Z})$ and $\bar{\rho}^{\prime}: \overline{\mathcal{M}^{\prime}} \rightarrow C^{*}\left(\overline{X^{\prime}} ; \mathbb{Z}\right)$ be 1-minimal models for the respective cochain algebras.

By Theorem 10.3, the induced maps on cochain algebras, $f^{\sharp}, q^{\sharp}$, and $q^{\prime \#}$, lift to maps between 1 -minimal models. For each $n \geq 1$, this leads to the following diagram, which commutes up to homotopy.


Since all the maps on the bottom row are 1-quasi-isomorphisms, the maps on the top row are isomorphisms, again by Theorem 10.3. Applying now the $H^{2}(-)$ functor to diagram (12.4), we obtain the following commuting diagram.


From the way the collapse map $q$ is defined, the induced homomorphism $H^{2}(q)$ may be identified with the natural inclusion $H^{2}(X, \mathbb{Z}) \rightarrow H^{2}(X, \mathbb{Z}) \oplus \mathbb{Z}^{I}$; thus the induced map on cokernels restricts to an isomorphism on torsion subgroups. Similar considerations apply to the map $H^{2}\left(q^{\prime}\right)$, while of course $H^{2}(f)$ is an isomorphism. It follows that $\operatorname{Tors}\left(\operatorname{coker}\left(H^{2}\left(\bar{\rho}_{n}\right)\right)\right) \cong \operatorname{Tors}\left(\operatorname{coker}\left(H^{2}\left(\bar{\rho}_{n}^{\prime}\right)\right)\right)$. This completes the proof of property (1) in the case where $X$ and $X^{\prime}$ are 2-dimensional.
Now let $X$ be an arbitrary $\Delta$-complex satisfying our assumptions. We let $\left(C^{*}(X ; \mathbb{Z}), \delta^{*}\right)$ be its simplicial cochain complex, $i: X^{(2)} \hookrightarrow X$ the inclusion of the 2 -skeleton into $X$, and $\rho^{(2)}: \mathcal{M}^{(2)} \rightarrow C^{*}\left(X^{(2)} ; \mathbb{Z}\right)$ a 1-minimal model for $X^{(2)}$. The induced homomorphism $H^{2}(i): H^{2}(X ; \mathbb{Z}) \rightarrow H^{2}\left(X^{(2)} ; \mathbb{Z}\right)$ may be identified with the natural inclusion $H^{2}(X ; \mathbb{Z}) \hookrightarrow$ $H^{2}(X ; \mathbb{Z}) \oplus \operatorname{im}\left(\delta^{2}\right)$. Since $\operatorname{im}\left(\delta^{2}\right) \subset C^{3}(X ; \mathbb{Z})$ is a free abelian group, the map induced by $H^{2}(i)$ on cokernels, $\operatorname{coker}\left(H^{2}\left(\rho_{n}\right)\right) \rightarrow \operatorname{coker}\left(H^{2}\left(\rho_{n}^{(2)}\right)\right)$, restricts to an isomorphism on torsion subgroups, and the proof of property (1) is complete.

Part (2) follows at once from Corollary 12.3.
12.3. Integral versus rational 1-minimal models. In this final section we give a family of links in the 3 -sphere that can be distinguished from each other using the integral 1-minimal model, but are not distinguished from each other using the corresponding


Figure 2. Generalized Borromean Rings
approach in rational homotopy theory. We begin with a quick review of triple Massey products and the cohomology of link complements.
We refer to [15] for the general definition and properties of Massey products. Here we restrict our attention to triple Massey products of elements in the first cohomology. Given a dga $(A, d)$ and elements $u_{1}, u_{2}, u_{3}$ in $H^{1}(A)$ with $u_{1} u_{2}=u_{2} u_{3}=0$, let $c_{i}$ be cocycles with cohomology classes $\left[c_{i}\right]=u_{i}$, and let $c_{i, j}$ for $i<j$ be elements in $A^{1}$ with $d c_{i, j}=c_{i} c_{j}$. Then $c_{1} c_{2,3}+c_{1,2} c_{3}$ is a 2-cocyle; let $\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ denote the subset of $H^{2}(A)$ consisting of the set of cohomology classes [ $c_{1} c_{2,3}+c_{1,2} c_{3}$ ] of this type. It follows that the difference between any two elements in $\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ is an element in $u_{1} \cup H^{1}(A)+H^{1}(A) \cup u_{3}$.
Now let $L=\bigcup_{i=1}^{n} L_{i}$ be an $n$-component link in the 3 -sphere, with complement $X_{L}=S^{3} \backslash$ $L$. This space has the homotopy type of a finite, 2-dimensional CW-complex; moreover, $H^{1}\left(X_{L} ; \mathbb{Z}\right)=\mathbb{Z}^{n}$, with basis elements corresponding by Alexander duality to generators of $H_{1}\left(L_{i} ; \mathbb{Z}\right)$, and with $H^{2}\left(X_{L} ; \mathbb{Z}\right)=\mathbb{Z}^{n-1}$ generated by the Lefschetz duals $\gamma_{i j}$ to paths from $L_{i}$ to $L_{j}$. A formula for the Massey products of elements in the first cohomology of a CW-complex in terms of the Magnus coefficients of the attaching maps of the 2-cells is given in [16], along with a proof that Massey products in the complement of a link and Milnor's $\bar{\mu}$-invariants of the link determine each other.

Consider now the following family of links in the 3 -sphere. For each $n \geq 1$, let $L(n)$ be the link pictured in Figure 2, where the pattern in the middle repeats $n$ times. Note that $L(1)$ is the well-known Borromean rings, and that the three components of $L(n)$ have pairwise linking numbers equal to 0 , for all $n \geq 1$. We denote by $X(n)=X_{L(n)}$ the complement of $L(n)$ in $S^{3}$, and we let $\rho(n): \mathcal{M}(n) \rightarrow A(n)$ be a 1-minimal model for the cochain algebra $A(n):=C^{*}(X(n) ; \mathbb{Z})$.

Proposition 12.5. For the links $L(n)$ with complement $X(n)$ described above, we have:
(1) The cokernel of $H^{2}\left(\rho_{2}(n)\right)$ is isomorphic to $\mathbb{Z}_{n} \oplus \mathbb{Z}_{n}$, and so $\kappa_{2}(X(n))=\mathbb{Z}_{n} \oplus \mathbb{Z}_{n}$.
(2) The cochain algebras $C^{*}(X(m) ; \mathbb{Z})$ and $C^{*}(X(n) ; \mathbb{Z})$ are not 2-step equivalent for $m \neq n$.
(3) The Sullivan algebras $A_{\mathrm{PL}}(X(m))$ and $A_{\mathrm{PL}}(X(n))$ are 2 -step equivalent for all positive integers $m$ and $n$.

Proof. (1) To compute the cokernel of $H^{2}\left(\rho_{2}(n)\right)$, we proceed as follows. First note that $\mathcal{M}_{1}(n)=\left(\mathrm{T}\left(\left\{x_{1}, x_{2}, x_{3}\right\}, d_{0}\right)\right.$, with $\rho_{1}(n)\left(x_{i}\right)$ equal to a cocycle whose cohomology class $u_{i}$ is Alexander dual to the generator of $H_{1}\left(L_{i} ; \mathbb{Z}\right)$ given by the orientation of $L_{i}$.

Since all linking numbers for $L(n)$ are zero, we have that all cup products of elements in $H^{1}(X(n) ; \mathbb{Z})$ are zero, and it follows that $\mathcal{M}_{2}(n)=\left(\mathrm{T}\left(\left\{x_{1}, x_{2}, x_{3}, x_{1,2}, x_{1,3}, x_{2,3}\right\}, d_{2}(n)\right)\right.$ where $d_{2}(n)\left(x_{i, j}\right)=x_{i} \otimes x_{j}$ and $\rho_{2}(n)$ is any extension of $\rho_{1}(n)$. In the spectral sequence of the Hirsch extension $\mathcal{M}_{1}(n) \hookrightarrow \mathcal{M}_{2}(n)$, it follows from Theorem 8.9 that the $E_{2}$ term is the tensor product of two exterior algebras,

$$
\begin{equation*}
E_{2}=\bigwedge\left(\left[x_{1}\right],\left[x_{2}\right],\left[x_{3}\right]\right) \otimes_{\mathbb{Z}} \wedge\left(\left[x_{1,2}\right],\left[x_{2,3}\right],\left[x_{1,3}\right]\right), \tag{12.6}
\end{equation*}
$$

where $[x]$ denotes the image of $x$ in $E_{2}$. The differential $d_{2}: E_{2} \rightarrow E_{2}$ is given by $d_{2}\left[x_{i}\right]=0$ and $d_{2}\left[x_{i, j}\right]=\left[x_{i} x_{j}\right]$. Then by direct computation of the $E_{\infty}^{p, q}$ terms with $p+q=2$, it follows that $H^{2}\left(\mathcal{M}_{2}(n)\right)=\mathbb{Z}^{8}$, with basis given by the triple Massey products

$$
\begin{align*}
&\left\langle\left[x_{1}\right],\right. {\left.\left[x_{1}\right],\left[x_{2}\right]\right\rangle,\left\langle\left[x_{1}\right],\left[x_{2}\right],\left[x_{2}\right]\right\rangle,\left\langle\left[x_{1}\right],\left[x_{1}\right],\left[x_{3}\right]\right\rangle,\left\langle\left[x_{1}\right],\left[x_{3}\right],\left[x_{3}\right]\right\rangle, } \\
& \quad\left\langle\left[x_{2}\right],\left[x_{2}\right],\left[x_{3}\right]\right\rangle,\left\langle\left[x_{2}\right],\left[x_{3}\right],\left[x_{3}\right]\right\rangle,\left\langle\left[x_{1}\right],\left[x_{2}\right],\left[x_{3}\right]\right\rangle,\left\langle\left[x_{1}\right],\left[x_{3}\right],\left[x_{2}\right]\right\rangle . \tag{12.7}
\end{align*}
$$

The correspondence between Massey products and elements in $E_{\infty}$ is indicated as follows. An element in $E_{\infty}^{1,1}$ such as $\left[x_{1}\right] \otimes\left[x_{1,2}\right]$ is represented by the element in the set $\left\langle\left[x_{1}\right],\left[x_{1}\right],\left[x_{2}\right]\right\rangle$ given by the cohomology class of the cocycle $x_{1} \otimes x_{1,2}-\zeta_{2}\left(x_{1}\right) \otimes x_{2}$ in $\mathcal{M}_{2}(n)$ and $\left\langle\left[x_{1}\right],\left[x_{2}\right],\left[x_{3}\right]\right\rangle$ is taken to be the cohomology class of the cocycle $x_{1} \otimes x_{2,3}+$ $x_{1,2} \otimes x_{3}$, which corresponds to the element $\left[x_{1}\right] \otimes\left[x_{2,3}\right]-\left[x_{3}\right] \otimes\left[x_{1,2}\right]$ in $E_{\infty}^{1,1}$.
We now determine the homomorphism $H^{2}\left(\rho_{2}(n)\right)$. From the naturality of Massey products, it follows that $H^{2}\left(\rho_{2}(n)\right)$ sends a Massey product $\left\langle\left[x_{i}\right],\left[x_{j}\right],\left[x_{k}\right]\right\rangle$ to $\left\langle u_{i}, u_{j}, u_{k}\right\rangle$. All cup products of elements in $H^{1}(X(n) ; \mathbb{Z})$ are zero, so each triple Massey product contains only one element. Since each 2 -component sublink of $L$ is equivalent to the unlink, it follows that each of the first six Massey products listed above maps to zero. From the computations in [16, Example 3, p. 46], we have that $\left\langle u_{1}, u_{2}, u_{3}\right\rangle=-n \gamma_{1,3}$ and $\left\langle u_{1}, u_{3}, u_{2}\right\rangle=n \gamma_{1,2}$. Since $\left\{\gamma_{1,3}, \gamma_{1,2}\right\}$ is a basis for $H^{2}(X(n) ; \mathbb{Z})$, the argument that coker $H^{2}\left(\rho_{2}(n)\right)=\mathbb{Z}_{n} \oplus \mathbb{Z}_{n}$ is complete.
Part (2) follows at once from Part (1) and Theorem 12.4.
(3) We now show that if the binomial cup-one dga $C^{*}(X(n) ; \mathbb{Z})$ is replaced by the cdga $B(n)=A_{\mathrm{PL}}(X(n))$ and the integral 1-minimal model $\mathcal{M}$ is replaced with the Sullivan rational 1-minimal model $\mathcal{M}_{\mathbb{Q}} \simeq \mathcal{M} \otimes \mathbb{Q}$ from [25], then $B(m)$ and $B(n)$ are 2-step equivalent for all positive integers $m$ and $n$.

Let $\rho_{\mathbb{Q}}(n):\left(\mathcal{M}_{\mathbb{Q}}, d\right) \rightarrow B(n)$ be the rational 1-minimal model for $B(n)$. Then $\mathcal{M}_{2, \mathbb{Q}}$ is the exterior algebra over $\mathbb{Q}$ given by $\bigwedge\left(y_{1}, y_{2}, y_{3}, y_{1,2}, y_{1,3}, y_{2,3}\right)$, with differential given by $d y_{i}=0$ and $d y_{i, j}=y_{i} \wedge y_{j}$. Let $v_{i} \in H^{1}(B(n))$ be the elements that correspond to the elements $u_{i} \in H^{1}(X(n) ; \mathbb{Q})$, and let $\omega_{i, j} \in H^{2}(B(n))$ be the elements that correspond to the elements $\gamma_{i, j} \in H^{2}(X(n) ; \mathbb{Q})$ via the zig-zag of quasi-isomorphisms

$$
\begin{equation*}
C^{*}(X(n) ; \mathbb{Q}) \longrightarrow C A(X(n)) \longleftarrow A_{\mathrm{PL}}(X(n))=B(n) . \tag{12.8}
\end{equation*}
$$

We can assume that $\rho_{1, \mathrm{Q}}\left(y_{i}\right)=v_{i}$ for $i \in\{1,2,3\}$. Since the maps in equation (12.8) are dga maps inducing isomorphisms on cohomology, it follows from the computation of Massey products in $C^{*}(X(n) ; \mathbb{Q})$ that in $H^{2}(B(n))$ we have $\left\langle v_{1}, v_{2}, v_{3}\right\rangle=-n \omega_{1,3}$ and $\left\langle v_{1}, v_{3}, v_{2}\right\rangle=n \omega_{1,2}$, while all triple products of the form $\left\langle v_{i}, v_{j}, v_{k}\right\rangle$ with $\{i, j, k\}$ a proper subset of $\{1,2,3\}$ are zero. It follows that $H^{2}\left(\rho_{2}\right)\left\langle\left[y_{1}\right],\left[y_{2}\right],\left[y_{3}\right]\right\rangle=-n \omega_{1,3}$ and $H^{2}\left(\rho_{2}\right)\left\langle\left[y_{1}\right],\left[y_{3}\right],\left[y_{2}\right]\right\rangle=n \omega_{1,2}$, while $H^{2}\left(\rho_{2}\right)\left\langle\left[y_{i}\right],\left[y_{j}\right],\left[y_{k}\right]\right\rangle=0$ if $\{i, j, k\} \varsubsetneqq\{1,2,3\}$.
Now given positive integers $n$ and $m$, we define a homomorphism $e_{n, m}: H^{2}(B(n)) \rightarrow$ $H^{2}(B(m))$ by $\omega_{i, j} \mapsto \frac{m}{n} \cdot \omega_{i, j}$ for $i=1$ and $j \in\{2,3\}$. Then $e_{n, m}$ is an isomorphism and the following diagram commutes,


The argument that over the rationals the link complements $X(m)$ and $X(n)$ are 2-step equivalent for all $m, n \geq 1$ is complete.

## References

[1] K.S. Brown, Cohomology of groups, Graduate texts in Mathematics, vol. 87, Springer, New York, Berlin, 1982. 2.3
[2] P.-J. Cahen and J.-L. Chabert, Integer-valued polynomials, Math. Surveys Monogr., vol. 48, Amer. Math. Soc., Providence, RI, 1997. 5.1, 5.3
[3] H. Cartan and S. Eilenberg, Homological algebra Princeton Univ. Press, Princeton, NJ, 1956. 2.3
[4] S. Eilenberg, S. Mac Lane, On the groups H(П, n). I., Ann. of Math. (2) 58 (1953), no. 1, 55-106. 2.3
[5] J. Elliott, Binomial rings, integer-valued polynomials, and $\lambda$-rings, J. Pure Appl. Algebra 207 (2006), no. 1, 165-185. 5.1, 5.1, 5.1, 5.3, 5.3
[6] Y. Félix, S. Halperin, and J.-C. Thomas, Rational homotopy theory, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. 1.1, 1.8, 3.3, 9.2, 10.5, 10.5, 11.2, 11.3
[7] Y. Félix, S. Halperin, and J.-C. Thomas, Rational homotopy theory II, World Scientific Publishing, Hackensack, NJ, 2015. 1.1
[8] G. Friedman, An elementary illustrated introduction to simplicial sets, Rocky Mountain J. Math. 42 (2012), no. 2, 353-423. 2.1, 3.2
[9] P. Griffiths, J. Morgan, Rational homotopy theory and differential forms, Second ed., Progr. Math., vol. 16, Springer, New York, 2013. 1.1, 10.5, 10.5, 11.3
[10] P. Hall, The Edmonton notes on nilpotent groups, Queen Mary College Mathematics Notes, Queen Mary College, 1976. 5.1
[11] S. Halperin, Lectures on minimal models, Mémoires de la Soc. Math. France 9-10 (1983), 1-261. 3.3
[12] A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002. 2.1, 3.2
[13] G. Hirsch, Quelques propriétés des produits de Steenrod, C. R. Acad. Sci. Paris 241 (1955), 923-925. 1.2, 4.1
[14] S. MacLane, Homology, Die Grundlehren der mathematischen Wissenschaften, Vol. 114, SpringerVerlag, Berlin-Göttingen-Heidelberg, 1963. 2.3
[15] J.P. May, Matric Massey products, J. Algebra, 12 (1969), 533-568. 12.3
[16] R. Porter, Milnor's $\bar{\mu}$-invariants and Massey products, Trans. Amer. Math. Soc. 257 (1980), no. 1, 39-71. 12.3, 12.3
[17] R.D. Porter and A.I. Suciu, Homology, lower central series, and hyperplane arrangements, Eur. J. Math. 6 (2020), nr. 3, 1039-1072. 1.9
[18] R.D. Porter and A.I. Suciu, Differential graded algebras, Steenrod cup-one products, binomial operations, and Massey products, Topology Appl. 313 (2022), Paper No. 107987, 37 pp. 1.3, 1.3, 1.4, 3.2, 3.6, 4.1, 4.4, 4.4, 4.7, 5.3, 5.5, 5.8, 5.9, 5.5, 5.6, 6.1, 6.1, 6.4, 6.8, 7.1, 7.6, 7.6
[19] R.D. Porter and A.I. Suciu, Groups associated to 1-minimal models for binomial cup-one algebras, draft (2023). 1.6, 1.9
[20] R.D. Porter and A.I. Suciu, Generalized triple Massey products, draft (2022). 1.9
[21] C.P. Rourke, B.J. Sanderson, $\Delta$-sets. I. Homotopy theory, Quart. J. Math. Oxford Ser. (2) 22 (1971), 321-338. 2.1, 3.2
[22] N.E. Steenrod, Products of cocycles and extensions of mappings, Ann. of Math. 48 (1947), 290-320. 1.2, 4.1, 4.1, 4.2
[23] A.I. Suciu, Formality and finiteness in rational homotopy theory, EMS Surveys in Mathematical Sciences (to appear), arXiv:2210.08310. 11.3
[24] A.I. Suciu and H. Wang, Formality properties of finitely generated groups and Lie algebras, Forum Math. 31 (2019), no. 4, 867-905. 11.3
[25] D. Sullivan, Infinitesimal computations in topology, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269-331. 1.8, 11.2, 11.3, 12.3
[26] C.T.C. Whitehead, Combinatorial homotopy. I., Bull. Amer. Math. Soc. 55 (1949), 213-245. 3.2, 3.2
[27] D. Yau, Lambda-rings, World Scientific, Hackensack, NJ, 2010. 5.3


[^0]:    ${ }^{1}$ Department of Mathematics, Northeastern University, Boston, MA 02115, USA
    E-mail addresses: r.porter@northeastern.edu, a.suciu@northeastern.edu.
    Date: June 22, 2023.
    2010 Mathematics Subject Classification. Primary 16E45. Secondary 13F20, 20F18, 20J05, 55N45, 55P62, 55S05, 55U10.

    Key words and phrases. Differential graded algebras, cochain algebras, Steenrod cup-i products, binomial rings, Hirsch extensions, minimal models, Massey products.
    ${ }^{2}$ Supported in part by the Simons Foundation Collaboration Grant for Mathematicians \#693825.

