TOPOLOGY OF HYPERPLANE ARRANGEMENTS

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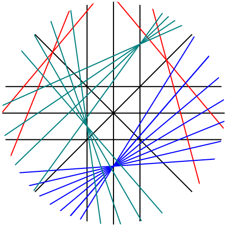
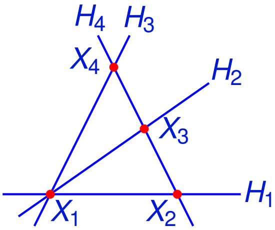
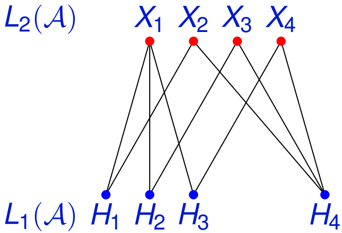


HYPERPLANE ARRANGEMENTS

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $\mathbb{C}^\ell$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.





- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$.

EXAMPLE (THE BOOLEAN ARRANGEMENT)

- $\mathcal{B}_n$: all coordinate hyperplanes $\{z_i = 0\}$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.

EXAMPLE (THE BRAID ARRANGEMENT)

- $\mathcal{A}_n$: all diagonal hyperplanes $\{z_i - z_j = 0\}$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for P_n, the pure braid group on n strings).

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i \colon \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$. Define an injective linear map

$$\iota \colon \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota \colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Hence, $M(\mathcal{A}) = \iota(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n$ is a Stein manifold.
- Therefore, $M = M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension ℓ.
- In fact, M has a minimal cell structure. Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.
- Let $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A})) = \mathbb{CP}^{\ell-1} \setminus \bigcup_{H\in\mathcal{A}} \mathbb{P}(H)$ be the projectivized complement. Then $M(\mathcal{A}) \cong U(\mathcal{A}) \times \mathbb{C}^*$.

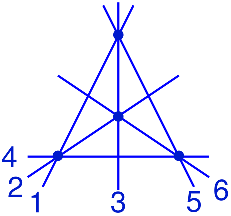
COHOMOLOGY RING

- The Betti numbers $b_q(M) := \operatorname{rank} H_q(M, \mathbb{Z})$ are given by

$$\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$

  where $\mu \colon L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function, defined recursively by $\mu(\mathbb{C}^{\ell}) = 1$ and $\mu(X) = -\sum_{Y \supsetneq X} \mu(Y)$.
- The logarithmic 1-forms $\omega_H = \frac{1}{2\pi \mathrm{i}} d \log f_H \in \Omega_{\mathrm{dR}}(M)$ are closed.
- Let E be the $\mathbb{Z}$-exterior algebra on the degree 1 cohomology classes $e_H = [\omega_H]$ dual to the meridians x_H around $H \in \mathcal{A}$.
- Let $\partial \colon E^* \to E^{*-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_X = \prod_{H \supseteq X} e_H$ for each $X \in L(A)$.
- The cohomology ring $A(\mathcal{A}) = H^*(M; \mathbb{Z})$ is isomorphic to the Orlik–Solomon algebra E/I, where $I = \langle \partial e_X : \operatorname{rank}(X) < |X| \rangle$.
- Hence, $A(\mathcal{A})$ is determined by $L(\mathcal{A})$.

EXAMPLE



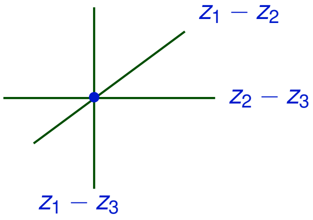
- $E = \bigwedge(e_1, \dots, e_6)$
- $I = \langle (e_1 - e_4)(e_2 - e_4), (e_1 - e_5)(e_3 - e_5), (e_2 - e_6)(e_3 - e_6), (e_4 - e_6)(e_5 - e_6) \rangle$

- The map $e_H \mapsto \omega_H$ extends to a cdga quasi-isomorphism, $(H^*(M_{\mathcal{A}}, \mathbb{R}), d = 0) \to \Omega^*_{\mathrm{dR}}(M_{\mathcal{A}})$. Therefore, $M(\mathcal{A})$ is formal.
- $M(\mathcal{A})$ is minimally pure (i.e., $H^k(M(\mathcal{A}), \mathbb{Q})$ is pure of weight $2k$, for all k), which again implies formality (Dupont 2016).
- D. Matei: For each prime p, there is an $\mathcal{A}$ such that $H^*(M; \mathbb{Z}_p)$ has non-vanishing Massey products, and so M is not $\mathbb{Z}_p$-formal.
- If $L(\mathcal{A})$ is supersolvable, then $A(\mathcal{A})$ admits a quadratic Gröbner basis, and thus it is a Koszul algebra. Does the converse hold?

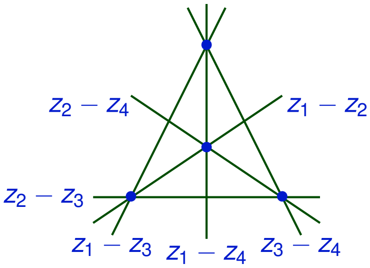
LINE ARRANGEMENTS

- Let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar slice of $\mathcal{A}$. Then the arrangement group, $G = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.
- So, for the purpose of studying π_1's, it is enough to consider arrangements of affine lines in $\mathbb{C}^2$, or projective lines in $\mathbb{CP}^2$.

EXAMPLE



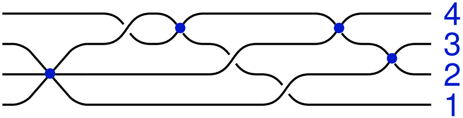
$G = P_3 \cong F_2 \times \mathbb{Z}$



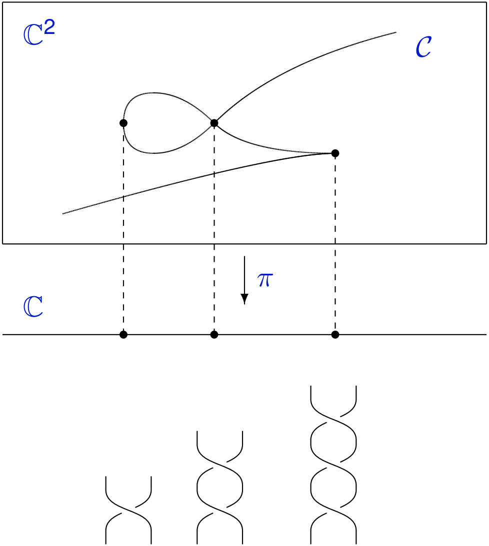
$G = P_4 \cong F_3 \rtimes P_3$

FUNDAMENTAL GROUPS OF ARRANGEMENTS

- Let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar section of $\mathcal{A}$. Then the arrangement group, $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.
- So let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$. Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $s = \#\{\text{multiple points}\}$ and the braids $\alpha_r \in P_n$ can be read off an associated braided wiring diagram,



- The group $G(\mathcal{A})$ has a presentation with meridional generators $x_1, \dots, x_n$ and commutator relators $x_i \alpha_j (x_i)^{-1}$.


$\mathbb{C}^2$
$\mathcal{C}$
π
$\mathbb{C}$

ASSOCIATED GRADED LIE ALGEBRA

- Let G be a group. The *lower central series* $\{\gamma_k(G)\}_{k\geqslant 1}$ is defined inductively by $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- Here, if $H, K < G$, then $[H, K]$ is the subgroup of G generated by $\{[a, b] := aba^{-1}b^{-1} \mid a \in H, b \in K\}$. If $H, K \lhd G$, then $[H, K] \lhd G$.
- The subgroups $\gamma_k(G)$ are, in fact, characteristic subgroups of G. Moreover, $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, $\forall k, \ell \geqslant 1$.
- In particular, it is a *central* series, i.e., $[G, \gamma_k(G)] \subseteq \gamma_{k+1}(G)$.
- In fact, it is the fastest descending central series for G.
- It is also a *normal* series, i.e., $\gamma_k(G) \lhd G$. Each quotient,

$$\operatorname{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$$

  lies in the center of $G/\gamma_{k+1}(G)$, and thus is an abelian group.
- If G is finitely generated, then so are its LCS quotients. Set $\phi_k(G) := \operatorname{rank} \operatorname{gr}_k(G)$.

- For a coefficient ring $\Bbbk$, we let $\mathrm{gr}(G;\Bbbk) = \bigoplus_{k\geq 1} \mathrm{gr}_k(G) \otimes \Bbbk$.
- This is a graded Lie algebra, with addition induced by the group multiplication and with Lie bracket $[\,,\,]\colon \mathrm{gr}_k \times \mathrm{gr}_\ell \to \mathrm{gr}_{k+\ell}$ induced by the group commutator.
- The construction is functorial. Write $\mathrm{gr}(G) = \mathrm{gr}(G;\mathbb{Z})$.
- Example: if F_n is the free group of rank n, then
  - $\mathrm{gr}(F_n)$ is the free Lie algebra $\mathrm{Lie}(\mathbb{Z}^n)$.
  - $\mathrm{gr}_k(F_n)$ is free abelian, of rank $\phi_k(F_n) = \frac{1}{k}\sum_{d|k}\mu(d)n^{\frac{k}{d}}$.
- $G/\gamma_k(G)$ is the maximal $(k-1)$-step nilpotent quotient of G.
- $G/\gamma_2(G) = G_{\mathrm{ab}}$, while $G/\gamma_3(G) \leftrightarrow H^{\leqslant 2}(G;\mathbb{Z})$.

CHEN LIE ALGEBRAS

- Let $G^{(i)}$ be the *derived series* of G, starting at $G^{(1)} = G'$, $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.
- The quotient groups, $G/G^{(i)}$, are solvable; $G/G' = G_{\mathrm{ab}}$, while G/G'' is the maximal metabelian quotient of G.
- The *i-th Chen Lie algebra* of G is defined as $\mathrm{gr}(G/G^{(i)}; \Bbbk)$.
- The projection $q_i \colon G \twoheadrightarrow G/G^{(i)}$, induces a surjection $\mathrm{gr}_k(G; \Bbbk) \twoheadrightarrow \mathrm{gr}_k(G/G^{(i)}; \Bbbk)$, which is an iso for $k \leqslant 2^i - 1$.
- Assuming G is finitely generated, write $\theta_k(G) = \operatorname{rank} \mathrm{gr}_k(G/G'')$ for the *Chen ranks*. We have $\phi_k(G) \geqslant \theta_k(G)$, with equality for $k \leqslant 3$.
- Example (K.-T. Chen 1951): $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, for $k \geqslant 2$.

HOLONOMY LIE ALGEBRA

- A quadratic approximation of the Lie algebra $\operatorname{gr}(G;\Bbbk)$, where $\Bbbk$ is a field, is the *holonomy Lie algebra* of G, defined as

$$\mathfrak{h}(G;\Bbbk) := \operatorname{Lie}(H_1(G;\Bbbk))/\langle \operatorname{im}(\mu_G^\vee)\rangle,$$

  where
  - $L = \operatorname{Lie}(V)$ the free Lie algebra on the $\Bbbk$-vector space $V = H_1(G;\Bbbk)$, with $L_1 = V$ and $L_2 = V \wedge V$;
  - $\mu_G^\vee\colon H_2(G;\Bbbk) \to V \wedge V$ is the dual of the cup product map $\mu_G\colon H^1(G;\Bbbk) \wedge H^1(G;\Bbbk) \to H^2(G;\Bbbk)$.

- There is natural epimorphism of graded Lie algebras, $\mathfrak{h}(G;\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$, which restricts to isos in degrees 1 and 2.

- For each $i \geqslant 2$, this morphism factors through epimorphisms $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G;\Bbbk)^{(i)} \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\Bbbk)$.

Lie algebras associated to arrangements

- The holonomy Lie algebra of $G = G(\mathcal{A})$ is determined by $L_{\leqslant 2}(\mathcal{A})$,

$$\mathfrak{h}(G) = \operatorname{Lie}(x_H : H \in \mathcal{A}) \Big/ \text{ideal}\left\{ \Big[x_H, \sum_{\substack{K \in \mathcal{A} \\ K \supset Y}} x_K \Big] : \begin{array}{l} H \in \mathcal{A}, Y \in L_2(\mathcal{A}) \\ H \supset Y \end{array} \right\}.$$

- Since M is formal, the group G is 1-formal. Hence, $\operatorname{gr}(G) \otimes \mathbb{Q}$ is determined by $H^{\leqslant 2}(M, \mathbb{Q})$, and thus, by $L_{\leqslant 2}(\mathcal{A})$.
- In fact, the surjection $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ induces an isomorphism, $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \operatorname{gr}(G) \otimes \mathbb{Q}$.
- (Papadima–S. 2004) The Chen ranks $\theta_k(G)$ are also determined by $L_{\leqslant 2}(\mathcal{A})$.
- Explicit combinatorial formulas for the LCS ranks $\phi_k(G)$ are known in some cases, but not in general.

- (Falk–Randell 1985) If $\mathcal{A}$ is supersolvable with exponents $d_1, \dots, d_q$, then $\phi_k(G) = \sum_{i=1}^{q} \phi_k(F_{d_i})$. (Also follows from Koszulity of $H^*(M, \mathbb{Q})$ and Koszul duality.)
- (Porter–S. 2020) The map $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism, but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- (S. 2002) The groups $\mathrm{gr}_k(G)$ may have non-zero torsion for $k \gg 0$. E.g., if $G = G(\text{MacLane})$, then $\mathrm{gr}_5(G) = \mathbb{Z}^{87} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_3$.
- (S. 2002): Is the torsion in $\mathrm{gr}(G)$ combinatorially determined?
- (Artal Bartolo, Guerville-Ballé, and Viu-Sos 2020): Answer: No!
- There are two arrangements of 13 lines, $\mathcal{A}^{\pm}$, each one with 11 triple points and 2 quintuple points, such that $\mathrm{gr}_k(G^+) \cong \mathrm{gr}_k(G^-)$ for $k \leqslant 3$, yet $\mathrm{gr}_4(G^+) = \mathbb{Z}^{211} \oplus \mathbb{Z}_2$ and $\mathrm{gr}_4(G^-) = \mathbb{Z}^{211}$.

NILPOTENT QUOTIENTS

- The quotient $G/\gamma_3(G)$ is determined by $L_{\leqslant 2}(\mathcal{A})$. Indeed, in the central extension,

$$0 \longrightarrow \mathrm{gr}_2(G) \longrightarrow G/\gamma_3(G) \longrightarrow G_{\mathrm{ab}} \longrightarrow 0,$$

  we have $\mathrm{gr}_2(G) = (I^2)^\vee$ and the k-invariant $H_2(G_{\mathrm{ab}}) \to \mathrm{gr}_2(G)$ is dual of the inclusion $I^2 \hookrightarrow E^2 = \bigwedge^2 G_{\mathrm{ab}}$.

- (G. Rybnikov 1994): $G/\gamma_4(G)$ is not always determined by $L_{\leqslant 2}(\mathcal{A})$.

- There are two arrangements of 13 lines, $\mathcal{A}^{\pm}$, each one with 15 triple points, such that $L(\mathcal{A}^+) \cong L(\mathcal{A}^-)$, and therefore $G^+/\gamma_3(G^+) \cong G^-/\gamma_3(G^-)$ and $\mathrm{gr}_3(G^+) \cong \mathrm{gr}_3(G^-)$, but $G^+/\gamma_4(G^+) \not\cong G^-/\gamma_4(G^-)$.

- The difference can be explained in terms of (generalized) Massey triple products over $\mathbb{Z}_3$.

DECOMPOSABLE ARRANGEMENTS

- For each flat $X \in L(\mathcal{A})$, let $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$ be the localization of $\mathcal{A}$ at X.
- The inclusions $\mathcal{A}_X \subset \mathcal{A}$ give rise to maps $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$. Restricting to rank 2 flats yields a map

$$j \colon M(\mathcal{A}) \longrightarrow \prod_{X \in L_2(\mathcal{A})} M(\mathcal{A}_X)\,.$$

- The induced homomorphism on fundamental groups, $j_\sharp$, defines a morphism of graded Lie algebras,

$$\mathfrak{h}(j_\sharp) \colon \mathfrak{h}(G) \longrightarrow \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}(G_X)\,.$$

THEOREM (PAPADIMA–S. 2006)

The map $\mathfrak{h}_k(j_\sharp)$ is a surjection for each $k \geqslant 3$ and an iso for $k = 2$.

DEFINITION

$\mathcal{A}$ is *decomposable* if the map $\mathfrak{h}_3(j_\sharp)$ is an isomorphism.

EXAMPLE

Let $\mathcal{A}(\Gamma) = \{z_i - z_j = 0 : (i,j) \in \mathsf{E}(\Gamma)\} \subset \mathcal{A}_n$ be a graphic arrangement. Then $\mathcal{A}(\Gamma)$ is decomposable if and only if Γ contains no K_4 subgraph.

THEOREM (PAPADIMA–S. 2006)

Let $\mathcal{A}$ be a decomposable arrangement, and let $G = G(\mathcal{A})$. Then

- *The map $\mathfrak{h}'(j_\sharp)\colon \mathfrak{h}'(G) \to \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}'(G_X)$ is an isomorphism of graded Lie algebras.*
- *The map $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ is an isomorphism*
- *For each $k \geqslant 2$, the group $\operatorname{gr}_k(G)$ is free abelian of rank $\phi_k(G) = \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})$.*

THEOREM (PORTER–S. 2020)

Let $\mathcal{A}$ and $\mathcal{B}$ be decomposable arrangements with $L_{\leqslant 2}(\mathcal{A}) \cong L_{\leqslant 2}(\mathcal{B})$. Then, for each $k \geqslant 2$,

$$G(\mathcal{A})/\gamma_k(G(\mathcal{A})) \cong G(\mathcal{B})/\gamma_k(G(\mathcal{B})).$$

Resonance varieties

- Let X be a connected, finite cell complex,
- Let $A = H^*(X, \Bbbk)$, where $\operatorname{char} \Bbbk \neq 2$. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- We thus get a cochain complex

$$(A, \cdot a)\colon \ A^0 \xrightarrow{\ a\ } A^1 \xrightarrow{\ a\ } A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are the jump loci for the cohomology of this complex

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\}$$

- E.g., $\mathcal{R}^1_1(X, \Bbbk) = \{a \in A^1 \mid \exists b \in A^1,\ b \neq \lambda a,\ ab = 0\}$.
- These loci are homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$. In general, they can be arbitrarily complicated.

RESONANCE VARIETIES OF ARRANGEMENTS

Work of Arapura, Falk, D.Cohen, A.S., Libgober, and Yuzvinsky, completely describes the varieties $\mathcal{R}_s(\mathcal{A}) = \mathcal{R}^1_s(M(\mathcal{A}), \mathbb{C})$.

- $\mathcal{R}_1(\mathcal{A})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A})$ is the union of those linear subspaces that have dimension at least $s+1$.
- Each k-*multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of $\mathcal{R}_1(\mathcal{A})$ of dimension $k-1$. Moreover, all components of $\mathcal{R}_1(\mathcal{A})$ arise in this way.

Multinets

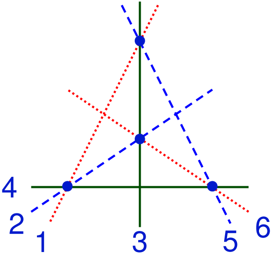
Definition (Falk and Yuzvinsky)

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, such that:

- $\exists\, d \in \mathbb{N}$ such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
- If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
- $\forall\, X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha : H \supset X} m_H$ is independent of α.
- $\left(\bigcup_{H \in \mathcal{A}_\alpha} H\right) \setminus \mathcal{X}$ is connected, for each α.

- Such a multinet is also called a (k, d)-multinet, or k-multinet.
- It is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$.
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.

EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from the triple points), and one essential component, from the above $(3,2)$-net:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

Characteristic varieties

- Let X be a connected, finite cell complex, let $G = \pi_1(X, x_0)$, and let $\mathrm{Hom}(G, \mathbb{C}^*)$ be the affine algebraic group of $\mathbb{C}$-valued, multiplicative characters on G.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}^q_s(X) = \{\rho \in \mathrm{Hom}(G, \mathbb{C}^*) \mid \dim H_q(X, \mathbb{C}_\rho) \geqslant s\}.$$

Here, $\mathbb{C}_\rho$ is the local system defined by ρ, i.e, $\mathbb{C}$ viewed as a $\mathbb{C}[G]$-module via $g \cdot x = \rho(g)x$, and $H_i(X, \mathbb{C}_\rho) = H_i(C_*(\widetilde{X}, \mathbb{C}) \otimes_{\mathbb{C}[G]} \mathbb{C}_\rho)$.

- These loci are Zariski closed subsets of the character group. In general, they can be arbitrarily complicated.
- The sets $\mathcal{V}^1_s(X)$ depend only on G/G''.

EXAMPLE (CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{C}\mathbb{Z} = \mathbb{C}[t^{\pm 1}]$. Then:

$$C_*(\widetilde{S^1}, \mathbb{C}) : \ 0 \longrightarrow \mathbb{C}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{C}[t^{\pm 1}] \longrightarrow 0 \ .$$

For $\rho \in \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*) = \mathbb{C}^*$, we get

$$C_*(\widetilde{S^1}, \mathbb{C}) \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}_\rho : \ 0 \longrightarrow \mathbb{C} \xrightarrow{\rho-1} \mathbb{C} \longrightarrow 0 \ ,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$.
Hence: $\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$ and $\mathcal{V}^i_s(S^1) = \emptyset$, otherwise.

EXAMPLE (PUNCTURED COMPLEX LINE)

Identify $\pi_1(\mathbb{C}\backslash\{n \text{ points}\}) = F_n$, and $\mathrm{Hom}(F_n, \mathbb{C}^*) = (\mathbb{C}^*)^n$. Then:

$$\mathcal{V}^1_s(\mathbb{C}\backslash\{n \text{ points}\}) = \begin{cases} (\mathbb{C}^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

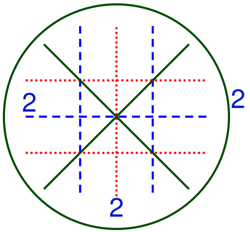
CHARACTERISTIC VARIETIES OF ARRANGEMENTS

- Let $\mathcal{A}$ be an arrangement of n hyperplanes, and let $\mathrm{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$ be the character torus.
- The characteristic variety $\mathcal{V}_1(\mathcal{A}) := \mathcal{V}_1^1(M(\mathcal{A}))$ lies in the subtorus $\{t \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\}$; it is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_1(\mathcal{A})$.
- All components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).
- In general, though, there are translated subtori in $\mathcal{V}_1(\mathcal{A})$, which are not *a priori* determined by $L(\mathcal{A})$.

(Denham–S. 2014)

- Suppose there is a multinet $\mathcal{M}$ on $\mathcal{A}$, and there is a hyperplane H for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.
- Then $\mathcal{V}_1(\mathcal{A} \setminus \{H\})$ has a component which is a 1-dimensional subtorus, translated by a character of order m_H.

EXAMPLE (THE DELETED B_3 ARRANGEMENT)



The B_3 arrangement supports a $(3,4)$-multinet; $\mathcal{X}$ consists of 4 triple points ($n_X = 1$) and 3 quadruple points ($n_X = 2$). So pick H with $m_H = 2$ to get a translated torus in $\mathcal{V}_1(\mathrm{B}_3 \setminus \{H\})$.

THE MILNOR FIBRATION(S) OF AN ARRANGEMENT

- Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{C}^\ell$.
- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel H.
- For each choice of multiplicities $m = (m_H)_{H\in\mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H^{m_H},$$

  a homogeneous polynomial of degree $N = \sum_{H\in\mathcal{A}} m_H$.
- The map $Q_m \colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.
- $F_m(\mathcal{A})$ is a Stein manifold. It has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, of dim $\ell - 1$.
- The *(geometric) monodromy* is the diffeomorphism

$$h\colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi \mathrm{i}/N} z.$$

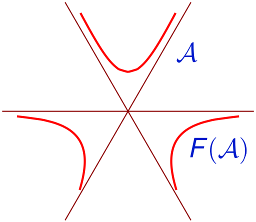
- If all $m_H = 1$, the polynomial $Q = Q(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

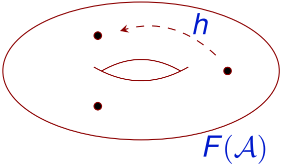
EXAMPLE

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then $M(\mathcal{A}) = \mathbb{C}^*$, $Q_m(\mathcal{A}) = z^m$, and $F_m(\mathcal{A}) = \{m\text{-roots of } 1\}$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:





More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

- Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(\mathbb{Q}_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential arrangement. The inclusion $\iota\colon M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{array}{ccccc} F_m(\mathcal{A}) & \longrightarrow & M(\mathcal{A}) & \xrightarrow{Q_m(\mathcal{A})} & \mathbb{C}^* \\ \downarrow & & \downarrow{\scriptstyle \iota} & & \| \\ F_m(\mathcal{B}_n) & \longrightarrow & M(\mathcal{B}_n) & \xrightarrow{Q_m(\mathcal{B}_n)} & \mathbb{C}^* \end{array}$$

- Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$

- (Zuber 2010) The mixed Hodge structure on $F = F(\mathcal{A})$ may be non-pure, and $\pi_1(F)$ may be non-1-formal.

TRIVIAL ALGEBRAIC MONODROMY

THEOREM (S. 2021)

Suppose $h_*\colon H_1(F;\mathbb{Z}) \to H_1(F;\mathbb{Z})$ *is the identity. Then*

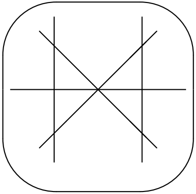
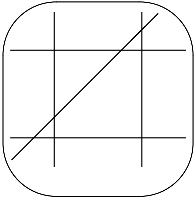
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)) \cong \operatorname{gr}_{\geqslant 2}(G)$.
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \cong \operatorname{gr}_{\geqslant 2}(G/G'')$.

THEOREM (S. 2021)

Suppose $h_*\colon H_1(F,\mathbb{Q}) \to H_1(F,\mathbb{Q})$ *is the identity. Then*

- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)) \otimes \mathbb{Q} \cong \operatorname{gr}_{\geqslant 2}(G) \otimes \mathbb{Q}$.
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \otimes \mathbb{Q} \cong \operatorname{gr}_{\geqslant 2}(G/G'') \otimes \mathbb{Q}$.
- $\phi_k(\pi_1(F)) = \phi_k(G)$ *and* $\theta_k(\pi_1(F)) = \theta_k(G)$ *for all* $k \geqslant 2$.

A pair of arrangements

$\mathcal{A}$   $\mathcal{A}'$

- Both $\mathcal{A}$ and $\mathcal{A}'$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$. Nevertheless, $M(\mathcal{A}) \simeq M(\mathcal{A}')$.
- Both Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- (S. 2017) $\pi_1(F) \not\cong \pi_1(F')$.
- The difference is picked by the depth-2 characteristic varieties: $\mathcal{V}_2^1(F) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2^1(F') = \{1\}$

THE HOMOLOGY OF THE MILNOR FIBER

- Let $(\mathcal{A}, m)$ be a multi-arrangement with $\gcd(m) = 1$. Set $N = \sum_{H\in\mathcal{A}} m_H$.
- The Milnor fiber $F_m(\mathcal{A})$ is a regular $\mathbb{Z}_N$-cover of the projectivized complement, $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))$, defined by the homomorphism

$$\delta_m \colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N, \quad x_H \mapsto m_H \bmod N.$$

- Let $\widehat{\delta_m} \colon \operatorname{Hom}(\mathbb{Z}_N, \mathbb{C}^*) \to \operatorname{Hom}(\pi_1(U(\mathcal{A})), \mathbb{C}^*)$ be the induced map between character groups.
- The dimension of $H_q(F_m(\mathcal{A}), \mathbb{C})$ may be computed by summing up the number of intersection points of $\operatorname{im}(\widehat{\delta_m})$ with the varieties $\mathcal{V}^q_s(U(\mathcal{A}))$, for all $s \geqslant 1$.

- We now consider the simplest non-trivial case: that of an arrangement $\mathcal{A}$ of n planes in $\mathbb{C}^3$, and its Milnor fiber, $F(\mathcal{A})$.
- Then $\operatorname{im}(\widehat{\delta}) \subset (\mathbb{C}^*)^n$ is generated by $(\zeta, \dots, \zeta)$, where $\zeta = e^{2\pi \mathrm{i}/n}$.
- Let $\Delta_{\mathcal{A}}(t) = \det(t \cdot \mathrm{id} - h_*)$ be the characteristic polynomial of the algebraic monodromy, $h_*\colon H_1(F(\mathcal{A}), \mathbb{C}) \to H_1(F(\mathcal{A}), \mathbb{C})$.
- Since $h_*^n = \mathrm{id}$, we may write

$$\Delta_{\mathcal{A}}(t) = \prod_{d \mid n} \Phi_d(t)^{e_d(\mathcal{A})}, \qquad (\star)$$

where $\Phi_d(t)$ is the d-th cyclotomic polynomial, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.

Problem

- *Is the polynomial $\Delta_{\mathcal{A}}$ (or, equivalently, the exponents $e_d(\mathcal{A})$) determined by the intersection lattice $L(\mathcal{A})$?*
- *In particular, is the first Betti number $b_1(F(\mathcal{A})) = \deg(\Delta_{\mathcal{A}})$ combinatorially determined?*

- By a transfer argument, $e_1(\mathcal{A}) = n - 1$.
- Not all divisors of n appear in ($\star$). E.g., if d does not divide at least one of the multiplicities $\geqslant 3$ of the intersection points, then $e_d(\mathcal{A}) = 0$.
- In particular, if $\mathcal{A}$ has only points of multiplicity 2 and 3, then $\Delta_{\mathcal{A}}(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

EXAMPLE

Let $\mathcal{A} = \mathcal{A}_4$ be the braid arrangement. Then $\mathcal{V}_1(\mathcal{A})$ has a single 'essential' component,

$$T = \{t \in (\mathbb{C}^*)^6 \mid t_1 t_2 t_3 = t_1 t_6^{-1} = t_2 t_5^{-1} = t_3 t_4^{-1} = 1\}.$$

Then $\mathrm{im}(\hat{\delta}) \cap T = \{(\omega, \dots, \omega)\}$, where $\omega = \zeta^2 = e^{2\pi \mathrm{i}/3}$. Hence, $\Delta_{\mathcal{A}}(t) = (t-1)^5(t^2+t+1)$.

Modular inequalities

- Let $A = H^\bullet(M(\mathcal{A}), \Bbbk)$, and let $\sigma = \sum_{H\in\mathcal{A}} e_H \in A^1$.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

Theorem (Cohen–Orlik 2000, Papadima–S. 2010)

$e_{p^m}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $m \geqslant 1$.

Theorem (Papadima–S. 2017)

- *Suppose $\mathcal{A}$ admits a k-net. Then $\beta_p(\mathcal{A}) = 0$ if $p \nmid k$ and $\beta_p(\mathcal{A}) \geqslant k-2$, otherwise.*
- *If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k-2$.*

COMBINATORICS AND MONODROMY

THEOREM (PS)

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then $\mathcal{A}$ admits a reduced 3-multinet iff $\mathcal{A}$ admits a 3-net iff $\beta_3(\mathcal{A}) \neq 0$. Moreover,

- $\beta_3(\mathcal{A}) \leqslant 2$.
- $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$*, and thus* $e_3(\mathcal{A})$ *is combinatorially determined.*

COROLLARY

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

THEOREM (PS)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$

CONJECTURE (PS)

The characteristic polynomial of the degree 1 algebraic monodromy for the Milnor fibration of an arrangement $\mathcal{A}$ of rank at least 3 is given by the combinatorial formula

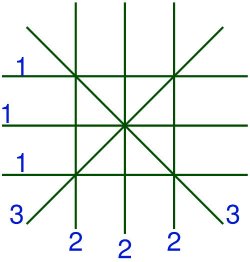
$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

- The conjecture has been verified for
  - All sub-arrangements of non-exceptional Coxeter arrangements (Măcinic, Papadima).
  - All complex reflection arrangements (Măcinic, Papadima, Popescu, Dimca, Sticlaru).
  - Certain types of complexified real arrangements (Yoshinaga, Bailet, Torielli, Settepanella).
- A counterexample was given by Yoshinaga (2020): there is an arrangement of 16 planes in $\mathbb{C}^3$ with $e_2 = 0$ but $\beta_2 = 1$.

Torsion in the homology of the Milnor fiber

Theorem (Cohen–Denham–S. 2003)

For every prime $p \geqslant 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion.



Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2 y(x^2 - y^2)^3 (x^2 - z^2)^2 (y^2 - z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We now can generalize and reinterpret these examples, as follows.

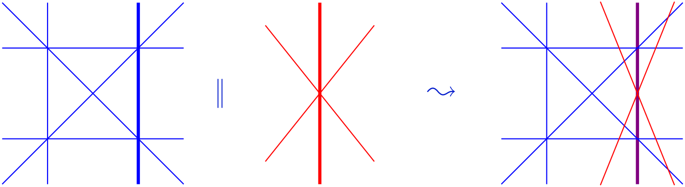
A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

THEOREM (DENHAM–S. 2014)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}_1^1(M(\mathcal{A}'), \Bbbk)$ varies with $\mathrm{char}(\Bbbk)$.

To produce p-torsion in the homology of the usual Milnor fiber, we use a “polarization" construction:



$(\mathcal{A}, m) \rightsquigarrow \mathcal{A}\|m$, an arrangement of $N = \sum_{H\in\mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank}\mathcal{A} + |\{H \in \mathcal{A}\colon m_H \geqslant 2\}|$.

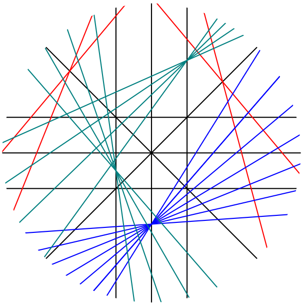
THEOREM (DS)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}'\|m'$ and $q = 1 + \left|\left\{K \in \mathcal{A}' : m'_K \geqslant 3\right\}\right|$.

COROLLARY (DS)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.



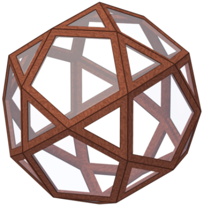
Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

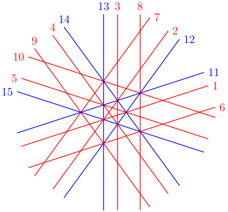
$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$

$$((x-y)^2-w_2^2)((x+y)^2-w_3^2)((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).

THE ICOSIDODECAHEDRAL ARRANGEMENT





- The icosidodecahedron is a quasiregular polyhedron in $\mathbb{R}^3$, with 20 triangular and 12 pentagonal faces, 60 edges, and 30 vertices, given by the even permutations of $(0, 0, \pm 1)$ and $\frac{1}{2}(\pm 1, \pm\phi, \pm\phi^2)$, where $\phi = (1 + \sqrt{5})/2$.
- One can choose 10 edges to form a decagon; there are 6 ways to choose these decagons, thereby giving 6 planes.
- Each pentagonal face has five diagonals; there are 60 such diagonals in all, and they partition in 10 disjoint sets of coplanar ones, thereby giving 10 planes, each containing 6 diagonals.

- These 16 planes form a arrangement $\mathcal{A}_{\mathbb{R}}$ in $\mathbb{R}^3$, whose complexification is the icosidodecahedral arrangement $\mathcal{A}$ in $\mathbb{C}^3$.
- The complement M is a $K(\pi, 1)$. Moreover, $P_U(t) = 1 + 15t + 60t^2$; thus, $\chi(U) = 36$ and $\chi(F) = 576$.
- In fact, $H_1(F, \mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. Thus, the algebraic monodromy of the Milnor fibration is trivial over $\mathbb{Q}$ and $\mathbb{Z}_p$ ($p > 2$), but not over $\mathbb{Z}$.
- Hence, $\mathrm{gr}(\pi_1(F)) \cong \mathrm{gr}(\pi_1(U))$, away from the prime 2. Moreover,
  - $\mathrm{gr}_1(\pi_1(F)) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$, $\mathrm{gr}_2(\pi_1(F)) = \mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$
  - $\mathrm{gr}_3(\pi_1(F)) = \mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$, $\mathrm{gr}_4(\pi_1(F)) = \mathbb{Z}^{1405} \oplus \mathbb{Z}_2^{?}$
- (Yoshinaga 2020) For this arrangement: $e_2 = 0$ but $\beta_2 = 1$.
- (Ishibashi, Sugawara, Yoshinaga 2022) For any arrangement $\mathcal{A}$: $e_2(\mathcal{A}) < \beta_2(\mathcal{A})$ if and only if $H_1(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion.