HOMOTOPY TYPE INVARIANTS

OF FOUR-DIMENSIONAL KNOT COMPLEMENTS

Alexandru Ion Suciu

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Abstract

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This thesis studies the homotopy type of smooth four dimensional knot complements. In contrast with the classical case, high-dimensional knot complements with fundamental group different from $\mathbb{Z}$ are never aspherical. The second homotopy group already provides examples of the way in which a knot in S^4 can fail to be determined by its fundamental group (C. McA. Gordon, S. P. Plotnick).

A natural class of knots to investigate is ribbon knots. They bound immersed disks with "ribbon singularities". A method is given for computing π_2 of such knot complements. I show that there are infinitely many ribbon knots in S^4 with isomorphic π_1 but distinct π_2 (viewed as $\mathbb{Z}\pi_1$-modules). They appear as boundaries of distinct ribbon disk pairs with the same exterior. These knots have the fundamental group of the spun trefoil, but none is a spun knot.

To a four-dimensional knot complement, one can associate a certain cohomology class, the first k-invariant of Eilenberg, MacLane and Whitehead. In a joint paper, Plotnick and I showed that there are arbitrarily many knots in S^4

whose complements have isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$ -modules), but distinct k-invariants. Here I prove this result using examples which are somewhat more natural and easier to produce. They are constructed from a fibered knot with fiber a punctured lens space and a ribbon knot by surgery.

The proofs involve writing down explicit cell complexes, computing twisted cohomology groups, combinatorial group theory and calculations in group rings.

Contents

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It is a great pleasure to thank Professor Steven Plotnick for his constant and invaluable guidance during the writing of this thesis. His ideas and the cooperation leading to our joint paper were decisive in the shaping of this work.

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And last but not least, I am grateful to my wife, Anda, for her patience and encouragement.

I. INTRODUCTION

The study of the homotopy type of knot complements started around 1910, when Dehn and Wirtinger gave a method for presenting the fundamental groups of knots in S^3. In 1956, Papakyriakopoulos [41] proved that classical knot complements are aspherical; and hence, that their homotopy type is completely determined by their "algebraic 2-type" (i.e. the fundamental group). The situation is completely different in higher dimensions. Andrews and Curtis [1] first gave an example of a knot in S^4 - the spun trefoil - whose complement has nontrivial π_2. D. B. A. Epstein [13] showed that the exterior of a spun knot is aspherical if and only if the knot is trivial. More generally, Dyer and Vasquez [11] proved that an aspherical knot complement in S^n $(n \geqslant 4)$ has fundamental group $\mathbb{Z}$. Combined with results of J. Stallings [50], J. Levine [34], J. Shaneson and C. T. C. Wall [51], this implies that the knot is trivial if $n \geqslant 5$. M. H. Freedman [16] shows that this is also true for $n = 4$, in the topological category.

The natural question to ask next is which homotopy type invariants can be used to distinguish among high-dimensional knot complements with the same fundamental group. Let $K = (S^{n+2}, S^n)$ be a (smooth) n-dimensional knot, and let X be its <u>exterior</u>, that is, the closure of the complement of a tubular neighborhood of S^n in S^{n+2}. By the usual abuse of language, we will call the homotopy type invariants of X the homotopy type invariants of the knot K. In [20], C. McA. Gordon gave examples of knots in S^4 with isomorphic π_1 but different π_2 (viewed as $\mathbb{Z}\pi_1$ - modules). S. P. Plotnick [43] generalizes this to arbitrarily many knots. In [44], he produces infinitely many examples in the TOP category, using the results of Freedman [15]. We somewhat improve Plotnick's result and get examples in the DIFF category (Theorem 1.3).

Now suppose we're given two knots in S^4 with isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$-modules). How can we distinguish their complements? A knot exterior $X = S^4 - S^2 \times \mathring{D}^2$ has the homotopy type of a 3-dimensional CW-complex. To any 3-complex X, one can associate an element $k(X) \in H^3(\pi_1 X, \pi_2 X)$, the first k-invariant of Eilenberg, MacLane and Whitehead [12], [39]. It is the first obstruction to a retraction $K(\pi_1 X, 1) \longrightarrow X$. To identify it, let

$C_3(\tilde{X}) \xrightarrow{\partial_3} C_2(\tilde{X}) \xrightarrow{\partial_2} C_1(\tilde{X}) \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ be the augmented chain complex of $\tilde{X}$, the universal cover of X, so that $C_i(\tilde{X})$ is a free $\mathbb{Z}\pi_1 X$-module, with rank equal to the number of i-cells of X. This complex may fail to be exact at $C_2(\tilde{X})$. Add to $C_3(\tilde{X})$ a free $\mathbb{Z}\pi_1(X)$-module $\overline{C}_3$ and map

$\overline{C}_3 \xrightarrow{\overline{\partial}_3} \ker \partial_2 \hookrightarrow C_2(\tilde{X})$ so as to kill $\pi_2 X$:

$$\begin{tikzcd}
\bar{C}_3 \oplus C_3(\tilde{X}) \arrow[r] \arrow[d, "\bar{\partial}_3"', "\partial_3"] \arrow[dd, bend right=60, "k"'] & C_2(\tilde{X}) \arrow[r, "\partial_2"] & C_1(\tilde{X}) \arrow[r, "\partial_1"] & C_0(\tilde{X}) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0. \\
\ker \partial_2 \arrow[ur, hook] \arrow[d, "p"] \\
\pi_2
\end{tikzcd}$$

This is now a partial free resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi_1 X$, and the map $k = p\bar{\partial}_3 : \bar{C}_3 \to \pi_2 X$ determines a well-defined class $k(X) = [k] \in H^3(\pi_1 X, \pi_2 X)$.

The triple $(\pi_1 X, \pi_2 X, k(X))$ is called the "algebraic 3-type" of X [39]. A map $(\pi_1 X, \pi_2 X, k(X)) \longrightarrow (\pi_1 X', \pi_2 X', k(X'))$ consists of a homomorphism $\alpha : \pi_1 X \to \pi_1 X'$ and an α-homomorphism $\beta : \pi_2 X \to \pi_2 X'$ satisfying $\alpha^*(k(X')) = \beta_*(k(X)) \in H^3(\pi_1 X, (\pi_2 X')_\alpha)$. When this condition holds, we say α and β preserve k-invariants. The complexes X and X' have the same algebraic 3-type when α and β are isomorphisms. A theorem of MacLane and Whitehead [39] asserts that there exists a map $f:X \to X'$ inducing α and β precisely when α and β preserve k-invariants. If α and β are isomorphisms, and $H_3(\tilde{X}) = H_3(\tilde{X}') = 0$, the Hurewicz and Whitehead theorems show that f is a homotopy equivalence. Knots in S^4 with exterior X satisfying $H_3(\tilde{X}) = 0$ are called quasi-aspherical. The reason, observed by S. J. Lomonaco, Jr., is that, in analogy with the classical case, the algebraic 3-type of a quasi-aspherical knot determines its homotopy type.

In [37], Lomonaco provides a way for computing the algebraic 3-type of a 2-knot from a motion picture. He asks (Problem 16) whether there are knots which are distinguished by their k-invariants. Plotnick [43] argues that the only reasonable candidates among fibered knots are, given the present status of 3-manifold theory, the 5-twist spin trefoil and its 2-fold cover. The question whether their k-invariants correspond reduces to a difficult problem about $\mathbb{Z}[SL(2,5)]$. Therefore, one should look at non-fibered knots for examples. The main theorem in [45] is:

<u>Theorem 1.1</u>: There are arbitrarily many knots in S^4 whose complements have isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$-modules), but distinct k-invariants.

We will prove Theorem 1.1 using examples which are somewhat more natural and easier to produce. In our paper [45], we relied on the fact that $\Sigma_1 \# \Sigma_2 \not\simeq (-\Sigma_1) \# \Sigma_2$ for Σ_i closed, orientable, aspherical 3-manifolds admitting no orientation reversing homotopy equivalence. Instead, we will use here J. H. C. Whitehead's classification of lens spaces up to homotopy type.

The knots constructed in [45] and here are quasi-aspherical, so the k-invariant is the last obstruction to a homotopy equivalence. But not all knots are quasiaspherical [18], [46]. Hence one might ask (Problem 1 in [37]):

does the algebraic 3-type of a knot exterior X determine its homotopy type? The invariants in the Postnikov tower of X to look at are $\pi_3 X$ and the second k-invariant $k_2(X) \in H^4(X_2, \pi_3 X)$, where $\pi_i(X_2) = \pi_i(X)$ for $i \leqslant 2$ and $\pi_i(X_2) = 0$ for $i > 2$. To answer this question seems a hard task.

Another question which arises is whether the exterior of a knot determines the knot. For classical knots the answer is not known. For multi-dimensional knots, a given exterior corresponds to at most two distinct knots [17],[6], [34]. Examples of inequivalent n-knots with the same complement were given by Cappell-Shaneson [9] for n = 3, 4 and C. McA. Gordon [22] for n = 2. One can ask the same question about disk knots $D^n \subset D^{n+2}$. Hitt-Sumners [30],[31] construct arbitrarily many examples of distinct disk knots $D^n \subset D^{n+2}$ with the same exterior for $n \geqslant 5$, and three examples for n = 4. S. P. Plotnick [44] produces infinitely many examples for $n \geqslant 3$. For n = 3, his proof requires Freedman's solution to the four-dimensional Poincaré conjecture, so he gets results in TOP. We give our own examples, which are somewhat simpler and also work in DIFF for n = 3.

Ribbon disks constitute a natural class of examples to work with, one which is interesting in its own right. They are obtained by pushing in D^{n+2} a D^n immersed in S^{n+1} with singularities of a certain type. Their exterior can be built with just 0-, 1- and 2-handles. Using a construction similar to that of Hitt-Sumners, we prove:

Theorem 1.2: There exist infinitely many distinct ribbon disks $D^n \subset D^{n+2}$, $n \geqslant 3$, with the same exterior.

A nice feature of these disk knots is that π_1 is the trefoil knot group. The difference comes from the fact that their meridians are not equivalent under any automorphism of π_1. The boundary of a ribbon disk is called a ribbon knot. Analyzing the boundaries of the examples provided by Theorem 1.2, we prove:

Theorem 1.3: There exist infinitely many ribbon knots in S^4 with isomorphic π_1 but distinct π_2 (as $\mathbb{Z}\pi_1$-modules).

The proof involves the study of the $\mathbb{Z}\pi_1$-module structure of π_2, and the reduction of the problem to a question about 2×2 matrices.

The simplest examples of ribbon knots are spun knots. We show in II.1 that, for most knots in S^3 (including torus knots), the fundamental group of the knot determines the spun knot. Combining this with the above theorem yields:

Corollary 1.4: There are infinitely many distinct knots in S^4 which are not spun but have the fundamental group of the spun trefoil.

This thesis is organized as follows. Chapter II studies ribbon disks and knots. In §1 we discuss several definitions and look at π_1. In §2 we give a method for computing π_2 of a ribbon 2-knot and derive some consequences. We prove Theorem 1.2 in §3 and Theorem 1.3 in §4.

Chapter III is devoted to the study of k-invariants, leading to the proof of Theorem 1.1. §1 interprets Whitehead's homotopy classification of lens spaces in terms of the k-invariant. §2 describes certain fibered knots with fiber punctured lens spaces. In §3, a construction based on surgery yields knots $K_{p,q}$, our examples for Theorem 1.1. In §4 we compute π_2 of the knot exteriors $X_{p,q}$. §5 describes a cell complex for $X_{p,q}$ and identifies $k(X_{p,q})$ on the cochain level. In §6 we compute $H^3(\pi_1,\pi_2)$ and locate the k-invariant. §7 completes the proof of Theorem 1.1. It studies automorphisms of π_1 and π_2 and their action on $H^3(\pi_1,\pi_2)$. A long computation in group rings wraps up the proof.

All $\mathbb{Z}\pi$ -modules are left-modules, unless otherwise stated. An element $u \in \mathbb{Z}\pi$ induces the $\mathbb{Z}\pi$ -module map $u : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ via right multiplication. Vectors in $(\mathbb{Z}\pi)^n$ are row vectors and matrices with entries in $\mathbb{Z}\pi$ act on the right. For any unexplained fact about cohomology of groups, see Brown's book [7].

II. RIBBON DISKS AND KNOTS

§1. Ribbon knots and π_1

In this section we introduce ribbon knots and disks and discuss several definitions. We recall a method for computing π_1 of a ribbon knot and show that a spun knot is determined by its fundamental group.

Ribbon n-knots were first defined by Fox [14], for n = 1, and Yajima [53], for n = 2. A knot $K = (S^{n+2}, S^n)$ is a <u>ribbon knot</u> if S^n bounds an immersed disk $D^{n+1} \hookrightarrow S^{n+2}$ with no triple points and such that the components of the singular set are n-disks whose boundary (n-1)-spheres either lie on S^n or are disjoint from S^n.

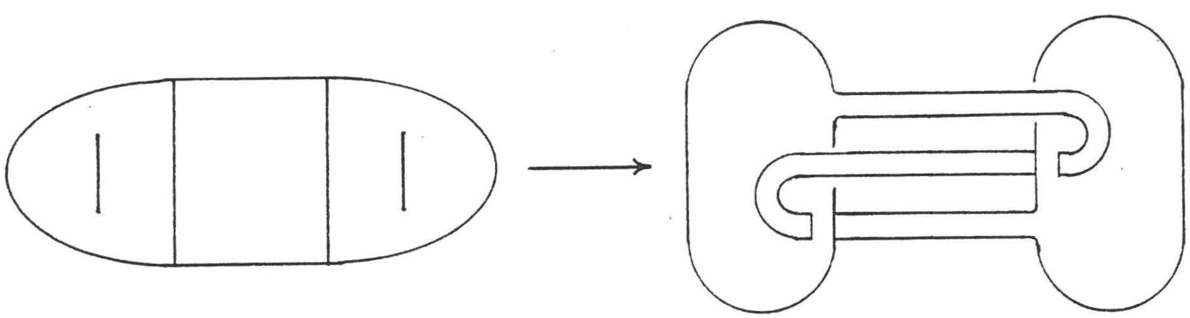


Figure 1

Pushing D^{n+1} into D^{n+3} produces a <u>ribbon disk</u> (D^{n+3}, D^{n+1}), with the ribbon knot (S^{n+2}, S^n) on its boundary. It can be pictured via a motion picture, as shown in Figure 2.

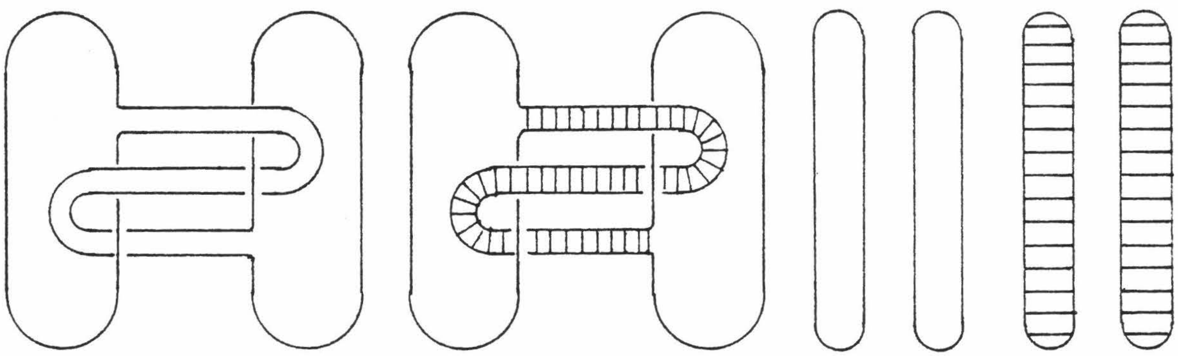


Figure 2

The double of a ribbon (n+1)-disk is an (n+1)-ribbon knot. Every (n+1)-ribbon knot is obtained in this manner.

Given a knot $K = (S^3, S^1)$, the <u>n-spin</u> of K, $n \geqslant 1$, is the (n+1)-knot $\sigma_n(K) = \partial(\overline{K} \times D^{n+1})$, where $\overline{K} = K$ - standard $(\mathring{D}^3, \mathring{D}^1)$. For n = 1, we get the usual spin of K. It is folklore that every n-spun knot is ribbon. One can see this from Figure 3.

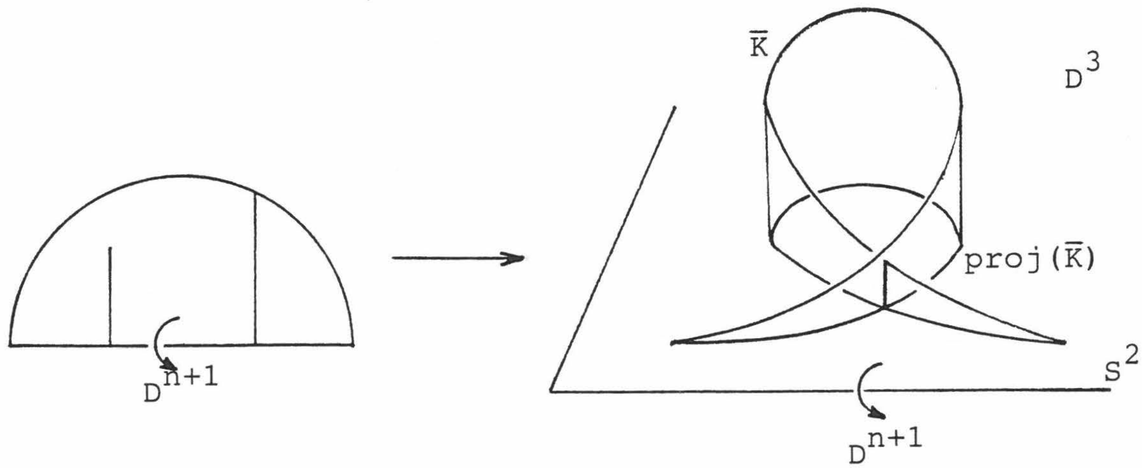


Figure 3

The spun trefoil is the double of the disk in Figure 2.

The fundamental group of a ribbon n-knot can be computed from a motion picture [14], [54]. The equatorial S^{n-1} consists of disjoint spheres $S_0^{n-1}, \dots, S_m^{n-1}$ (with meridians x_i) joined together by m bands running from S_0^{n-1} to S_i^{n-1} $(1 \leq i \leq m)$. Then $\pi_1(S^{n+2} - S^n) =$

$$\pi_1(D_+^{n+2} - D_+^n) = (x_0, x_1, \dots x_m \mid x_0 = w_i x_i w_i^{-1},\ 1 \leq i \leq m),$$

where $w_i = \Pi\, x_{k_j}^{\varepsilon_{ij}}$,

$$\text{and} \qquad \varepsilon_{ij} = \begin{cases} +1, & \text{if the i-th band goes over } x_{k_j} \\ -1, & \text{if the i-th band goes under } x_{k_j} \\ 0, & \text{otherwise.} \end{cases}$$

We call such a presentation of π_1 a "ribbon presentation". For example, the spun trefoil, with equatorial section the square knot (Figure 4), has

$\pi_1(S^4 - S^2) = (t,x \mid t = (xt)\, x\, (xt)^{-1}) = (t,x \mid txt = xtx).$

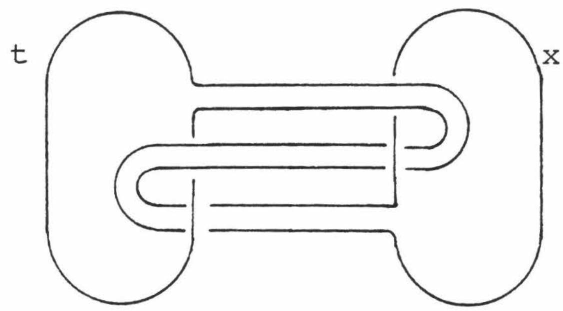


Figure 4

Does the fundamental group of a ribbon knot determine the knot? We will see in §4 that this is not the case. For spun knots, however, it is reasonable to conjecture an affirmative answer to this question. We can prove this for a large class of knots:

<u>Theorem 1.1</u>: Let K_1 and K_2 be knots in S^3 with $\pi_1 X_1 \cong \pi_1 X_2$. Assume K_i are not (p,q)-cables, $|p| \leqslant 2$, of a non-trivial knot. Then $\sigma_n(K_1) = \sigma_n(K_2)$.

<u>Proof</u>: Results of Johannson, Feustel, Whitten, Burde and Zieschang (see [23, p. 9-10]), imply that either (i) K_i are prime knots, with $X_1 = X_2$ or (ii) K_i are composite knots, with the prime factors equal, up to orientations. In case (i), $\sigma_n(X_1) = \sigma_n(X_2)$, and by Gluck [17], for n = 1 and Cappell [8], for n > 1, $\sigma_n(K_1) = \sigma_n(K_2)$. In case (ii), the argument in Gordon [21] yields the equivalence of $\sigma_n(K_i)$. $\square$

<u>Remark</u>: The asphericity of classical knots implies that spun knots with isomorphic π_1 have homotopy equivalent exteriors.

Not every ribbon 2-knot is spun. For example, the ribbon 2-knot with cross section the stevedore knot (Figure 5) has $G = \pi_1 (S^4 - S^2) = (t,a \mid t = (at^{-1})\ a\ (at^{-1})^{-1}) \underset{x=at^{-1}}{=} (t,x \mid txt^{-1} = x^2)$ (compare [14, p. 136]).

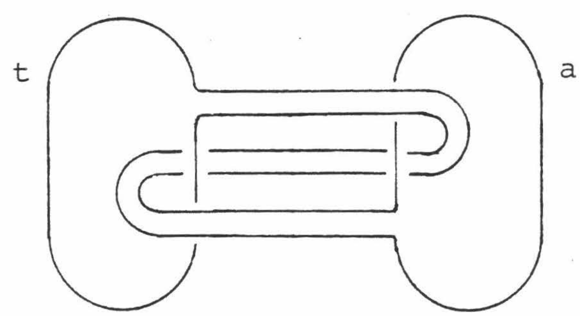


Figure 5

But G is not a 3-manifold group [25], [26], so the knot is not spun. This knot will play an important role in Chapter III. Even more striking, if we let the band on Figure 5 wrap k times around a and t $(k > 1)$, we get a ribbon 2-knot with $\pi_1(S^4 - S^2) =$
$(t,a \mid t = (at^{-1})^k\, a\, (at^{-1})^{-k}) \underset{x=at^{-1}}{=} (t,x \mid tx^k t^{-1} = x^{k+1})$,

a Baumslag-Solitar non-residually finite group. But, according to Thurston, 3-manifold groups <u>are</u> residually finite.

Ribbon knots are, by definition, slice knots. Whether the converse is true is a famous question in classical knot theory, but in higher dimensions counterexamples abound [29], [48], [10]. All 2-knots are slice [31], but no m-twist spun 2-knot $(m > 1)$ is ribbon [10]. This fact follows easily from [27, Cor. to Thm. 1], but unfortunately there is a gap in that paper.

Here is another construction of ribbon disks and knots. Start with (D^{n+3}, D^{n+1}), the standard disk pair, with meridian t. Add 1-handles h_i^1 $(1 \leqslant i \leqslant m)$ to D^{n+3}, with core circles x_i, and 2-handles h_i^2 along curves r_i, with $r_i \cap D^{n+1} = \emptyset$ and r_i isotopic to x_i in $D^{n+3} \cup h_1^1 \ldots \cup h_m^1$. By the handle cancelling theorem, $D^{n+3} \cup \{h_i^1\} \cup \{h_i^2\} = D^{n+3}$ and we get a new disk pair (D^{n+3}, D^{n+1}), with $\pi_1(D^{n+3} - D^{n+1}) = (t, x_1, \ldots, x_m \mid r_1, \ldots, r_m)$. The procedure is illustrated in the figure below.

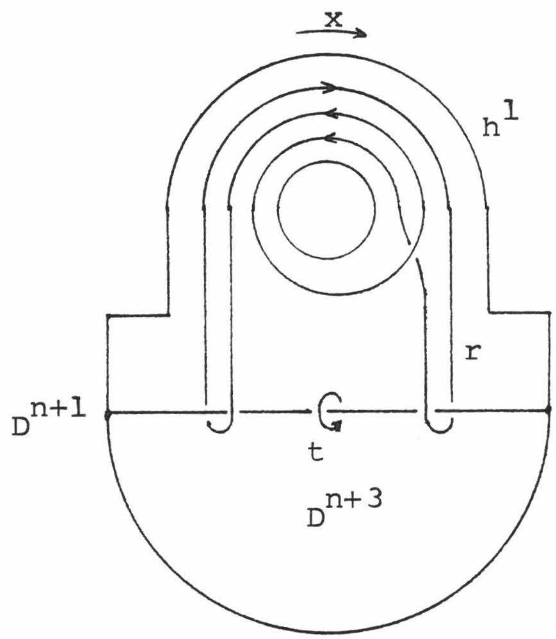


Figure 6

<u>Theorem 1.2</u> ([28],[5]): A disk pair is ribbon if and only if it can be obtained from the standard disk pair by adding 1- and 2-handles in the above manner.

In practice, one passes from the above presentation of π_1 to a ribbon presentation through Andrews-Curtis moves [2], and then draws the ribbon knot prescribed by this presentation. An example of the procedure is given in §3. The exterior of the ribbon knot $(S^{n+2},S^n) =$ $\partial(D^{n+3},D^{n+1})$ is obtained from $S^1 \times D^{n+1} \# (\overset{m}{\underset{1}{\#}} S^1 \times S^{n+1})$

by performing surgery on the curves r_i. For example, the knot in Figure 4 is the boundary of the disk pair in Figure 6, and can be constructed by surgery on $r = txt^{-1}x^{-2}$ in $S^1 \times D^3 \# S^1 \times S^3$:

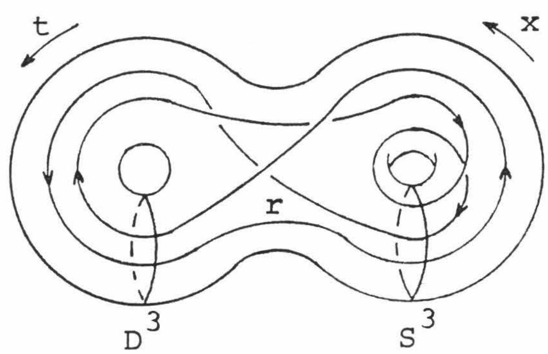


Figure 7

§2. π_2 of a ribbon 2-knot

This section gives a method for calculating π_2 of a ribbon 2-knot as a $\mathbb{Z}\pi_1$-module. This method, briefly sketched in [45], yields explicit results for one-relator ribbon knots and spun knots. Let X be the exterior of a one-relator ribbon knot. It is obtained by surgery on a simple closed curve r in $S^1 \times D^3 \# S^1 \times S^3$, where $\pi_1(S^1 \times D^3) = \mathbb{Z}(t)$, $\pi_1(S^1 \times S^3) = \mathbb{Z}(x)$, and $r(t,x)$ has exponent sum ± 1 in x. We write $\pi = \pi_1 X = (t,x \mid r)$, $\pi_2 = \pi_2 X$. Let M be the cover of $S^1 \times D^3 \# S^1 \times S^3$ corresponding to the kernel of $\mathbb{Z} * \mathbb{Z} \longrightarrow (\pi = \mathbb{Z} * \mathbb{Z}/\langle r\rangle)$. If we perform equivariant surgery on the lifts of r in M, we get $\tilde{X}$. M consists of copies of $\mathbb{R} \times D^3$, indexed by the cosets $\pi/\mathbb{Z}(t)$, and copies of $\mathbb{R} \times S^3$, indexed by the cosets $\pi/\mathbb{Z}(x)$, tubed together by "connectors" $S^3 \times I$, indexed by π. Figure 8 depicts the cover, together with three lifts of the surgery curve $r = txt^{-1}x^{-2}$. The lifts of the "fiber" S^3, $gS^3 = S^3_g$, are indexed by π. The lifts of r are indexed by their basepoint $g \in \pi$.

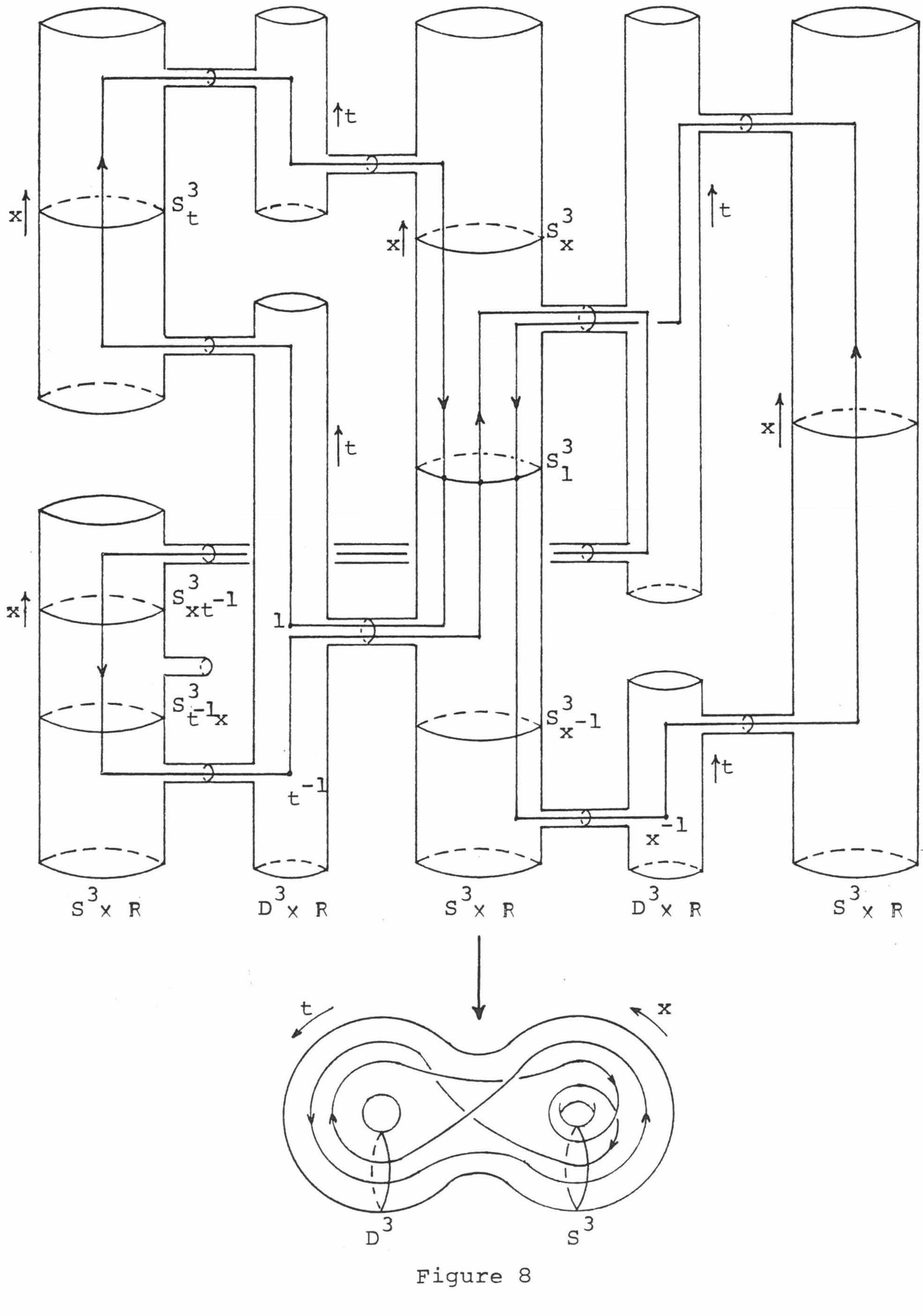

x
S^3_t
t
x
S^3_x
t
t
x
S^3_1
x
$S^3_{xt^{-1}}$
1
$S^3_{t^{-1}x}$
$S^3_{x^{-1}}$
t
t^{-1}
x^{-1}
$S^3 \times R$
$D^3 \times R$
$S^3 \times R$
$D^3 \times R$
$S^3 \times R$
t
x
D^3
S^3

Figure 8

Let
$$M = M_0 \coprod_{\pi} S^1 \times S^2 \cup (\coprod_{\pi} S^1 \times D^3),$$
$$\tilde{X} = M_0 \coprod_{\pi} S^1 \times S^2 \cup (\coprod_{\pi} D^2 \times S^2).$$

The Mayer-Vietoris sequences corresponding to these decompositions yield:

$$0 \to \oplus H_3(S^1 \times S^2) \to H_3(M_0) \longrightarrow H_3(M) \xrightarrow{\phi} \oplus H_2(S^1 \times S^2)$$

$$\to H_2(M_0) \to H_2(M) \to 0 \to H_1(M_0) \xrightarrow{\simeq} H_1(M) \to 0$$

and

$$0 \to \oplus H_3(S^1 \times S^2) \longrightarrow H_3(M_0) \longrightarrow H_3(\tilde{X}) \to \oplus H_2(S^1 \times S^2)$$

$$\to \oplus H_2(D^2 \times S^2) \oplus H_2(M_0) \longrightarrow H_2(\tilde{X}) \longrightarrow \oplus H_1(S^1 \times S^2) \xrightarrow{\psi} H_1(M_0) \to 0.$$

Notice that $H_2(M) = 0$ and $H_3(M) = \mathbb{Z}\pi$, generated by the lifts of S^3. These sequences simplify to give:

$$H_3(\tilde{X}) = \ker\,(\mathbb{Z}\pi \xrightarrow{\phi} \mathbb{Z}\pi)$$

$$0 \to \operatorname{coker}\phi \longrightarrow H_2(\tilde{X}) \longrightarrow \ker\psi \to 0.$$

Let $X_r = e^0 \cup e^1_t \cup e^1_x \cup e^2_r$ be the 2-complex associated to the presentation $\pi = (t,x \mid r)$. The reduced chain complex of its universal cover is (see [7, p. 45-46]):

$$\mathbb{Z}\pi \xrightarrow{\partial_2 = (\frac{\partial r}{\partial t}\ \frac{\partial r}{\partial x})} \mathbb{Z}\pi \oplus \mathbb{Z}\pi \xrightarrow{\partial_1 = \binom{t-1}{x-1}} \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \quad (*)$$

where ∂_2 is the matrix of Fox derivatives. The relation r is not a proper power, since the exponent sum of x is ± 1. Hence, by Lyndon's theorem [38], X_r is aspherical, that is, $\partial_2^{\tilde{X}_r} = (\frac{\partial r}{\partial t}\ \frac{\partial r}{\partial x})$ is a monomorphism.

To compute ϕ, note first that the "fiber" S^3 is a dual cycle to x. Hence, the algebraic sum of the lifts of S^3 cut by the lift of r at 1 equals $(\partial r/\partial x)\cdot S^3$. Therefore $\phi(S^3_1)$, which is the algebraic sum of the lifts of r which intersect S^3_1, equals $\overline{\partial r/\partial x}$, where $\overline{\Sigma n_g g} = \Sigma n_g g^{-1}$. This is to say, $\phi = \cdot\overline{\partial r/\partial x} : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$. For example, if $r = txt^{-1}x^{-2}$, $\phi(1) = t^{-1} - x^{-1} - 1$, which can be seen directly in Figure 8. We need the following lemma, which will be used repeatedly in Chapter III:

<u>Lemma 2.1</u>: Let $g \in G$ be an element of infinite order in a group G. Then $\mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G$ is a monomorphism.

<u>Proof</u>: Suppose $(\Sigma n_h h)\cdot(g-1) = 0$. Then $n_{hg^{-1}} - n_h = 0$, and so $n_h = n_{hg^{-1}} = n_{hg^{-2}} = \ldots$, an infinite sequence of equalities. Hence, $n_h = 0$.

□

The exact sequence (*) gives $\partial r/\partial t\cdot(t-1) + \partial r/\partial x\cdot(x-1) = 0$. From the lemma and the injectivity of $(\partial r/\partial t\ \ \partial r/\partial x)$ we deduce that $\partial r/\partial x$ is injective. Hence, ϕ is a monomorphism, and $H_3(\tilde{X}) = 0$.

Lyndon's theorem [38] also shows that the relation module $H_1(M_0)$ is freely generated by the lifts of r, so that $\psi : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ is an isomorphism. Hence $\ker\psi = 0$. We sum up our computations in:

induces a ring homomorphism $\mathbb{Z}\pi \longrightarrow \mathbb{Z}Z$, which takes the Jacobian matrix of Fox derivatives to the Alexander matrix. We thus recover the classical fact that the Alexander matrix of a knot in S^3 is hermitian, up to trivial units [47, p. 208].

Proposition 2.2: One relator ribbon 2-knots are quasi-aspherical, with $\pi_2 = \mathbb{Z}\pi/(\overline{\partial r/\partial x})$, where $\pi_1 = (t,x \mid r)$.

$\square$

For example, the knot in Figures 5 and 7, with $\pi_1 \equiv G = (t,x \mid txt^{-1} = x^2)$, has $\pi_2 = \mathbb{Z}G/(1 + x^{-1} - t^{-1})$, whereas the spun trefoil (Figure 4), with $\pi_1 = (t,x \mid txt = xtx)$, has $\pi_2 = \mathbb{Z}\pi/(1 + t^{-1}x^{-1} - t^{-1})$. These calculations check the ones in [37, Appendix B].

Proposition 2.3: Let X_1 and X_2 be the exteriors of one relator ribbon 2-knots. If $\pi_1 X_1 \cong \pi_1 X_2$ and $\pi_2 X_1 \cong \pi_2 X_2$ (as $\mathbb{Z}\pi_1$-modules), then $X_1 \simeq X_2$.

Proof: We saw during the proof of Proposition 2.2 that π_1 has a 2-dimensional $K(\pi_1,1)$. Therefore $H^3(\pi_1, \pi_2) = 0$ and the k-invariant vanishes. As X_i are quasi-aspherical, the theorem of Lomonaco [37] mentioned in the introduction implies $X_1 \simeq X_2$.

$\square$

The exterior of a ribbon disk is homotopy equivalent to the 2-complex associated to the presentation of its fundamental group. Hence, by Lyndon's theorem,

Proposition 2.4: One-relator ribbon disk exteriors are aspherical.

$\square$

It is claimed in [5] that Proposition 2.4 is valid

§3. Meridians and ribbon disks

In this section we produce the examples for Theorem I.1.2. The (n-2)-spun trefoil, $n \geqslant 3$, is a fibered knot with fiber $(S^1 \times S^{n-1} \# S^1 \times S^{n-1}) - \mathring{D}^n$ [4]. If u and v generate π_1 of the fiber, the monodromy σ is given by $\sigma(u) = v$, $\sigma(v) = u^{-1}v$. This knot bounds a fibered ribbon disk pair $D_0 = (D^{n+2}, D^n)$, with fiber $V^{n+1} = S^1 \times D^n \natural S^1 \times D^n$ and monodromy σ. The exterior $V \underset{\sigma}{\times} S^1$ has meridian t and π_1 the trefoil knot group $(t,u,v \mid tut^{-1} = v,\ tvt^{-1} = u^{-1}v)$.

We now construct other disk pairs D_k, with the same exterior, but different meridians. Add a 2-handle h^2 to $V \underset{\sigma}{\times} S^1$ along a simple closed curve representing $t_k = u^k t$, with either framing. Since t_k is homologous to t in $V \underset{\sigma}{\times} S^1$, the Mayer-Vietoris sequence shows that the resulting manifold $\mathcal{B}^{n+2}$ is acyclic. Its fundamental group is Andrews-Curtis equivalent to the trivial group:

$$\begin{aligned}\pi_1(\mathcal{B}^{n+2}) &= (t,u,v \mid tut^{-1} = v,\ tvt^{-1} = u^{-1}v,\ u^k t = 1) \\ &= (u,v \mid u^{-k}uu^k = v,\ u^{-k}vu^k = u^{-1}v) \\ &= (v \mid v^{-k}v\,v^k = v^{-1}v) \\ &= (v \mid v = 1).\end{aligned}$$

By a standard argument [2], $\mathcal{B}^{n+2}$ is diffeomorphic to D^{n+2}.

Then $(\mathcal{B}^{n+2}$, cocore of 2-handle) is a knotted disk pair $D_k = (D^{n+2}, D^n)$ with exterior $V \underset{\sigma}{\times} S^1$, and meridian t_k. The fundamental group is $\pi_1 = \pi_1(V \underset{\sigma}{\times} S^1) = (t,u,v \mid tut^{-1} = v,\ tvt^{-1} = u^{-1}v)$, which is A-C equivalent to:

for arbitrary ribbon disks. The proof rests on an unproved assertion, erroneously attributed to Lomonaco. It amounts to proving $\ker \psi = 0$, for an arbitrary ribbon 2-knot. The asphericity of ribbon disks is implied by the Whitehead Conjecture (see [10, Question 2]).

Let $\pi = (t, x_1, \ldots, x_m \mid r_1, \ldots, r_m)$ be a Wirtinger presentation of the group of a classical knot K . The spin of K is a ribbon knot with exterior X_1 obtained from $S^1 \times D^3 \;\#\; (\#_1^m S^1 \times S^3)$ by surgery on the curves r_i . With notation as before, we compute $\phi = \overline{\left(\frac{\partial r_i}{\partial x_j}\right)} : (\mathbb{Z}\pi)^m \longrightarrow (\mathbb{Z}\pi)^m$. The exterior X of K is an aspherical 2-complex [41], with $\partial \tilde{X}_2 = \left(\frac{\partial r_i}{\partial t} \;\; \frac{\partial r_i}{\partial x_j}\right)$. As in the proof of Proposition 2.2, ϕ is a monomorphism and $H_3(\tilde{X}_1) = 0$. It also follows that the map $\left(\frac{\partial r_i}{\partial x_j}\right) : H_1(M_0) \longrightarrow (\mathbb{Z}\pi)^m$ ([7, p. 43-46]), is a $\mathbb{Z}\pi$-isomorphism. Therefore ψ is an isomorphism. We have proved:

<u>Proposition 2.5</u>: Spun 2-knots are quasi-aspherical, with

$$\pi_2 = (\mathbb{Z}\pi)^m \Big/ \overline{\left(\frac{\partial r_i}{\partial x_j}\right)} \text{ , where } \pi_1 = (t, x_1, \ldots, x_m \mid r_1, \ldots, r_m). \qquad \square$$

This complements Andrews' and Lomonaco's computation $\pi_2 = (\mathbb{Z}\pi)^m \Big/ \left(\frac{\partial r_i}{\partial x_j}\right)^t$ [3], [36]. The abelianization map $\pi \to \mathbb{Z}$

$$(t,u,v,t_k \mid tut^{-1} = v\ ,\ tvt^{-1} = u^{-1}v\ ,\ t_k = u^k t)$$

$$= (t_k,u,v \mid u^{-k}t_k u t_k^{-1}u^k = v\ ,\ u^{-k}t_k v t_k^{-1}u^k = u^{-1}v)$$

$$= (t_k,u \mid u^{-k}t_k u^{-k}t_k u t_k^{-1}u^k\ t_k^{-1}u^k = u^{-1}u^{-k}t_k u t_k^{-1}u^k)$$

$$=(t_k,u \mid t_k^{-1}ut_k u^{-k}t_k u t_k^{-1}u^{k-1} = 1)$$

Hence $V \underset{\sigma}{\times} S^1$ has a handle decomposition $h^0 \cup h^1_{t_k} \cup h^1_u \cup h^2_r$, with h^2_r attached along a simple closed curve representing $r = r(t_k,u) = t_k^{-1}ut_k u^{-k}t_k u t_k^{-1}u^{k-1}$ with the property $r(1,u) = u$. Now $D^{n+2} = V \underset{\sigma}{\times} S^1 \cup h^2 = (h^0 \cup h^1_{t_k} \cup h^2) \cup h^1_u \cup h^2_r = D_0^{n+2} \cup h^1_u \cup h^2_r$, with D^n = cocore h^2 = standard n-disk in $D_0^{n+2} = h^0 \cup h^1_{t_k} \cup h^2$ and r isotopic to u in $D_0^{n+2} \cup h^1_u$. By Theorem 1.2, D_k is a ribbon disk pair.

The pairs D_0 and D_1 are equivalent, since the conjugation map $\mu_v : V \to V$ extends to a diffeomorphism of $V \underset{\sigma}{\times} S^1$ taking t to t_1. The boundary of $D_0 = (D^5,D^3)$ is the spun trefoil (Figure 4). In order to picture the other disk pairs, we give here a ribbon presentation of π_1.

$$\pi_1 = (t_k,u \mid t_k^{-1}ut_k\ u^{-k}t_k u^k\ u^{-k+1}t_k^{-1}u^{k-1} = 1)$$

$$= \left(t_k,u,c,d \;\middle|\; \begin{matrix} t_k^{-1}\ u\ t_k\ c\ d^{-1} = 1\ ,\ c = u^{-k}t_k u^k, \\ d = u^{-k+1}t_k u^{k-1} \end{matrix}\right)$$

$$= \left(t_k, c, d \;\middle|\; \begin{array}{l} d = t_k(dc^{-1})^{-k+1} \cdot t_k \cdot (dc^{-1})^{k-1}\, t_k^{-1} \\ \\ c = t_k c d^{-1} t_k^{-1} \cdot d \cdot t_k d c^{-1} t_k^{-1} \end{array} \right) .$$

Figure 9 depicts the boundary of $D_2 = (D^5, D^3)$ - a ribbon knot in S^4 with its equatorial cross section drawn.

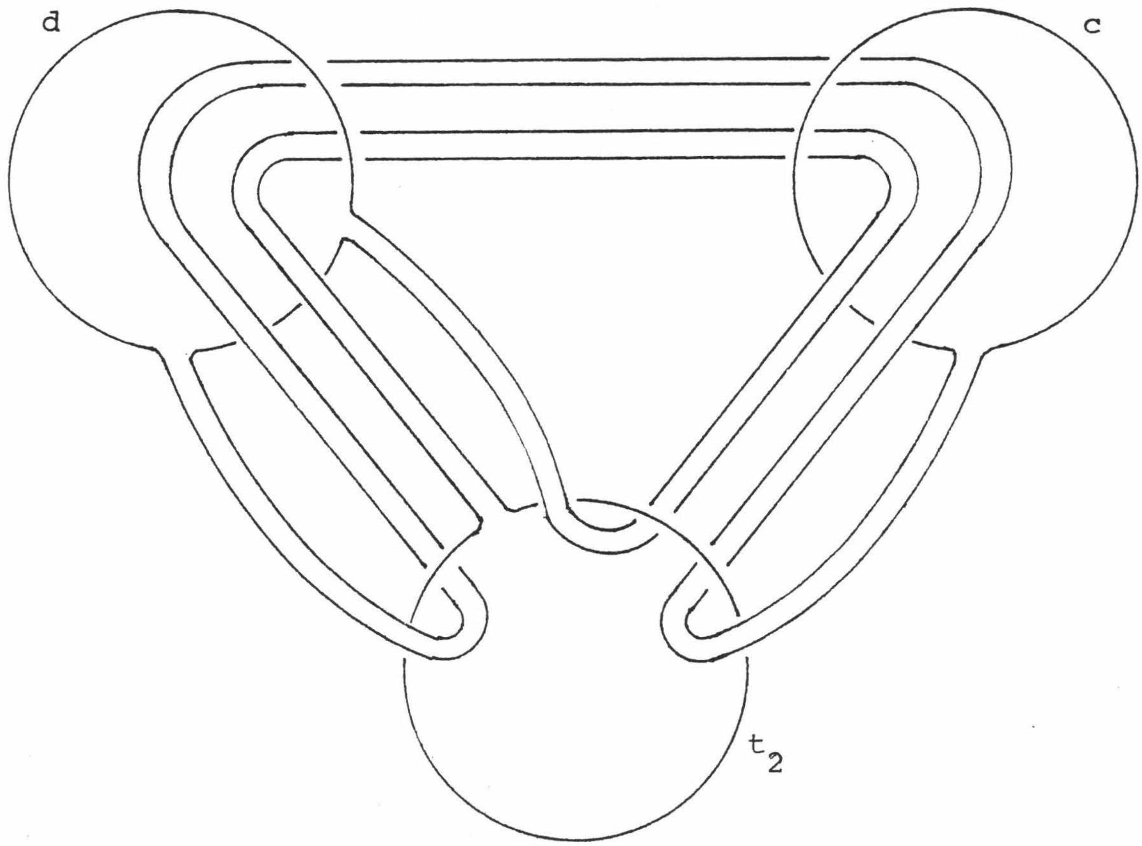


Figure 9

The ribbon disk pairs $D_k = (D^{n+2}, D^n)$, $k \geqslant 1$, have the same exterior $V \underset{\sigma}{\times} S^1$. To prove Theorem I.1.2, we have to show that they are all distinct. A diffeomorphism of pairs $D_k \longrightarrow D_\ell$ restricts to a diffeomorphism of $V \times_\sigma S^1$ preserving meridians, thus taking t_k to $t_\ell^{\pm 1}$. It induces an automorphism of $\pi_1 = \pi_1(V \times_\sigma S^1)$ taking t_k to $t_\ell^{\pm 1}$. Rewriting π_1 as:

$$\pi_1 = (t,u,v \mid tut^{-1} = v,\ tvt^{-1} = u^{-1}v) \underset{\substack{u=t^{-1}x \\ v=xt^{-1}}}{\equiv} (t,x \mid txt = xtx) \underset{\substack{t=b^{-1}a \\ x=ab^{-1}}}{\equiv} (a,b \mid a^2 = b^3)\ ,$$

gives $t_k = u^k t = (a^{-1}b\ a\ b^{-1})^k\ b^{-1}a$. It is well known that $\pi/Z(\pi) \cong PSL(2,\mathbb{Z})$, under the isomorphism $a \longmapsto A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b \longmapsto B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. The center being characteristic, we are left with proving:

<u>Lemma 3.1</u>: Let $T_k = (A^{-1}B\ A\ B^{-1})^k\ B^{-1}A \in PSL(2,\mathbb{Z})$. There is no automorphism of $PSL(2,\mathbb{Z})$ taking T_k to $T_\ell^{\pm 1}$ for $k, \ell \geq 1$, $k \neq \ell$.

<u>Proof</u>: We compute $A^{-1}BAB^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, $(A^{-1}BAB^{-1})^k = \begin{pmatrix} a_{2k} & -a_{2k-1} \\ -a_{2k-1} & a_{2k-2} \end{pmatrix}$, where $a_0 = a_1 = 1$, $a_k = a_{k-1} + a_{k-2}$ are the Fibonacci numbers. Therefore $T_k = \begin{pmatrix} a_{2k} & a_{2k-2} \\ -a_{2k-1} & -a_{2k-3} \end{pmatrix}$ and $tr(T_k^{\pm 1}) = a_{2k} - a_{2k-3} = 2a_{2k-2}$. As the automorphisms of $PSL(2,\mathbb{Z})$ come from conjugations by matrices in $GL(2,\mathbb{Z})$ (see Lemma 4.4) , we are done.

$\square$

§4. Ribbon knots with different π_2

In the previous section we produced ribbon disk pairs $D_k = (D^5, D^3)$. The boundary of D_k is a ribbon knot $K_k = (S^4, S^2)$. We show in this section that the knots K_k provide the examples for Theorem I.1.3. The exterior X_k is obtained from $(S^1 \times S^2 \# S^1 \times S^2) \times_\sigma S^1$ by deleting a neighborhood of the curve $t_k = u^k t$. Actually, X_k is fibered over S^1, with fiber $S^1 \times S^2 \# S^1 \times S^2 - \mathring{D}^3$ and monodromy $\sigma_k = \mu_{u^k}\sigma$. As explained in [44], we "untwist" the deleted curve, thereby "twisting" the monodromy. The fundamental group is

$$\pi_1 = \pi_1 X_k = (u, v, t_k \mid t_k u t_k^{-1} = u^k v u^{-k}\ ,\ t_k v t_k^{-1} = u^{k-1} v u^{-k})$$

$$= (t_k, u \mid r_k)\ ,$$

where $r_k = u t_k u^{-k} t_k u t_k^{-1} u^{k-1} t_k^{-1}$. We saw that $\pi_1 X_k \cong \pi =$ $(t, u, v \mid tut^{-1} = v\ ,\ tvt^{-1} = u^{-1}v)$, the trefoil knot group.

Proposition 2.2 gives the following presentation for $\pi_2 X_k$:

$$0 \longrightarrow \mathbb{Z}\pi \xrightarrow{\overline{\frac{\partial r_k}{\partial u}}} \mathbb{Z}\pi \longrightarrow \pi_2 X_k \longrightarrow 0,$$

where $w_k = \overline{\dfrac{\partial r_k}{\partial u}} =$

$$\overline{= 1 - u t_k (u^{-1} + \ldots + u^{-k}) + u t_k u^{-k} t_k + t_k u^{-k+1} (1 + \ldots + u^{k-2})}$$

$$= 1 - (u+\dots+u^k)t_k^{-1}u^{-1} + t_k^{-1}u^k t_k^{-1}u^{-1} + (u+\dots+u^{k-1})t_k^{-1}$$

$$= u[u^k - (1+\dots+u^{k-1})t^{-1} + u^{-1}t^{-2} + (1+\dots+u^{k-2})t^{-1}u]u^{-k-1}.$$

We have the following result, which proves Theorem I.1.3:

<u>Lemma 4.1</u>: Let $\alpha : \pi_1 X_k \longrightarrow \pi_1 X_\ell$ be an isomorphism, $k, \ell \geqslant 1$, $k \neq \ell$. There is no α-isomorphism $\beta : \pi_2 X_k \longrightarrow \pi_2 X_\ell$.

<u>Proof</u>: We start by studying the automorphisms of π. α induces

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} * \mathbb{Z} & \longrightarrow & \pi_1 X_k & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow \pm 1 & & \\
1 & \longrightarrow & \mathbb{Z} * \mathbb{Z} & \longrightarrow & \pi_1 X_\ell & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 .
\end{array}$$

<u>Lemma 4.2</u>: There is a diffeomorphism $F: X_k \longrightarrow X_k$ inducing -1 on $\mathbb{Z}$.

<u>Proof</u>: We define $f \in \mathrm{Aut}(\mathbb{Z} * \mathbb{Z})$ via $f \begin{cases} u \to vu^{-1} \\ v \to uvu^{-1} \end{cases}$.

We check that $f = \sigma_k f \sigma_k$:

$$\sigma_k f \sigma_k(u) = \sigma_k f(u^k v u^{-k}) = \sigma_k((vu^{-1})^k uvu^{-1}(uv^{-1})^k)$$

$$= \sigma_k((vu^{-1})^{k-1}v(uv^{-1})^{k-1}) = u^{-k+1}u^{k-1}vu^{-k}u^{k-1} = vu^{-1}$$

$$\sigma_k f \sigma_k(v) = \sigma_k f(u^{k-1}vu^{-k}) = \sigma_k((vu^{-1})^{k-1}uvu^{-1}(uv^{-1})^k)$$

$$= \sigma_k((vu^{-1})^{k-2}v(uv^{-1})^{k-1}) = u^{-k+2}u^{k-1}vu^{-k}u^{k-1} = uvu^{-1}.$$

f can be realized by a diffeomorphism of $S^1 \times D^3 \natural S^1 \times D^3$ by handle slides and inversions (see [33, Lemma 2]). Up to conjugation, σ_k consists of handle slides and inversions also; hence $f = \sigma_k f \sigma_k$ geometrically. Restrict f to the boundary $S^1 \times S^2 \# S^1 \times S^2$. We can assume that f fixes a ball D^3 and that the relation still holds. The required diffeomorphism is $F(x,t) = (f(x), 1-t)$.

$\square$

Hence, replacing α by $\alpha \circ F_*$ if need be, we may assume that α induces $+1$ on $\mathbb{Z}$.

We have the central extension

$$\begin{array}{ccccccc} 1 \to & \mathbb{Z} & \longrightarrow & \pi & \longrightarrow & SL(2,\mathbb{Z}) & \to 1. \\ & \| & & \| & & \| & \\ & (a^4) & & (a,b \mid a^2=b^3) & & (a,b \mid a^2=b^3,\ a^4=1) & \end{array}$$

The automorphism α of π induces $\bar{\alpha} \in \mathrm{Aut}(SL(2,\mathbb{Z}))$.

Lemma 4.3: $\bar{\alpha}$ is an inner automorphism.

Assuming the lemma, we finish the proof. $\bar{\alpha} = \mu_h$ extends to an automorphism of $\mathbb{Z}(SL(2,\mathbb{Z}))$. Define the ring homomorphism $\Phi : \mathbb{Z}(SL(2,\mathbb{Z})) \longrightarrow \mathfrak{M}(2,\mathbb{Z})$ by adding up the matrices in the formal sum. $\bar{\alpha}$ extends via Φ to $\mu_h \in \mathrm{Aut}(\mathfrak{M}(2,\mathbb{Z}))$.

We now turn to studying isomorphisms of π_2. Given $\beta : \pi_2 X_k \longrightarrow \pi_2 X_\ell$ an α-isomorphism, with inverse the α^{-1}-isomorphism β^{-1}, they lift to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\pi \arrow[rr, "w_k"] \arrow[d, bend right, "\bar{c}"'] & & \mathbb{Z}\pi \arrow[rr] \arrow[d, bend right, "c"'] & & \mathbb{Z}\pi/(w_k) \arrow[r] \arrow[d, bend right, "\beta"'] & 0 \\
0 \arrow[r] & \mathbb{Z}\pi \arrow[rr, "w_\ell"] \arrow[u, bend right, "\bar{d}"'] & & \mathbb{Z}\pi \arrow[rr] \arrow[u, bend right, "d"'] & & \mathbb{Z}\pi/(w_\ell) \arrow[r] \arrow[u, bend right, "\beta^{-1}"'] & 0 ,
\end{tikzcd}$$

where $c, d, \bar{c}, \bar{d} \in \mathbb{Z}\pi$. From the commutativity of the diagram, we find

$$\begin{cases} \alpha(w_k)\cdot c = \bar{c}\cdot w_\ell \\ \alpha^{-1}(w_\ell)\cdot d = \bar{d}\cdot w_k \\ \alpha(d)\cdot c = y\cdot w_\ell + 1 \\ \alpha^{-1}(c)\cdot d = z\cdot w_k + 1 , \end{cases}$$

for some $y, z \in \mathbb{Z}\pi$. Projecting these equations to $\mathbb{Z}(SL(2,\mathbb{Z}))$, and then mapping them to $\mathfrak{m}(2,\mathbb{Z})$ via Φ , we find

$$\begin{cases} hW_k h^{-1}\cdot C = \bar{C}\cdot W_\ell \\ h^{-1}W_\ell h\cdot D = \bar{D}\cdot W_k \\ hDh^{-1}\cdot C = Y\cdot W_\ell + I \\ h^{-1}Ch\cdot D = Z\cdot W_k + I . \end{cases}$$

These equations provide the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[rr, "W_k"] \arrow[d, bend right, "h^{-1}\bar{C}"'] & & \mathbb{Z}^2 \arrow[rr] \arrow[d, bend right, "h^{-1}C"'] & & \mathbb{Z}^2/(W_k) \arrow[r] \arrow[d, bend right, "B"'] & 0 \\
\mathbb{Z}^2 \arrow[rr, "W_\ell"] \arrow[u, bend right, "h\bar{D}"'] & & \mathbb{Z}^2 \arrow[rr] \arrow[u, bend right, "hD"'] & & \mathbb{Z}^2/(W_\ell) \arrow[r] \arrow[u, bend right, "B^{-1}"'] & 0 ,
\end{tikzcd}$$

showing that

$$\mathbb{Z}^2/(W_k) \cong \mathbb{Z}^2/(W_\ell)\,. \tag{*}$$

Now recall that the projection $\pi \longrightarrow SL(2,\mathbb{Z})$ takes $t = b^{-1}a$ to $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u^k = (a^{-1}\, b\, a\, b^{-1})^k$ to $U^k = \begin{pmatrix} a_{2k} & -a_{2k-1} \\ -a_{2k-1} & a_{2k-2} \end{pmatrix}$. Hence

$$\begin{aligned}
U^{-1}W_kU^{k+1} &= U^{k-1}(U-T^{-1}) - (I+U+\ldots+U^{k-2})T^{-1}(I-U) + U^{-1}T^{-2} \\
&= \begin{pmatrix} a_{2k-2} & -a_{2k-3} \\ -a_{2k-3} & a_{2k-4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} a_{2k-3} & 1-a_{2k-4} \\ 1-a_{2k-4} & 1+a_{2k-5} \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2a_{2k-1} & -1-a_{2k-3} \\ 2-2a_{2k-2} & -1+a_{2k-4} \end{pmatrix},
\end{aligned}$$

which gives

$$\begin{aligned}
\det W_k &= 2[a_{2k-1}(a_{2k-4}-1) + 1 - a_{2k-2} + a_{2k-3} - a_{2k-2}a_{2k-3}] \\
&= 2[(a_{2k-1}-1)(a_{2k-4}-1) - a_{2k-2}a_{2k-3}] \\
&= 2[(a_{2k-2}+a_{2k-3}-1)(a_{2k-2}-a_{2k-3}-1)-(a_{2k-1}-a_{2k-3})a_{2k-3}] \\
&= 2(a_{2k-2}^2 - 2a_{2k-2} + 1 - a_{2k-1}a_{2k-3}) \\
&= 4(1-a_{2k-2}),
\end{aligned}$$

where we used

$$\begin{aligned}
a_k^2 - a_{k+1}a_{k-1} &= a_k^2 - (a_k + a_{k-1})a_{k-1} = a_k(a_k - a_{k-1}) - a_{k-1}^2 \\
&= -(a_{k-1}^2 - a_ka_{k-2}) = \begin{cases} +1\ , & \text{if } k \text{ is even} \\ -1\ , & \text{if } k \text{ is odd}\ . \end{cases}
\end{aligned}$$

This contradicts $(*)$, thus proving Lemma 4.1.

$\square$

In order to prove Lemma 4.3, we need two (presumably well known) lemmas about the outer-automorphism groups of $PSL(2,\mathbb{Z})$ and $SL(2,\mathbb{Z})$.

Lemma 4.4: $Out(PSL(2,\mathbb{Z})) \cong \mathbb{Z}_2$, generated by conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof: $PSL(2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2(a) * \mathbb{Z}_3(b)$. From the stucture theorem for subgroups of free products (see [40]), we infer that, up to conjugation, an automorphism of $PSL(2,\mathbb{Z})$ has the form

$$\alpha \begin{cases} a \to a \\ b \to b' = wb^{\pm 1}w^{-1} \end{cases}, \text{ with } w = b^{\pm 1}a\ldots b^{\pm 1}a .$$

Since α is onto, we can write $w^{-1} = a^{\varepsilon}(b')^{\pm 1}a\ldots(b')^{\pm 1}a$, with $\varepsilon = 0$ or 1. Hence

$$b = w^{-1}(b')^{\pm 1}w = a^{\varepsilon}wb^{\pm 1}w^{-1}a\ldots awb^{\pm 1}w^{-1}a\ldots wb^{\mp 1}w^{-1}a^{\varepsilon} .$$

As there are no cancellations, we conclude that $w = 1$. This shows that the outer-automorphism group is generated by $\begin{cases} a \to a \\ b \to b^{-1} \end{cases}$, which is conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

□

Lemma 4.5: $Out(SL(2,\mathbb{Z})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, represented by

$$\begin{cases} a \to a^{\pm 1} \\ b \to b^{\pm 1} \end{cases}.$$

Proof: We have the central extension

$$\begin{array}{ccccccc} 1 \to & \mathbb{Z}_2 & \longrightarrow & SL(2,\mathbb{Z}) & \longrightarrow & PSL(2,\mathbb{Z}) & \to 1 , \\ & & & \ni & & \ni & \\ & & & g & \longrightarrow & \bar{g} & \end{array}$$

where $\mathbb{Z}_2 = \{\pm 1\}$. Given $\alpha \in Aut(SL(2,\mathbb{Z}))$, it induces $\bar{\alpha} \in Aut(PSL(2,\mathbb{Z}))$. Conjugating by an element of $GL(2,\mathbb{Z})$, we get $\bar{\alpha} = id$. Hence $\alpha(g) = \pm g = \lambda(\bar{g})\cdot g$, where $\lambda: PSL(2,\mathbb{Z}) \longrightarrow \{\pm 1\}$. As $\{\pm 1\}$ is central, we see that λ is a homomorphism $\mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow \mathbb{Z}_2$. The only nontrivial such homomorphism is the projection $\begin{cases} a \to -1 \\ b \to 1 \end{cases}$. As $-a = a^{-1}$, we get $\alpha \begin{cases} a \to a^{-1} \\ b \to b \end{cases}$.

□

Proof of Lemma 4.3: We are given $\alpha \in Aut\,\pi$ inducing $+1$ on $\pi/\pi' \cong \mathbb{Z}$. Up to conjugation, the induced automorphism $\bar{\alpha} \in Aut(SL(2,\mathbb{Z}))$ has the form $\bar{\alpha} \begin{cases} a \to a^{\pm 1} \\ b \to b^{\pm 1} \end{cases}$, by Lemma 4.5. Hence α has the form $\alpha \begin{cases} a \to a^{\pm 1} a^{4i} \\ b \to b^{\pm 1} a^{4j} \end{cases}$. The abelianization map $\gamma : \pi \longrightarrow \mathbb{Z}$ is given by $\gamma(a) = 3$, $\gamma(b) = 2$. Since α induces $+1$ on $\mathbb{Z}$, we get

$$1 = \gamma(\alpha(b^{-1}a)) = \gamma(b^{\mp 1}a^{-4j}a^{\pm 1}a^{4i}) = \mp 2 \pm 3 + 12(i-j),$$

which rules out everything except $\bar{\alpha} = id$.

□

III. k- INVARIANTS OF KNOTS IN S^4

§1. Lens spaces

From now on, p and q will be coprime integers, p odd, $0 < q < p$. The lens space $L(p,q)$ is the 2-fold branched cover of S^3, branched over a 2-bridge knot $B_{p,q}$ (Schubert [49]). $L(p,q)$ can be expressed as two solid tori sewn together along their boundaries by the matrix $\begin{pmatrix} q & r \\ p & s \end{pmatrix}$, where $qs - pr = 1$. The branched covering involution τ is a 180° rotation in the axes shown below. See Rolfsen [47, p. 303] for a picture of the branched set downstairs.

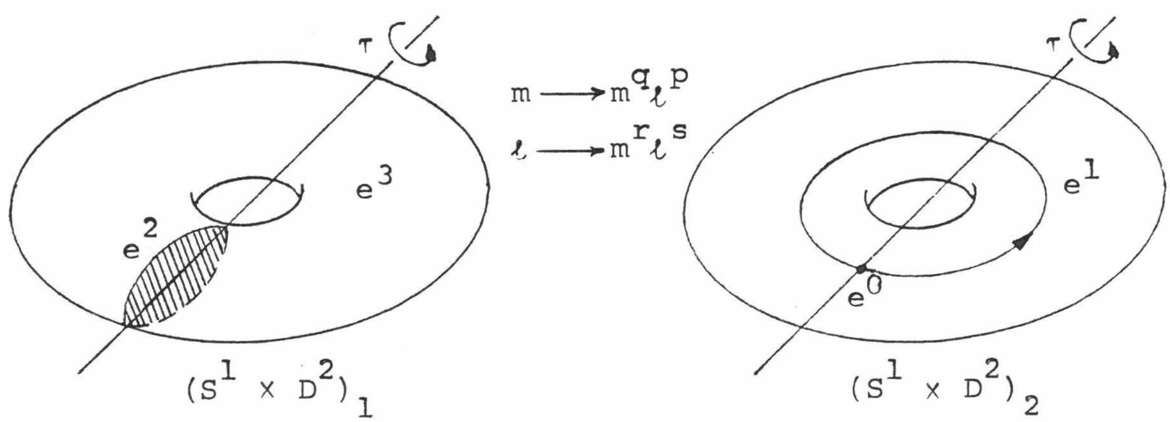


Figure 10

A cell decomposition for $L(p,q)$ consists of e^0, the core e^1 of $(S^1 \times D^2)_2$, a meridinal disk e^2 of

$(S^1 \times D^2)_1$, and e^3. The cell e^2 is attached to e^1 by a map of degree p. The fundamental group is $\mathbb{Z}_p = (a \mid a^p = 1)$, where $a = e^1$. Note, for further use, that $\tau_*(a) = a^{-1}$. The resulting augmented chain complex for $\widetilde{L(p,q)} = S^3$ is

$$0 \longrightarrow \mathbb{Z}\mathbb{Z}_p \xrightarrow{a^s - 1} \mathbb{Z}\mathbb{Z}_p \xrightarrow{N} \mathbb{Z}\mathbb{Z}_p \xrightarrow{a-1} \mathbb{Z}\mathbb{Z}_p \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 ,$$

where $N = 1 + a + \ldots + a^{p-1}$ is the norm element.

Let $\mathring{L}(p,q)$ be the punctured lens space, with cell decomposition $e^0 \cup e^1 \cup e^2$. Since $\widetilde{\mathring{L}(p,q)} = S^3 - \bigcup_1^p D^3$, $0 \longrightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z}\mathbb{Z}_p \longrightarrow \pi_2(\mathring{L}(p,q)) \longrightarrow 0$ is an exact sequence of $\mathbb{Z}\mathbb{Z}_p$-modules. This gives $\pi_2(\mathring{L}(p,q)) = \mathbb{Z}\mathbb{Z}_p/N$, generated by the boundary 2 sphere $b = (a^s - 1)e^2$. To identify the k-invariant of $\mathring{L}(p,q)$, we use the partial free resolution of $\mathbb{Z}$ over $\mathbb{Z}\mathbb{Z}_p$ provided by $C_*(\widetilde{L(p,q)})$ and pull it back to the standard resolution:

$$\begin{tikzcd}
\mathbb{Z}\mathbb{Z}_p \arrow[r, "N"] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "a-1"] \arrow[d, "\tilde{k} = 1 + a^s + \ldots + a^{s(q-1)}"'] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "N"] \arrow[d, equal] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "a-1"] \arrow[d, equal] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "\epsilon"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbb{Z}\mathbb{Z}_p \arrow[r, "a^s - 1"] \arrow[dr, "k"'] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "N"] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "a-1"] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0 \\
 & & \pi_2 = \ker N = \mathbb{Z}\mathbb{Z}_p/N \arrow[u, hook] & & & &
\end{tikzcd}$$

Since $H^i(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p) = H^i(L^\infty(p), \mathbb{Z}\mathbb{Z}_p) = H^i_f(S^\infty, \mathbb{Z}) = 0$ for $i > 0$, the exact sequence defining π_2 gives

$0 \to H^3(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p/N) \xrightarrow[\simeq]{\delta} H^4(\mathbb{Z}_p, \mathbb{Z}) \to 0$. From the diagram

$$\begin{tikzcd}
 & & \tilde{k} \arrow[d, phantom, "\cap"] \arrow[rr] & & k \arrow[d, phantom, "\cap"] & \\
0 \arrow[r] & C^3(\mathbb{Z}_p,\mathbb{Z}) \arrow[r, "N"] \arrow[d, "d^3"] & C^3(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p) \arrow[rr] \arrow[d, "d^3"] & & C^3(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p/N) \arrow[r] \arrow[d, "d^3"] & 0 \\
0 \arrow[r] & C^4(\mathbb{Z}_p, \mathbb{Z}) \arrow[r] & C^4(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p) \arrow[rr] & & C^4(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p/N) \arrow[r] & 0
\end{tikzcd} ,$$

we have $d^3\tilde{k}(1) = \tilde{k}(N) = N \cdot (1+a^s+\ldots+a^{s(q-1)}) = q \cdot N$, so that δk is $\mathbb{Z}\mathbb{Z} \xrightarrow{q \cdot \varepsilon} \mathbb{Z}$. This means that the connecting homomorphism δ takes $[k]$ to $q \cdot [\varepsilon]$. Since $[\varepsilon]$ generates $H^4(\mathbb{Z}_p, \mathbb{Z}) \simeq \mathbb{Z}_p$, we conclude

$$\begin{tikzcd}
H^3(\pi_1 \mathring{L}(p,q), \pi_2\mathring{L}(p,q)) \arrow[rr, "\delta", "\simeq"'] \arrow[d, phantom, sloped, "\ni"] & & H^4(\mathbb{Z}_p,\mathbb{Z}) \simeq \mathbb{Z}p \arrow[d, phantom, sloped, "\ni"] \\
k(\mathring{L}(p,q)) \arrow[rr] & & q
\end{tikzcd}$$

(Compare with Plotnick [42]).

The punctured lens spaces $\mathring{L}(p,q)$ have the same 2-skeleton, independent of q , and so are homotopy equivalent for all q . A homotopy equivalence $\mathring{L}(p,q) \simeq \mathring{L}(p,q')$ (rel ∂) extends to $L(p,q) \simeq L(p,q')$. A well known theorem of J.H.C. Whitehead [52] asserts that $L(p,q) \simeq L(p,q')$ if, and only if, $\pm qq'$ is a quadratic residue (mod p). In fact, given a homotopy equivalence $f: L(p,q) \longrightarrow L(p,q')$, f induces $n: \mathbb{Z}_p \to \mathbb{Z}_p$ on π_1 , with $(n,p) = 1$, and, via duality and cup product, f induces $n^2: \mathbb{Z}_p \to \mathbb{Z}_p$ on H^4 . We can assume $f = \pm id$ on

$\partial \mathring{L}(p,q)$. Hence f induces ± 1 on π_2. As f preserves k-invariants, $n^2 q' = \pm q \pmod p$. The converse follows from the theorem of MacLane-Whitehead [39] by reversing the argument (or directly, by constructing a chain equivalence of the chain complexes of the universal covers).

§2. 2-twist spun 2-bridge knots

In this section we analyze the algebraic 3-type of certain fibered knots with fiber $\mathring{L}(p,q)$. We show that the k-invariant of the knot exterior restricts to the k-invariant of the fiber. Although that can be seen via a spectral sequence argument [43], we will need this specific approach in proving Theorem I.1.1. Besides, the calculations here serve as a good warmup for the later sections.

Let K be a knot in S^3 and m a positive integer. The <u>m-twist spin</u> of K, K^m, is a fibered knot in S^4 whose fiber is the punctured m-fold cyclic branched cover of (S^3, K), and whose monodromy is the canonical branched covering transformation [55]. With the notation from §1, the exterior of the 2-twist spin of the 2-bridge knot $B_{p,q}$ is $B^2_{p,q} = \mathring{L}(p,q) \times_{\tau} S^1$. The boundary of $\mathring{L}(p,q)$ is a 2-sphere b, intersecting the branch set in two points. If $*$ is one of them, then $* \times S^1$ is a meridian of $B^2_{p,q}$. The fundamental group is $H \equiv \pi_1(B^2_{p,q}) = (a,x \mid a^p = 1,\ xax^{-1} = \tau_*(a)) = (a,x \mid a^p = 1,\ xax^{-1} = a^{-1}) = \mathbb{Z}_p \rtimes \mathbb{Z}$. As a $\mathbb{Z}\mathbb{Z}_p$-module, $\pi_2(B^2_{p,q})$ is just $\pi_2(\mathring{L}(p,q)) = \mathbb{Z}\mathbb{Z}_p/N$, generated by the boundary sphere b. As a $\mathbb{Z}H$-module, $\pi_2(B^2_{p,q}) = \operatorname{coker}(\mathbb{Z}H/N \xrightarrow{\cdot(x-1)} \mathbb{Z}H/N) = \mathbb{Z}H/(x-1)$, by a result of Andrews-Sumners [4]. In fact,

$\widetilde{B}^2_{p,q} = \widetilde{\mathring{L}(p,q)} \times \mathbb{R}$ and π_2 is generated by b, subject to the relations $Nb = 0$, $(x-1)b = 0$.

A cell decomposition for $B^2_{p,q}$ consists of e^0, e^1_a, e^2 (the cells of $\mathring{L}(p,q)$), $e^1_x = e^0 \times S^1$, $e^1_a \times S^1$, and $e^2 \times S^1$. The boundary maps in the augmented chain complex of $\widetilde{B}^2_{p,q}$ can be computed from the picture below:

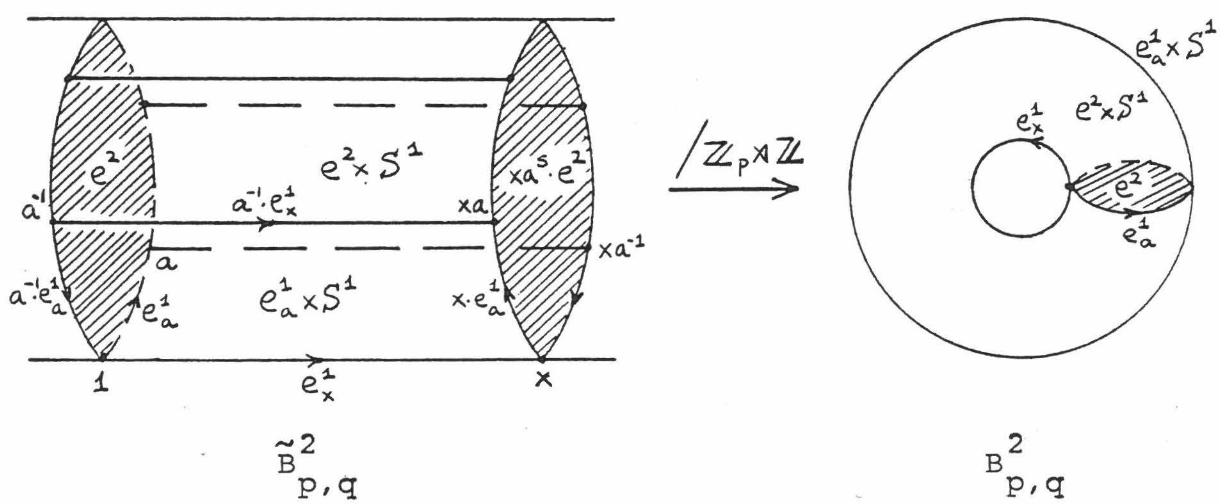


Figure 11

We have that

$$\partial_2(e^1_a \times S^1) = (1-a^{-1}) \cdot e^1_x + (a^{-1} + x) \cdot e^1_a \quad \text{(Fox derivatives)}$$

$$\partial_3(e^2 \times S^1) = N \cdot e^1_a \times S^1 - (1 + xa^s)e^2 .$$

This gives $C_*(\widetilde{B}^2_{p,q})$:

$$\mathbb{Z}H \xrightarrow{(N,-(1+xa^s))} (\mathbb{Z}H)^2 \xrightarrow{\begin{pmatrix} 1-a^{-1} & a^{-1}+x \\ 0 & N \end{pmatrix}} (\mathbb{Z}H)^2 \xrightarrow{\begin{pmatrix} x-1 \\ a-1 \end{pmatrix}} \mathbb{Z}H \xrightarrow{\varepsilon} \mathbb{Z} \to 0 .$$

Note that $\partial_3(b \times S^1) = \partial_3((a^s-1)e^2 \times S^1) = -(a^s-1)(1+xa^s)e^2 = (x-1)(a^s-1)e^2 = (x-1)b$, confirming the computation of π_2.

We now fit π_2 into an exact sequence, similar to that defining $\pi_2(\mathring{L}(p,q))$.

<u>Lemma 2.1</u>: $\pi_2(B^2_{p,q})$ is given by the following exact sequence of $\mathbb{Z}H$-modules:

$$0 \to \mathbb{Z}(H/\mathbb{Z}_p) \xrightarrow{(-N,x-1)} \mathbb{Z}H \oplus \mathbb{Z}(H/\mathbb{Z}_p) \xrightarrow{\begin{pmatrix} x-1 \\ N \end{pmatrix}} \mathbb{Z}H \to \pi_2 \to 0 .$$

<u>Proof</u>: $\mathbb{Z}(H/\mathbb{Z}_p) = \mathbb{Z}\mathbb{Z}$ is the induced module $\mathbb{Z}H \otimes_{\mathbb{Z}\mathbb{Z}_p} \mathbb{Z}$. The abelianization map $H \to H/\mathbb{Z}_p = \mathbb{Z}$ induces a ring homomorphism $\mathbb{Z}H \to \mathbb{Z}\mathbb{Z}$, $\xi \mapsto \bar{\xi}$. The homomorphism $\mathbb{Z}(H/\mathbb{Z}_p) \xrightarrow{N} \mathbb{Z}H$, $\bar{\xi} \mapsto \xi N$, is induced from $\mathbb{Z} \xrightarrow{N} \mathbb{Z}\mathbb{Z}_p$, and so is injective. Lemma II.2.1 shows that the homomorphism $\mathbb{Z}\mathbb{Z} \xrightarrow{(x-1)} \mathbb{Z}\mathbb{Z}$, $\bar{\xi} \mapsto \overline{\xi(x-1)}$, is injective.

The standard free $\mathbb{Z}\mathbb{Z}_p$-resolution of $\mathbb{Z}$ induces a free $\mathbb{Z}H$-resolution, which gives $\ker(a-1) = \operatorname{Im}N$, $\ker N = \operatorname{Im}(a-1)$. All that is left to prove is exactness at $\mathbb{Z}H \oplus \mathbb{Z}(H/\mathbb{Z}_p)$.

Let $(\mu, \bar{\nu}) \in \ker(\mathbb{Z}H \oplus \mathbb{Z}(H/\mathbb{Z}_p) \xrightarrow{\begin{pmatrix} x-1 \\ N \end{pmatrix}} \mathbb{Z}H)$. From $\mu(x-1) + \nu N = 0$, we get $0 = \mu(x-1)(a-1) = \mu(1-a)(1+a^{-1}x)$. Since $a^{-1}x$ has infinite order, $\mu(1-a) = 0$, and so $\mu = \xi N$. Now $(\xi(x-1) + \nu)N = 0$ implies $\nu = -\xi(x-1) + \xi'(a-1)$ and hence $\bar{\nu} = \overline{-\xi(x-1)}$, which shows $(\mu, \bar{\nu}) = (-\bar{\xi}) \cdot (-N, x-1)$. $\square$

We now describe $k(B^2_{p,q})$. To $C_*(\tilde{B}^2_{p,q})$ add a free summand $\bar{C}_3(B) = \mathbb{Z}H(b)$ and map $\bar{C}_3 \xrightarrow{\bar{\partial}_3} C_2(\tilde{B})$ so as to kill π_2 : $\bar{\partial}_3(b) = b$. The natural inclusion $C_*(\widetilde{\mathring{L}(p,q)}) \hookrightarrow C_*(\tilde{B}^2_{p,q})$ extends to $C_*(\widetilde{L(p,q)}) \hookrightarrow \bar{C}_3 \oplus C_*(\tilde{B}^2_{p,q})$:

$$
\begin{tikzcd}
\mathbb{Z}\mathbb{Z}_p \arrow[rr, "a^s-1"] \arrow[ddd, hook] \arrow[ddr, "k(\mathring{L}(p,q))"'] & & \mathbb{Z}\mathbb{Z}_p \arrow[r, "N"] \arrow[ddd, hook] & \mathbb{Z}\mathbb{Z}_p \arrow[r, "a-1"] \arrow[ddd, hook] & \mathbb{Z}\mathbb{Z}_p \arrow[r] \arrow[ddd, hook] & \mathbb{Z} \arrow[r] \arrow[ddd, equal] & 0 \\
 & \ker N \arrow[ur, hook] \arrow[d, equal] & & & & & \\
 & \pi_2\mathring{L}(p,q) = \mathbb{Z}\mathbb{Z}p/N \arrow[ddd, bend right] & & & & & \\
\mathbb{Z}H \oplus C_3(\tilde{B}) \arrow[rr, "(\bar{\partial}_3{,}\partial_3)"] \arrow[ddr, "k(B^2_{p,q})"'] & & C_2(\tilde{B}) \arrow[r, "\partial_2"] & C_1(\tilde{B}) \arrow[r, "\partial_1"] & C_0(\tilde{B}) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \\
 & \ker\partial_2 \arrow[ur, hook] \arrow[d] & & & & & \\
 & \pi_2(B^2_{p,q}) = \mathbb{Z}H/\left(\begin{smallmatrix}x-1\\ N\end{smallmatrix}\right) & & & & &
\end{tikzcd}
$$

We see from this diagram that the cocycle representing $k(B^2_{p,q})$ restricts to the cocycle representing $k(\mathring{L}(p,q))$.

To identify $k(B^2_{p,q})$ as an element of $H^3(\pi_1(B^2_{p,q}),$ $\pi_2(B^2_{p,q})) = H^3(H,\pi_2)$, we need some facts about the cohomology of H . Note that H has $L^\infty(p) \times S^1$ as a $K(H,1)$, so $H^i(H,\mathbb{Z}H) = H^i(L^\infty(p) \underset{\tau}{\times} S^1 , \mathbb{Z}H) = H^i_f(S^\infty \times R, \mathbb{Z}) = 0$, for $i>1$. (Since $[H,\mathbb{Z}] = p < \infty$, this also follows from Shapiro's lemma: $H^i(H,\mathbb{Z}H) = H^i(\mathbb{Z},\mathbb{Z}\mathbb{Z}))$. To compute the cohomology with coefficients in $\mathbb{Z}(H/\mathbb{Z}p)$, first note that

$\mathbb{Z}(H/\mathbb{Z}_p) = \mathbb{Z}H \underset{\mathbb{Z}\mathbb{Z}_p}{\otimes} \mathbb{Z} = \underset{\mathbb{Z}}{\oplus} \mathbb{Z}$, as $\mathbb{Z}\mathbb{Z}_p$-modules. Since $K(\mathbb{Z}_p,1) = L^\infty(p)$ has one cell in each dimension, the cohomology of $\mathbb{Z}_p$ commutes with direct sums. Hence

$$H^i(\mathbb{Z}_p,\mathbb{Z}(H/\mathbb{Z}_p)) = H^i(\mathbb{Z}_p, \underset{\mathbb{Z}}{\oplus} \mathbb{Z}) = \underset{\mathbb{Z}}{\oplus} H^i(\mathbb{Z}_p,\mathbb{Z}) = \begin{cases} \underset{\mathbb{Z}}{\oplus}\mathbb{Z}_p \,; i \text{ even}>0 \\ \\ 0;\ i \text{ odd}. \end{cases}$$

The Wang sequence for the fibration $L^\infty(p) \longrightarrow K(H,1) \longrightarrow S^1$ with coefficients $\mathbb{Z}(H/\mathbb{Z}_p)$ yields

$$\begin{array}{l} 0 \longrightarrow H^4(H,\mathbb{Z}(H/\mathbb{Z}_p)) \longrightarrow \underset{\underset{\mathbb{Z}}{\oplus}\mathbb{Z}_p}{\underset{\wr\wr}{H^4(\mathbb{Z}_p,\mathbb{Z}(H/\mathbb{Z}_p))}} \xrightarrow{\cdot(x-1)} \underset{\underset{\mathbb{Z}}{\oplus}\mathbb{Z}_p}{\underset{\wr\wr}{H^4(\mathbb{Z}_p,\mathbb{Z}(H/\mathbb{Z}_p))}} \\ \qquad \longrightarrow H^5(H,\mathbb{Z}(H/\mathbb{Z}_p)) \longrightarrow 0 . \end{array}$$

Since x acts trivially on $H^4(\mathbb{Z}_p,\mathbb{Z})$, the action of x on $\underset{\mathbb{Z}}{\oplus}\mathbb{Z}_p$ is just the permutation of the cosets $\mathbb{Z} = H/\mathbb{Z}_p$. Hence $H^4(H,\mathbb{Z}(H/\mathbb{Z}_p)) = 0$ and $H^5(H,\mathbb{Z}(H/\mathbb{Z}_p)) \simeq \mathbb{Z}_p$.

The exact sequence from Lemma 2.1 breaks into the short exact sequences

$$0 \longrightarrow A \longrightarrow \mathbb{Z}H \longrightarrow \pi_2 \longrightarrow 0 \qquad (*)$$

and

$$0 \longrightarrow \mathbb{Z}(H/\mathbb{Z}_p) \xrightarrow{(-N,x-1)} \mathbb{Z}H \oplus \mathbb{Z}(H/\mathbb{Z}_p) \longrightarrow A \longrightarrow 0 \quad (**)$$

These two coefficient sequences yield:

$$H^3(H,\pi_2) \xrightarrow[\cong]{\delta} H^4(H,A)$$

and

$$0 \to H^4(H,A) \xrightarrow{\delta} H^5(H,\mathbb{Z}(H/\mathbb{Z}_p)) \xrightarrow{(x-1)} H^5(H,\mathbb{Z}(H/\mathbb{Z}_p)) \ .$$

As $H^5(H,\mathbb{Z}(H/\mathbb{Z}_p)) = \operatorname{coker}(x-1)$, multiplication by $(x-1)$ induces the zero map on it. Hence

$$H^3(H,\pi_2) \cong H^4(H,A) \cong H^5(H, \mathbb{Z}(H/\mathbb{Z}_p)) \cong \mathbb{Z}_p \ .$$

<u>Remark 1</u> The Mayer-Vietoris sequence of the extension $H = \mathbb{Z}_p \rtimes \mathbb{Z}$ (= the Wang sequence for the fibration $L^\infty(p) \to K(H,1) \to S^1$) ends in

$$H_1(H,\mathbb{Z}H) \to H_0(\mathbb{Z}_p,\mathbb{Z}H) \xrightarrow{\cdot(x-1)} H_0(\mathbb{Z}_p,\mathbb{Z}H) \to H_0(H,\mathbb{Z}H) \to 0 \ ,$$

which gives the short exact sequence $0 \to \mathbb{Z}(H/\mathbb{Z}_p) \xrightarrow{\cdot(x-1)} \mathbb{Z}(H/\mathbb{Z}_p) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$. Comparing the long exact sequences associated to this sequence and (**) , we find $H^4(H,A) \simeq H^4(H,\mathbb{Z})$. The Mayer-Vietoris sequence of the extension $H = \mathbb{Z}p \rtimes \mathbb{Z}$ gives $H^4(H,\mathbb{Z}) \simeq H^4(\mathbb{Z}p,\mathbb{Z}) \simeq \mathbb{Z}p$. We thus recover the above result.

<u>Remark 2</u> The $\mathbb{Z}H$-module $\mathbb{Z}(H/\mathbb{Z}_p) = \mathbb{Z}H \otimes_{\mathbb{Z}\mathbb{Z}_p} \mathbb{Z}$ is an induced module which is <u>not</u> coinduced. For if it were, Shapiro's lemma would give $H^*(H,\mathbb{Z}H \otimes_{\mathbb{Z}\mathbb{Z}_p} \mathbb{Z}) \cong H^*(\mathbb{Z}_p,\mathbb{Z})$, which is not the case.

As a consequence of these computations we get the following proposition, which is a particular case of a theorem of Plotnick [43]:

Proposition 2.2: The inclusion $\pi_1(\mathring{L}(p,q)) \hookrightarrow \pi_1(B^2_{p,q})$ induces an isomorphism $H^3(\pi_1\mathring{L}(p,q), \pi_2\mathring{L}(p,q)) \xleftarrow{\simeq} H^3(\pi_1 B^2_{p,q}, \pi_2 B^2_{p,q})$ under which k-invariants correspond, namely

$$\begin{array}{ccc} H^3(H, \pi_2) \xrightarrow{\cong} H^3(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p/N) & \simeq & \mathbb{Z}_p \\ \Psi & & \Psi \\ k(B^2_{p,q}) & \longrightarrow & q \end{array}$$

$\square$

The knot exteriors $B^2_{p,q}$ have the same 3-skeleton, independent of q, and so are homotopy equivalent for all q. An algebraic way of seeing this will be given at the end of §7. On the other hand, $B^2_{p,q} \simeq B^2_{p,q'}$ $(\mathrm{rel}\,\partial)$ if, and only if, $L(p,q) \simeq L(p,q')$ [43].

§3. The knots $K_{p,q}$

We now describe our examples of knots with different k-invariants. We start by giving a general construction of "companion" knots, and interpret this construction in terms of surgery.

Let $K_i = (S^{n+2}, S^n)$, $n > 1$, $i=1,2$, be knots with exteriors X_i and meridians t_i. Let c be a simple closed curve in X_1. If we remove a tubular neighborhood of c from X_1 we get a space Z_1 diffeomorphic to $S^n \times D^2 - N(K_1)$. Let $X = Z_1 \underset{S^1 \times S^n}{\cup} X_2$. Since

$$X \underset{S^1 \times S^n}{\cup} D^2 \times S^n = (Z_1 \cup D^2 \times S^n) \cup X_2 = S^{n+2} ,$$

the space X is the exterior of a knot K in S^{n+2}, with meridian t_1.

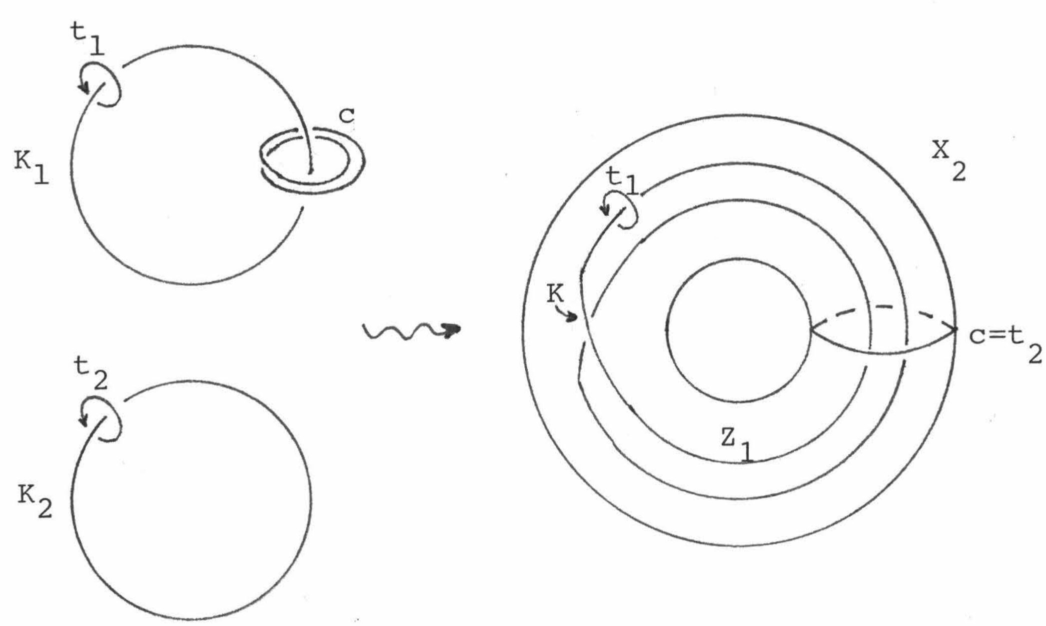


Figure 12

The knot K has fundamental group

$$\pi_1 X = \pi_1 X_1 \underset{\mathbb{Z}/|c|}{*} (\pi_1 X_2 / \langle t_2^{|c|} = 1 \rangle),$$

where $|c|$ is the order of c in $\pi_1 X_1$. If c has infinite order, $\pi_1 X = \pi_1 X_1 * \pi_1 X_2$.

Another construction is the following. Surgery on K_2 yields $Y_2 = X_2 \cup_{S^1 \times S^n} S^1 \times D^{n+1}$. Form $X_1 \# Y_2$ and do surgery on the curve $c^{-1} t_2$. Call the resulting manifold X' (see Figure 13). It is the exterior of a knot K' in S^{n+2}. Indeed, adding $D^n \times S^2$ to X' along $S^1 \times D^{n+1}$ kills t_1. The result is $S^{n+2} \# Y_2$ with surgery performed on the curve t_2, which is precisely S^{n+2}.

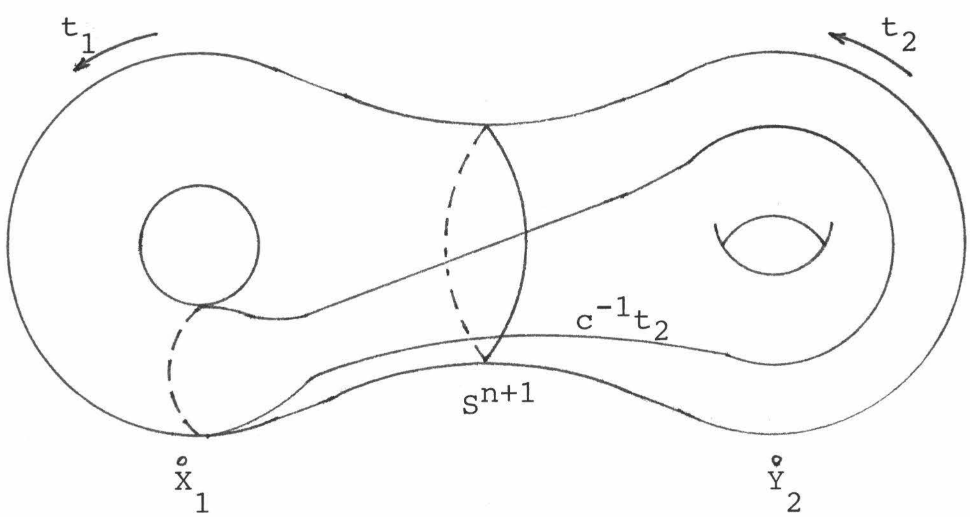


Figure 13

<u>Proposition 3.1</u>: The knot exteriors X and X' are diffeomorphic.

Proof: Note first that $X_1 \# Y_2$ with a tubular neighborhood of $c^{-1}t_2$ removed is $Z_1 \underset{I \times S^n}{\cup} X_2$. Hence

$$X' = (Z_1 \underset{I \times S^n}{\cup} X_2) \underset{S^1 \times S^n}{\cup} D^2 \times S^n =$$

$$= Z_1 \underset{S^1 \times S^n}{\cup} X_2 = X.$$

$\square$

Remark 1: If $c = t_1$, the knot K' is just $K_1 \# K_2$. The above argument is similar to the one showing that (integral) surgery on a composite classical knot is $X_1 \cup X_2$ [24, pp. 700-701].

Remark 2: If we choose the right framings, we actually get $K = K'$.

Our examples of knots with different k-invariants are obtained as follows. Start with $S^1 \times D^3 \# S^1 \times S^3$, and let $\pi_1(S^1 \times D^3) = \mathbb{Z}(t)$, $\pi_1(S^1 \times S^3) = \mathbb{Z}(y)$. Perform surgery on the curve $tyt^{-1}y^{-2}$ to get the (ribbon) knot exterior X, with $\pi_1 X \cong G = (t,y \mid tyt^{-1} = y^2)$ (see II, §1). Let $Y_{p,q} = B^2_{p,q} \underset{S^2 \times S^1}{\cup} D^3 \times S^1 = L(p,q) \underset{\tau}{\times} S^1$ be surgery on the 2-twist spun 2-bridge knot from §2. $\pi_1(B^2_{p,q}) = \pi_1(Y_{p,q})$ is the semi-direct product $H = \mathbb{Z}_p \rtimes \mathbb{Z}$ $= (a,x \mid a^p = 1,\ xax^{-1} = a^{-1})$. Apply the above construction with $X_1 = X$, $X_2 = B^2_{p,q}$, $c = y$, to get a knot $K_{p,q}$ in S^4, with exterior $X_{p,q}$. According to Proposition 3.1, $X_{p,q}$ is the result of surgery on the curve $y^{-1}x$ in

$X \# Y_{p,q}$.

Notice that $S^1 \times S^3 \# Y_{p,q}$, with surgery on the curve $y^{-1}x$ is just $Y_{p,q}$. Hence $X_{p,q}$ can be described as the manifold obtained from $S^1 \times D^3 \# Y_{p,q}$ by performing surgery on the curve $txt^{-1}x^{-2}$ (Figure 14).

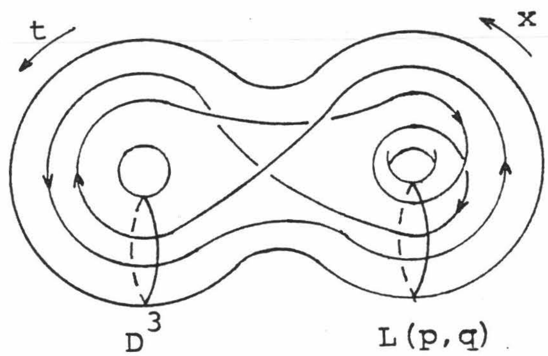


Figure 14

We write:

$$\pi_1 = \pi_1(X_{p,q}) = \mathbb{Z} * H / \langle txt^{-1}x^{-2} \rangle =$$

$$= (t,a,x \mid a^p = 1,\ xax^{-1} = a^{-1}\ ,\ txt^{-1} = x^{-2}) = G \underset{\mathbb{Z}(x)}{*} H\ ,$$

$$\pi_2 = \pi_2(X_{p,q}).$$

The classification of the knot exteriors $X_{p,q}$ mirrors the classification of the "fibers" $L(p,q)$. We will prove in §7 the following theorem:

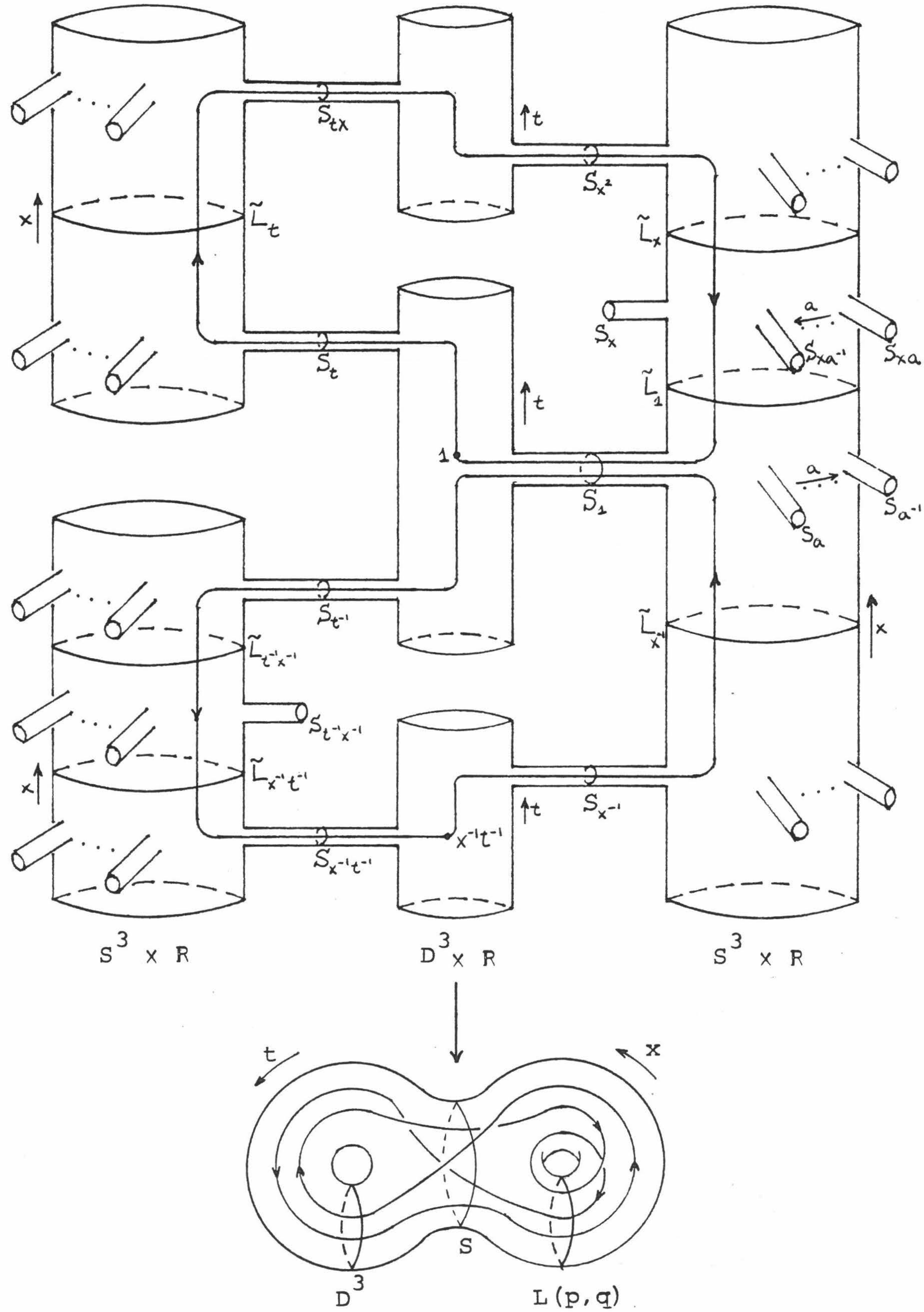
<u>Theorem 3.2</u>: Assume $\pm qq'$ is not a quadratic residue mod p. Then the knots $K_{p,q}$ and $K_{p,q'}$ have isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$ -modules), but there is no map $f: X_{p,q} \longrightarrow X_{p,q'}$ realizing an isomorphism on π_1 and π_2.

By choosing p to contain enough distinct primes of the form 4i+1 in its factorization, we can produce arbitrarily many pairs q, q' with $\pm qq' \not\equiv n^2 \pmod{p}$, thus proving Theorem I.1.1.

<u>Remark 3</u>: If we take $X_1 = B^2_{p,q}$, $X_2 = X$ and c = a, we get the González-Acuña and Montesinos knots [18], with $\pi_1 = (t,a,x \mid a^p = 1,\ tat^{-1} = a^{-1},\ axa^{-1} = x^2)$ having infinitely many ends. If we perform surgery on one of these knots, and then remove a neighborhood of the curve xt^{-1}, we get the exterior of a quasi-aspherical knot with infinitely many ends [19].

§4. Computation of $\pi_2(X_{p,q})$

Let M be the cover of $S^1 \times D^3 \# Y_{p,q}$ corresponding to the kernel of $\mathbb{Z}*H \longrightarrow \mathbb{Z}*H/\langle txt^{-1}x^{-2}\rangle = \pi$. Equivariant surgery on the lifts of $txt^{-1}x^{-2}$ in M produces $\tilde{X}_{p,q}$. M consists of copies of $\tilde{Y}_{p,q} = S^3 \times R$, indexed by the cosets π/H , and copies of $D^3 \times R$, indexed by the cosets $\pi/\mathbb{Z}(t)$, tubed together by "connectors" $S^3 \times I$, indexed by π . Here is a schematic picture of the covering $M \longrightarrow S^1 \times D^3 \# Y_{p,q}$, together with two lifts of the surgery curve:



t
x
$\tilde{L}_t$
S_{tx}
S_{x^2}
$\tilde{L}_x$
S_x
$S_{xa^{-1}}$
S_{xa}
a
S_t
$\tilde{L}_1$
1
S_1
$S_{a^{-1}}$
S_a
$S_{t^{-1}}$
$\tilde{L}_{t^{-1}x^{-1}}$
$S_{t^{-1}x^{-1}}$
$\tilde{L}_{x^{-1}}$
$\tilde{L}_{x^{-1}t^{-1}}$
$S_{x^{-1}}$
$x^{-1}t^{-1}$
$S_{x^{-1}t^{-1}}$
$S^3 \times R$
$D^3 \times R$
$S^3 \times R$
S
D^3
L(p,q)

Figure 1 5

Let $M = M_0 \underset{\amalg_\pi S^1 \times S^2}{\cup} (\amalg_\pi S^1 \times D^3)$, $\widetilde{X}_{p,q} = M_0 \underset{\amalg_\pi S^1 \times S^2}{\cup} (\amalg_\pi D^2 \times S^2)$.

The Mayer-Vietoris sequences corresponding to these decompositions (see II, §2) yield:

$$H_3(\widetilde{X}_{p,q}) = \ker\, (H_3(M) \xrightarrow{\varphi} \oplus_\pi H_2(S^1 \times S^2))$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{coker}\varphi \arrow[r] & H_2(M_0) \arrow[r] \arrow[d] & H_2(M) \arrow[r] & 0 \\
 & & H_2(\widetilde{X}_{p,q}) \arrow[d] & & \\
 & & \ker\psi \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $\psi\colon \oplus_\pi H_1\ (S^1 \times S^2) \longrightarrow H_1(M_0)$.

We saw during the proof of Proposition II.2.2 that $\psi_G\colon \mathbb{Z}G \longrightarrow \mathbb{Z}G$ is an isomorphism. Hence $\psi = 1 \otimes \psi_G$: $\mathbb{Z}\pi \otimes_{\mathbb{Z}G} \mathbb{Z}G \longrightarrow \mathbb{Z}\pi \otimes_{\mathbb{Z}G} \mathbb{Z}G$ is also an isomorphism and $\ker\psi = 0$. Also, $H_2(M) = \mathbb{Z}\pi \otimes_{\mathbb{Z}H} H_2(\widetilde{Y}_{p,q}) = 0$. Therefore $\pi_2 = H_2(\widetilde{X}_{p,q}) = \operatorname{coker}\varphi$.

$C_3(M)$ is generated by the lifts $\{\widetilde{L}_g;\ g \in \pi/\mathbb{Z}_p\}$ of the "fiber" $L(p,q)$. Notice that $\widetilde{L}_1 - \widetilde{L}_{x^{-1}} = S_1 + S_a + \ldots + S_{a^{p-1}}$ in $H_3(M)$. The proof of Lemma 2.1, together with $\mathbb{Z}\pi = \mathbb{Z}\pi \otimes_{\mathbb{Z}H} \mathbb{Z}H$, yields the short exact sequence

$$0 \longrightarrow \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{(-N,\ 1-x^{-1})} \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p) \longrightarrow H_3(M) \longrightarrow 0 . \qquad (1)$$

To compute φ, examine the lifts of $txt^{-1}x^{-2}$ which cut through S_1^3 and $\widetilde{L}_1$ (Figures 8 and 15) and get $\varphi(S_1^3) = 1 + x^{-1}t^{-1} - t^{-1} - x^{-2} = (1 - x^{-1})(1 + x^{-1} - t^{-1})$ and $\varphi(\widetilde{L}_1) = N(1 + x^{-1} - t^{-1})$. This means that φ lifts to

$$\begin{tikzcd}
\mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p) \arrow[r] \arrow[dr, "{((1-x^{-1})(1+x^{-1}-t^{-1}),\ N(1+x^{-1}-t^{-1}))}"'] & H_3M \arrow[r] \arrow[d, "\varphi"] & 0 \\
& \mathbb{Z}\pi &
\end{tikzcd} \tag{2}$$

The map $(1+x^{-1}-t^{-1})\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$ is injective, since $(1+x^{-1}-t^{-1})(1-x^{-1}) = (1-x^{-2})(1-t^{-1})$, and the maps $(1-x^{-1})$, $(1-x^{-2})$ and $(1-t^{-1})$ are injective by Lemma II.2.1. Tensoring the exact sequence given by Lemma 2.1 with $\mathbb{Z}\pi$ gives the exact sequence of $\mathbb{Z}\pi$-modules

$$0 \longrightarrow \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{(-N,\,x-1)} \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{\begin{pmatrix} x-1 \\ N \end{pmatrix}} \mathbb{Z}\pi .$$

Therefore the sequence

$$0 \longrightarrow \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{(-N,\,x-1)} \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{\begin{pmatrix} (x-1)\cdot(1+x^{-1}-t^{-1}) \\ N\ \cdot(1+x^{-1}-t^{-1}) \end{pmatrix}} \mathbb{Z}\pi \tag{3}$$

is exact. Hence $H_3(\widetilde{X}_{p,q}) = \ker \varphi = 0$, and π_2 is given by the exact sequence

$$0 \longrightarrow H_3M \xrightarrow{\ \varphi\ } \mathbb{Z}\pi \longrightarrow \pi_2 \longrightarrow 0 \ . \tag{4}$$

Splicing together the sequences (2)-(4) yields

<u>Proposition 4.1</u>: The knot $K_{p,q}$ is quasiaspherical, with $\pi_2(X_{p,q})$ given by the following exact sequence of $\mathbb{Z}\pi$-modules:

$$0 \to \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{(-N,x-1)} \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{\begin{pmatrix} (x-1)(1+x^{-1}-t^{-1}) \\ N\ (1+x^{-1}-t^{-1}) \end{pmatrix}} \mathbb{Z}\pi \to \pi_2 \to 0, \qquad \square$$

<u>Remark 1</u>: The map $\cdot(1+x^{-1}-t^{-1}) : \pi_2(B^2_{p,q}) \longrightarrow \pi_2(X_{p,q})$ extends to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}(H/\mathbb{Z}_p) \arrow[r, "{(-N,x-1)}"] \arrow[d, hook] & \mathbb{Z}H \oplus \mathbb{Z}(H/\mathbb{Z}_p) \arrow[r, "{\begin{pmatrix} x-1 \\ N \end{pmatrix}}"] \arrow[d, hook] & \mathbb{Z}H \arrow[r] \arrow[d, "\cdot(1+x^{-1}-t^{-1})"'] & \pi_2 B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}(\pi/\mathbb{Z}_p) \arrow[r] & \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p) \arrow[r, "{\begin{pmatrix} (x-1)(1+x^{-1}-t^{-1}) \\ N\ (1+x^{-1}-t^{-1}) \end{pmatrix}}"] & \mathbb{Z}\pi \arrow[r] & \pi_2 \arrow[r] & 0
\end{tikzcd}$$

<u>Remark 2</u>: The quasi-asphericity of $K_{p,q}$ can also be seen directly from $\pi_1 = G *_{\mathbb{Z}} H$. The group $H = \mathbb{Z}_p \rtimes \mathbb{Z}$ has two ends, whereas $G = \mathbb{Z} *_{\mathbb{Z}} \mathfrak{J}$ has one end (write the Mayer-Vietoris sequence with $\mathbb{Z}G$ coefficients). The Mayer-Vietoris sequence for the decomposition $G *_{\mathbb{Z}} H$ with $\mathbb{Z}\pi$ coefficients shows that π has one end. Hence $K_{p,q}$ is quasiaspherical [46], [19].

§5. A cell complex for $X_{p,q}$

Recall $X_{p,q}$ is obtained from surgery on $txt^{-1}x^{-2}$ in $S^1 \times D^3 \# Y_{p,q}$. Figure 16 shows $S^1 \times D^3 \# Y$, the surgery curve and its intersections with two fibers of Y.

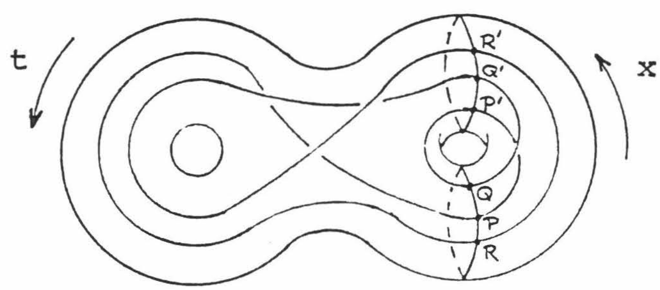


Figure 16

Remove a neighborhood of $txt^{-1}x^{-2}$ from $S^1 \times D^3 \# Y$. The resulting space consists of $(S^1 \times D^3)^{\circ}$ and $\mathring{Y}$-neighborhood of arcs, glued together along the four-times punctured "connector" S^3 :

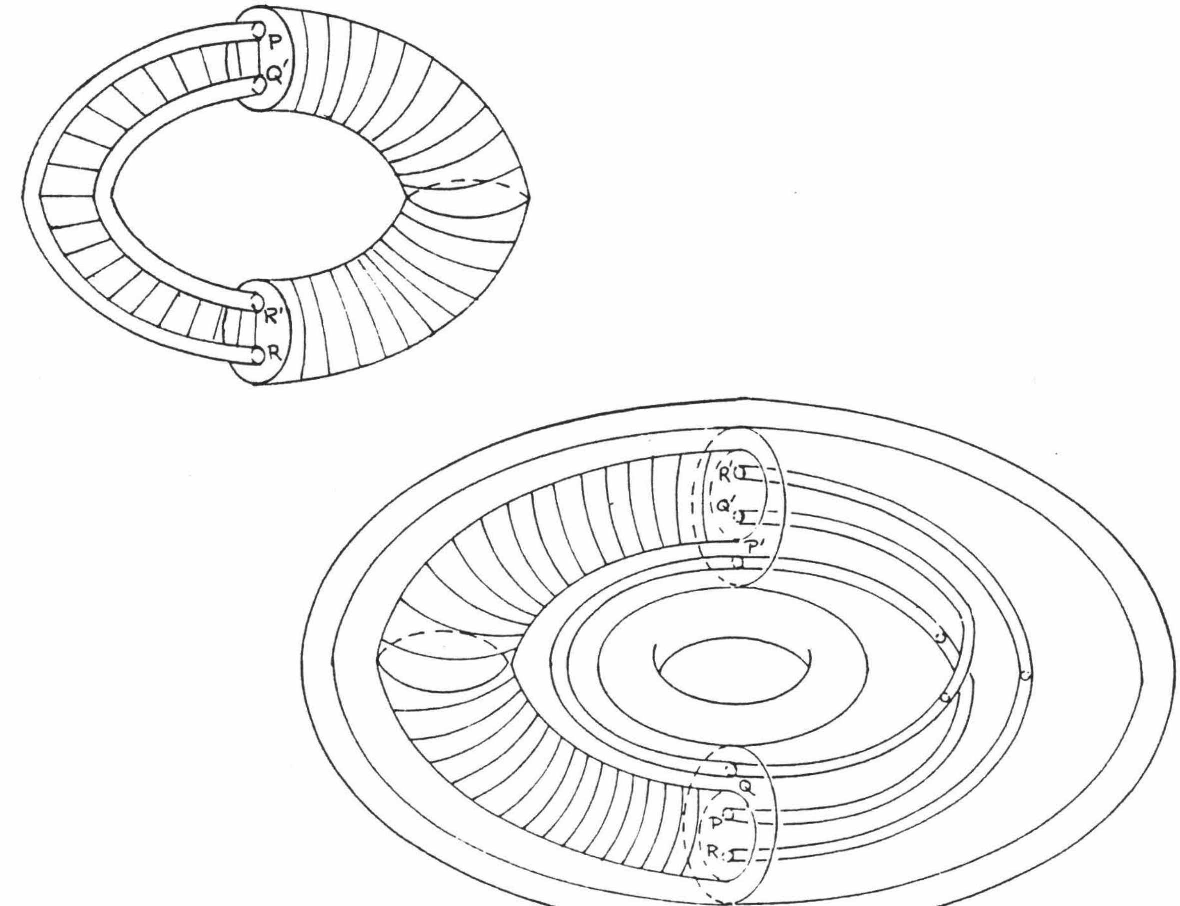
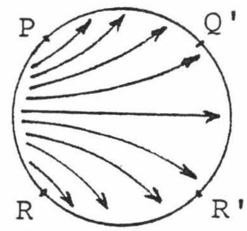


Figure 17

The outer shell of Y is $B^2_{p,q} = \mathring{L}(p,q) \times_{\tau} S^1$. Notice also that $S^1 \times D^3$-neighborhoods of arcs has been deformed to S^3 with two 1-handles, plus a 2-cell e^2_c connecting PR to $Q'R'$.

Now glue in $S^2 \times D^2$ along the boundary of a neighborhood of the surgery curve. A deformation retraction of $S^2 \times D^2$ onto $S^2 \times I$ via



brings us to Figure 18, where e_c^2 is attached along $txt^{-1}x^{-2}$.

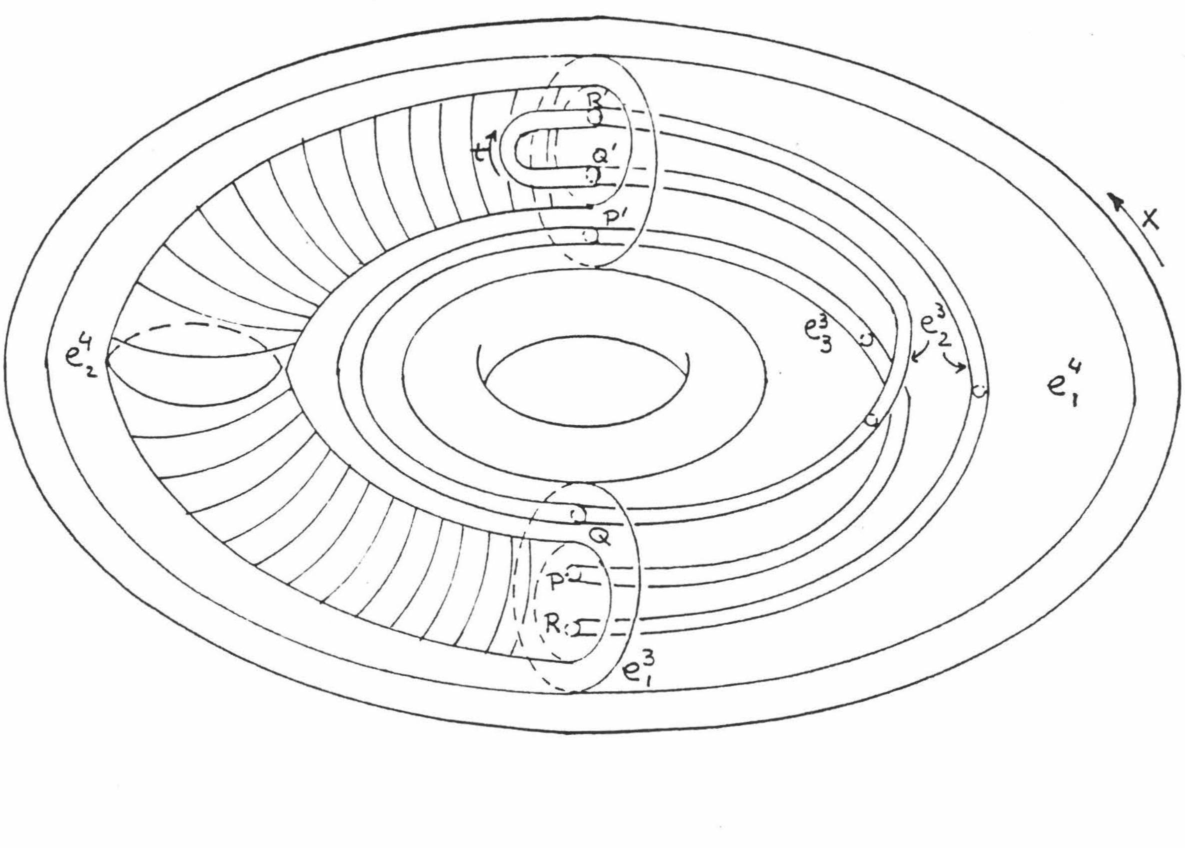


Figure 1 8

We have that

$$\partial_3 e_1^3 = b - (P + Q + R) \qquad \text{(see Figure 19)}$$

$$\partial_3 e_2^3 = Q + x^{-1}tR$$

$$\partial_3 e_3^3 = P - x\,Q\ .$$

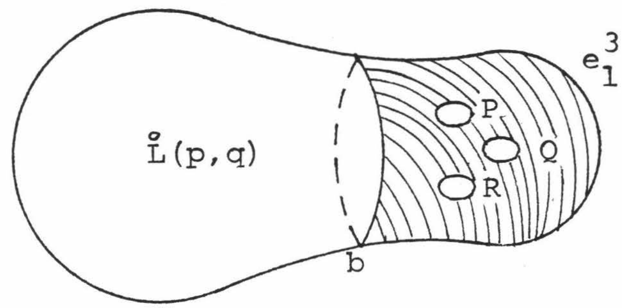


Figure 19

Finally, collapse e_1^4 and e_2^4, and cancel e_2^3, e_3^3 against Q,R, replacing e_1^3 by e^3. This gives $C_*(\widetilde{X}_{p,q})$:

$$\mathbb{Z}\pi(e^3) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_3(\widetilde{B})) \xrightarrow{\partial_3} \mathbb{Z}\pi(P) \oplus \mathbb{Z}\pi(e_c^2) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_2(\widetilde{B}))$$

$$\xrightarrow{\partial_2} \mathbb{Z}\pi(e_t^1) \oplus (\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_1(\widetilde{B})) \xrightarrow{\partial_1} \mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_0(\widetilde{B}) \xrightarrow{\varepsilon} \mathbb{Z} \to 0\ ,$$

where

$$\partial_i \Big|_{\mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_i(\widetilde{B})} = 1 \otimes \partial_i^{\widetilde{B}}$$

$$\partial_3(e^3) = b - (1+x^{-1}-t^{-1})P$$

$$\partial_2(P) = 0$$

$$\partial_2(e_c^2) = (t-x-1)\cdot e_x^1 + (1-x^2)\cdot e_t^1 \qquad \text{(Fox derivatives).}$$

Putting this together with the explicit description of $C_*(\widetilde{B}^2_{p,q})$ from §2 expresses $C_*(\widetilde{X}_{p,q})$ as:

$$(\mathbb{Z}\pi)^2 \xrightarrow{\begin{pmatrix} t^{-1}-x^{-1}-1 & 0 & 0 & a^s-1 \\ 0 & 0 & N & -(1+xa^s) \end{pmatrix}} (\mathbb{Z}\pi)^4 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 \\ 1-x^2 & t-x-1 & 0 \\ 0 & 1-a^{-1} & a^{-1}+x \\ 0 & 0 & N \end{pmatrix}} (\mathbb{Z}\pi)^3$$

$$\xrightarrow{\begin{pmatrix} t-1 \\ x-1 \\ a-1 \end{pmatrix}} \mathbb{Z}\pi \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \ .$$

From the proof of Proposition 4.1 (or from Lyndon's theorem), we deduce that e_c^2 does not contribute to ker ∂_2.

We know from §2 that $\ker \partial_2^{\widetilde{B}} = \mathbb{Z}H(b)$, where $N\, b = 0$ and $(x-1)\, b \in \operatorname{Im} \partial_3^{\widetilde{B}}$. Hence $\pi_2 X = H_2 \widetilde{X}$ is generated by P , subject to the relations $(x-1)\,(1+x^{-1}-t^{-1})\; P = 0$, $N(1+x^{-1}-t^{-1})P = 0$. Therefore, $\pi_2(X_{p,q}) =$

$$\mathbb{Z}\pi(P) \Big/\!\!\Big/ \begin{pmatrix} (x-1)\cdot(1+x^{-1}-t^{-1})P = 0 \\ N \cdot (1+x^{-1}-t^{-1})P = 0 \end{pmatrix} .$$

We now describe $k(X_{p,q})$. Add a free module $\overline{C}_3(X) = \mathbb{Z}\pi(P)$ to $C_3(\widetilde{X})$ and map $\overline{C}_3 \xrightarrow{\overline{\partial}_3} \ker \partial_2 \subset C_2$ to kill the 2-cycles: $\overline{\partial}_3(P) = P$. In §2 we defined $\overline{C}_3(B) = \mathbb{Z}H$, $\overline{\partial}_3(b) = b$. The natural inclusion $C_*(\widetilde{B}) \hookrightarrow \mathbb{Z}\pi \otimes_{\mathbb{Z}H} C_*(\widetilde{B})$ extends to a chain map $C_*(\widetilde{B}) \oplus \overline{C}_3(B) \longrightarrow C_*(\widetilde{X}) \oplus \overline{C}_3(X)$ by defining

$$\begin{aligned} \overline{C}_3(B) &\longrightarrow \overline{C}_3(X) \oplus \mathbb{Z}\pi(e^3) \\ b &\longmapsto (1+x^{-1}-t^{-1})P + e^3 \end{aligned}$$

We collect the information obtained in this section in the following diagram

$$
\begin{tikzcd}[column sep=small]
\bar{C}_3(B)\oplus C_3(\widetilde{B}) \arrow[rrr, "\partial_3"] \arrow[dddd] & & & C_2(\widetilde{B}) \arrow[r, "\partial_2"] \arrow[dddd, hook] & C_1(\widetilde{B}) \arrow[r, "\partial_1"] \arrow[dddd, hook] & \mathbb{Z}H \arrow[r, "\varepsilon"] \arrow[dddd, hook] & \mathbb{Z} \arrow[r] \arrow[dddd, equal] & 0 \\
 & \bar{C}_3(B) = \mathbb{Z}H \arrow[dr, "k(B^2_{p,q})"] \arrow[dd, "1+x^{-1}-t^{-1}"] & \ker\partial_2 = \mathbb{Z}H(b) \arrow[ur, hook] \arrow[d] & & & & & \\
 & & \pi_2 B = \mathbb{Z}H\big/\begin{pmatrix} x-1 \\ N \end{pmatrix} \arrow[ddddr, bend right, "1+x^{-1}-t^{-1}"'] & & & & & \\
 & \bar{C}_3(X) = \mathbb{Z}\pi \arrow[dddrr, bend left, "k(X_{p,q})"'] & & & & & & \\
\bar{C}_3(X)\oplus\mathbb{Z}\pi(e^3)\oplus(\mathbb{Z}\pi\otimes_{\mathbb{Z}H} C_3(\widetilde{B})) \arrow[rrr, "\partial_3"] & & & \mathbb{Z}\pi(P)\oplus\mathbb{Z}\pi(e^2)\oplus(\mathbb{Z}\pi\otimes_{\mathbb{Z}H} C_2(\widetilde{B})) \arrow[r, "\partial_2"] & \mathbb{Z}\pi\otimes_{\mathbb{Z}H} C_1(\widetilde{B}) \arrow[r, "\partial_1"] & \mathbb{Z}\pi \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \\
 & & & \ker\partial_2 = \mathbb{Z}\pi(P)\oplus\mathbb{Z}\pi(b) \arrow[u, hook] \arrow[d] & & & & \\
 & & & \pi_2 = \mathbb{Z}\pi\Big/\begin{pmatrix} (x-1)(1+x^{-1}-t^{-1}) \\ N\ (1+x^{-1}-t^{-1}) \end{pmatrix} & & & &
\end{tikzcd}
$$

This shows that the cocycle representing $k(X_{p,q})$ restricts to the cocycle representing $k(B^2_{p,q})$, followed by the map $\cdot(1+x^{-1}-t^{-1}): \pi_2 B^2_{p,q} \longrightarrow \pi_2 X_{p,q}$.

§6. Computation of the k-invariant

We now identify $k(X_{p,q})$ as an element of $H^3(\pi_1,\pi_2)$. In addition to the facts about the cohomology of H mentioned in §2 we need the following:

(1) The cohomological dimension of G is 2 by Lyndon's theorem or from the Mayer-Vietoris sequence for the HNN extension $G = \mathbb{Z}*_{\mathbb{Z}}\circlearrowleft$;

(2) $H = \mathbb{Z}_p \rtimes \mathbb{Z}$ has a $K(H,1)$ with finitely many cells in each dimension, so $H^*(H, \oplus M_i) = \oplus H^*(H,M_i)$.

The Mayer-Vietoris sequence for the amalgamation $\pi = G*_{\mathbb{Z}}H$ yields, for $i \geq 3$:

$$H^i(\pi,\mathbb{Z}\pi) = H^i(H,\mathbb{Z}\pi) = H^i(H, \bigoplus_{H\backslash\pi} \mathbb{Z}H) = \bigoplus_{H\backslash\pi} H^i(H,\mathbb{Z}H) = 0 .$$

We also have to compute $H^i(\pi,\mathbb{Z}(\pi/\mathbb{Z}p))$, for $i = 4,5$. In order to do that, we need the following lemma from [45], the proof of which we include for completeness:

<u>Lemma 6.1</u>: Given a free product with amalgamation $A*_C B$, let $a \in A$ be such that there is no $a' \in A$ with $a'aa'^{-1} \in C$. Then $waw^{-1} \in A$ implies $w \in A$.

<u>Proof</u>: Recall that each $w \in A*_C B$ has a unique normal form $w = cd_1\ldots d_n$, where the d_i are chosen alternately

from fixed coset representatives for $C\setminus A$ and $C\setminus B$. We say that any w as above has length n [40]. Elements of A and B have length ≤ 1. Suppose $waw^{-1} \in A$. If $d_n \notin A$, $waw^{-1} = cd_1 \ldots d_n a\, d_n^{-1} \ldots d_1^{-1} c^{-1}$ has length $2n+1 > 1$, a contradiction. Thus $d_n \in A$. Since $d_n a\, d_n^{-1} \notin C$, waw^{-1} has length $2n-1$. Hence, $n=1$, and $w=cd_n \in A$. $\square$

$\mathbb{Z}(\pi/\mathbb{Z}_p)$ is the induced module $\mathbb{Z}\pi \otimes_{\mathbb{Z}\mathbb{Z}_p} \mathbb{Z}$. The projection $\pi \longrightarrow \pi/\mathbb{Z}_p$, $g \longmapsto \bar{g}$ induces a homomorphism $\mathbb{Z}\pi \longrightarrow \mathbb{Z}(\pi/\mathbb{Z}_p)$, $\xi \longmapsto \bar{\xi}$. H acts on $\mathbb{Z}(\pi/\mathbb{Z}_p)$ via $h\cdot\bar{\xi} = \overline{h\xi}$. Suppose $h\cdot\bar{g} = \bar{g}$, for some $g \in \pi$, $h \in H$. Then $g^{-1}hg \in \mathbb{Z}_p$, so h is a torsion element and thus equals $a^j \in \mathbb{Z}_p$. As a^j cannot be conjugated into a power of x, Lemma 6.1 gives $g \in H$. Hence, for $g \in \pi - H$, $\bigoplus_{h\in H} \mathbb{Z}[\overline{hg}] \cong \mathbb{Z}H$. As for $g \in H$, $\bigoplus_{h\in H}\mathbb{Z}[\overline{hg}] \simeq \mathbb{Z}(H/\mathbb{Z}_p)$. We thus get the direct sum decomposition of $\mathbb{Z}H$-modules

$$\mathbb{Z}(\pi/\mathbb{Z}_p) \cong \Big(\bigoplus_{H\backslash\pi - H\cdot 1} \mathbb{Z}H\Big) \oplus \mathbb{Z}(H/\mathbb{Z}_p) .$$

The Mayer-Vietoris sequence for the amalgamation $\pi = G *_{\mathbb{Z}} H$, together with the above direct sum decomposition, facts (1) and (2), and the cohomology computations from §2, yield (for $i=4,5$)

$$H^i(\pi,\mathbb{Z}(\pi/\mathbb{Z}_p)) \cong H^i(H,\mathbb{Z}(\pi/\mathbb{Z}_p)) \cong \bigoplus_{H\backslash\pi - H\cdot 1} H^i(H,\mathbb{Z}H) \oplus H^i(H,\mathbb{Z}(H/\mathbb{Z}_p))$$

$$\cong H^i(H,\mathbb{Z}(H/\mathbb{Z}_p)) \cong \begin{cases} 0 & ; \quad i = 4 \\ \mathbb{Z}_p & ; \quad i = 5 \,. \end{cases}$$

Recall from § 4 the short exact sequences of $\mathbb{Z}\pi$-modules

$$0 \longrightarrow H_3(M) \longrightarrow \mathbb{Z}\pi \longrightarrow \pi_2 \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{(-N,x-1)} \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p) \longrightarrow H_3(M) \longrightarrow 0 \,.$$

The long exact sequences for these coefficient sequences yield

$$H^3(\pi,\pi_2) \xrightarrow[\cong]{\delta} H^4(\pi,H_3(M))$$

$$0 \longrightarrow H^4(\pi,H_3(M)) \xrightarrow{\delta} H^5(\pi, \mathbb{Z}(\pi/\mathbb{Z}_p)) \xrightarrow{\cdot(x-1)} H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}_p)) \,.$$

As in the proof of Proposition 2.2, we see that $\cdot(x-1)$ is the zero map on $H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}_p))$, and so

$$H^3(\pi,\pi_2) \cong H^4(\pi,H_3(M)) \cong H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}_p)) \cong \mathbb{Z}_p \,.$$

This, together with Proposition 2.2, Remark 1 from §4, and the results from §5, proves

<u>Proposition 6.2</u>: The inclusions $\pi_1(\mathring{L}(p,q)) \hookrightarrow \pi_1(B^2_{p,q}) \hookrightarrow \pi_1(X_{p,q})$ induce isomorphisms $H^3(\pi_1\mathring{L}(p,q), \pi_2\mathring{L}(p,q)) \xleftarrow{\simeq} H^3(\pi_1 B^2_{p,q}, \pi_2 B^2_{p,q}) \xleftarrow{\simeq} H^3(\pi_1 X_{p,q}, \pi_2 X_{p,q})$ under which k-invariants correspond, namely

$$\begin{array}{ccccccc}
H^3(\pi,\pi_2) & \xrightarrow{\simeq} & H^3(H,\pi_2 B^2_{p,q}) & \xrightarrow{\simeq} & H^3(\mathbb{Z}_p, \mathbb{Z}\mathbb{Z}_p/N) & \xrightarrow{\simeq} & \mathbb{Z}_p \\
\Cup & & \Cup & & \Cup & & \Cup \\
k(X_{p,q}) & \longrightarrow & k(B^2_{p,q}) & \longrightarrow & k(\mathring{L}(p,q)) & \longrightarrow & q
\end{array}$$

$\square$

§7. Proof of the Theorem

We now prove Theorem 3.2. Assume there is a map $f: X_{p,q} \longrightarrow X_{p,q'}$ inducing an isomorphism $\alpha: \pi_1 X_{p,q} \longrightarrow \pi_1 X_{p,q'}$ and an α-isomorphism $\beta: \pi_2 X_{p,q} \longrightarrow \pi_2 X_{p,q'}$. Then α and β preserve k-invariants. If we precede f by a map $X_{p,q} \longrightarrow X_{p,q}$, or follow f by a map $X_{p,q'} \longrightarrow X_{p,q'}$, the composed maps on π_1 and π_2 still preserve the k-invariants. Recall that $\pi = G *_{\mathbb{Z}} H$, where $G = (t,x \mid txt^{-1} = x^2)$ and $H = \mathbb{Z}_p \rtimes \mathbb{Z} = (a,x \mid a^p = 1,\ xax^{-1} = a^{-1})$.

<u>Proposition 7.1</u>: Let $\alpha \in \operatorname{Aut} \pi$. Up to conjugation, α has the form $\alpha: \begin{cases} a \longrightarrow a^n \\ x \longrightarrow x^{\pm 1} \end{cases}$, $(n,p) = 1$.

<u>Proof</u>: First note that G is torsion-free and that the only subgroup of $H = \mathbb{Z}_p \rtimes \mathbb{Z}$ isomorphic to $\mathbb{Z}_p$ is $\mathbb{Z}_p(a)$. From the structure theorem for subgroups of amalgamated products [40, p. 243], $\alpha(\mathbb{Z}_p)$ is a conjugate of $\mathbb{Z}_p$. Thus, up to conjugation, $\alpha(\mathbb{Z}_p) = \mathbb{Z}_p$. Since x normalizes $\mathbb{Z}_p$, $\alpha(x) \cdot a \cdot \alpha(x)^{-1} \in \mathbb{Z}_p$. Applying Lemma 6.1, we see that $\alpha(x) \in H$, so $\alpha(H) \subset H$. The same argument gives $\alpha^{-1}(H) \subset H$.

Suppose there is no $h \in H$ with $h\alpha(x)h^{-1} \in \mathbb{Z}(x)$. Then, since $\alpha(t) \cdot \alpha(x) \cdot \alpha(t)^{-1} = \alpha(txt^{-1}) = \alpha(x^2) \notin H$, Lemma 6.1 implies $\alpha(t) \notin H$. Since $H_1(\pi) = \mathbb{Z}(t)$ and

$H \subset [\pi,\pi]$, this is a contradiction. Hence, up to conjugation by an element $h \in H$, $\alpha(x) \in \mathbb{Z}(x)$. Since $\alpha|_H \in \mathrm{Aut}\, H$ and $H_1(H) = \mathbb{Z}(x)$, $\alpha(x) = x^{\pm 1}$. $\square$

Following f by a map $X_{p,q'} \longrightarrow X_{p,q'}$ which realizes conjugation by a suitable element, we may assume α has the above form. If $\alpha(x) = x^{-1}$, construct a fiber-preserving diffeomorphism

$$F\colon L(p,q) \underset{\tau}{\times} S^1 \longrightarrow L(p,q) \underset{\tau}{\times} S^1 ,$$

with $F_*(a) = a$, $F_*(x) = x^{-1}$. (This is possible since the monodromy τ is an involution). We may assume F preserves a $D^3 \times S^1$, in which we take connected sum with $S^1 \times D^3$ and do surgery, so that F extends to a homotopy equivalence $\tilde{F}\colon X_{p,q} \longrightarrow X_{p,q}$. Replacing f by $f \circ \tilde{F}$ permits us to assume that α has the form

$$\alpha\colon \begin{cases} a \longrightarrow a^n \\ \\ x \longrightarrow x \qquad , \ (n,p) = 1 . \end{cases}$$

Note that $\alpha(t)$ can be quite complicated. For example, we might have $\alpha(t) = t^{-1}xt^2$, or $\alpha(t) = a^m t$, or a composition of such. Later on we will further modify α until it projects to id_G .

Now we examine k-invariants. Since $\alpha(H) = H$ and $\alpha(\mathbb{Z}_p) = \mathbb{Z}_p$, we get the following commuting diagram of isomorphisms:

$$\begin{tikzcd}
k(X_{p,q'}) \arrow[r, phantom, "\in"] \arrow[dddd, bend right] & H^3(\pi,\pi_2) \arrow[r, "\alpha^*"] \arrow[d] & H^3(\pi,(\pi_2)_\alpha) \arrow[d] & H^3(\pi,\pi_2) \arrow[l, "\beta_*"'] \arrow[d] \arrow[r, phantom, "\ni"] & k(X_{p,q}) \arrow[dddd, bend left] \\
& H^3(H,\pi_2 B^2_{p,q'}) \arrow[r, "\alpha^*"] \arrow[d] & H^3(H,(\pi_2 B^2_{p,q'})_\alpha) \arrow[d] & H^3(H,\pi_2 B^2_{p,q}) \arrow[l, "\beta_*"'] \arrow[d] & \\
& H^3(\mathbb{Z}_p,\mathbb{Z}\mathbb{Z}_p/N) \arrow[r, "\alpha^*"] \arrow[d] & H^3(\mathbb{Z}_p,(\mathbb{Z}\mathbb{Z}_p/N)_\alpha) \arrow[d] & H^3(\mathbb{Z}_p,\mathbb{Z}\mathbb{Z}_p/N) \arrow[l, "\beta_*"'] \arrow[d] & \\
& H^4(\mathbb{Z}_p,\mathbb{Z}) \arrow[r, "\alpha^*"] \arrow[d] & H^4(\mathbb{Z}_p,\mathbb{Z}) \arrow[d] & H^4(\mathbb{Z}_p,\mathbb{Z}) \arrow[l, "\beta_*"'] \arrow[d] & \\
q' \arrow[r, phantom, "\in"] & \mathbb{Z}_p \arrow[r, "\cdot n^2"] & \mathbb{Z}_p & \mathbb{Z}_p \arrow[l, "\beta_*"'] \arrow[r, phantom, "\ni"] & q
\end{tikzcd}$$

As $\pm qq'$ is not a quadratic residue mod p, all we have to show in order to derive a contradiction is $\beta_* = \pm 1$.

Before proceeding with the proof, let us collect some facts about $\mathbb{Z}\pi$ that will be needed.

(i) $\mathbb{Z}\pi \xrightarrow{\cdot(g-1)} \mathbb{Z}\pi$ is a monomorphism, for $g \in \pi$ of infinite order (Lemma II 2.1).

(ii) $\mathbb{Z}\pi \xrightarrow{\cdot(1+x^{-1}-t^{-1})} \mathbb{Z}\pi$ is a monomorphism (proof of Proposition 4.1).

(iii) Tensoring the standard left (right) $\mathbb{Z}\mathbb{Z}_p$-resolution of $\mathbb{Z}$ with $\mathbb{Z}\pi$ on the left (right) induces the free left (right) $\mathbb{Z}\pi$-resolution of $\mathbb{Z}$

$\longrightarrow \mathbb{Z}\pi \xrightarrow{a-1} \mathbb{Z}\pi \xrightarrow{N} \mathbb{Z}\pi \xrightarrow{a-1} \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$. This gives $\ker N = \operatorname{Im}(a-1)$ and $\ker(a-1) = \operatorname{Im}N$, where the maps are multiplication on the right (left).

(iv) The augmentation ideal $I\pi = \ker(\mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z})$ has presentation

$$(\mathbb{Z}\pi)^3 \xrightarrow{\begin{pmatrix} 1-x^2 & t-x-1 & 0 \\ 0 & 1-a^{-1} & x+a^{-1} \\ 0 & 0 & N \end{pmatrix}} (\mathbb{Z}\pi)^3 \xrightarrow{\begin{pmatrix} t-1 \\ x-1 \\ a-1 \end{pmatrix}} I\pi \longrightarrow 0 ,$$

where the map $(\mathbb{Z}\pi)^3 \longrightarrow (\mathbb{Z}\pi)^3$ is the "Jacobian matrix" of Fox derivatives associated to the presentation of π (see [7,p. 45-46], or look at $C_*(\tilde{X}_{p,q})$ in §5).

We now start the study of the action of β_* on cohomology. The α-map β lifts to an α-map $\theta\colon \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ which induces $\bar{\theta}\colon (\mathbb{Z}\pi)^2 \longrightarrow (\mathbb{Z}\pi)^2$:

$$\begin{tikzcd}[column sep=large]
\mathbb{Z}\pi \oplus \mathbb{Z}\pi \arrow[r, "{\begin{pmatrix} (x-1)\cdot(1+x^{-1}-t^{-1}) \\ N\cdot(1+x^{-1}-t^{-1}) \end{pmatrix}}"] \arrow[d, "{\bar{\theta} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}}"'] & \mathbb{Z}\pi \arrow[r] \arrow[d, "\theta=(u)"] & \pi_2 \arrow[r] \arrow[d, "\beta"] & 0 \\
\mathbb{Z}\pi \oplus \mathbb{Z}\pi \arrow[r] & \mathbb{Z}\pi \arrow[r] & \pi_2 \arrow[r] & 0 ,
\end{tikzcd} \qquad (*)$$

with $u, \bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{Z}\pi$. There is considerable freedom in choosing θ; we can replace u by $u + \mu(x-1)(1+x^{-1}-t^{-1}) + \nu N(1+x^{-1}-t^{-1})$, with $\mu, \nu \in \mathbb{Z}\pi$. We will do this repeatedly towards the end of the proof. As for now, we fix θ and try to find a convenient lift $\bar{\theta}$, namely one which projects to a map $\bar{\theta} : \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}p) \longrightarrow \mathbb{Z}\pi \oplus \mathbb{Z}(\pi/\mathbb{Z}_p)$. This will permit us to analyze the action of β_* on $H^3(\pi_1,\pi_2)$.

From the commutativity of the diagram (*), we find

$$\begin{cases} (x-1)(1+x^{-1}-\alpha(t^{-1}))\cdot u = (\bar{a}(x-1) + \bar{b}N)(1+x^{-1}-t^{-1}) & \text{(1a)} \\ N(1+x^{-1}-\alpha(t^{-1}))\cdot u = (\bar{c}(x-1) + \bar{d}N)(1+x^{-1}-t^{-1}) \,. & \text{(1b)} \end{cases}$$

Since β is an isomorphism, there is $u' \in \mathbb{Z}\pi$ satisfying equations (1') analogous to (1) and

$$\begin{cases} \alpha^{-1}(u)\cdot u' = (u_0(x-1) + u_1N)(1+x^{-1}-t^{-1}) + 1 & (2) \\ \alpha(u')\cdot u = (u_0'(x-1) + u_1'N)(1+x^{-1}-t^{-1}) + 1 \,. & (2') \end{cases}$$

Left-multiplying equation (1b) by (a-1) gives

$$(a-1)(\bar{c}(x-1) + \bar{d}N)(1+x^{-1}-t^{-1}) = 0 \,,$$

or, by (ii)

$$(a-1)(\bar{c}(x-1) + \bar{d}N) = 0 \,.$$

Hence, by (iii), there is $\bar{u} \in \mathbb{Z}\pi$ such that

$$\bar{c}(x-1) + \bar{d}N = N\,\bar{u}\,. \tag{3}$$

Combining equations (1b) and (3) yields

$$N\,(1+x^{-1} - \alpha(t^{-1}))\,u = N\,\bar{u}\,(1+x^{-1}-t^{-1})\,. \tag{4}$$

From (1'b) we deduce an analogous equation (4') .

We use the information gained so far to improve further the map α . Consider the projection $\lambda\colon \pi = G *_{\mathbb{Z}} H \longrightarrow G$. This induces a ring homomorphism $\lambda\colon \mathbb{Z}\pi \longrightarrow \mathbb{Z}G$. Also, $\alpha \in \operatorname{Aut}\pi$ induces $\bar{\alpha} \in \operatorname{Aut} G$ with $\lambda\alpha = \bar{\alpha}\lambda$. Equations (4), (4') , (2) ,(2'), when projected to $\mathbb{Z}G$, become

$$\begin{cases} (1+x^{-1} - \bar{\alpha}(t^{-1}))\lambda(u) = \lambda(\bar{u})(1+x^{-1}-t^{-1}) & (5) \\ (1+x^{-1} - \bar{\alpha}(t^{-1}))\lambda(u') = \lambda(\bar{u}')(1+x^{-1}-t^{-1}) & (5') \\ \bar{\alpha}^{-1}(\lambda(u))\;\lambda(u') = w(1+x^{-1}-t^{-1}) & (6) \\ \bar{\alpha}(\lambda(u'))\;\lambda(u) = w'(1+x^{-1}-t^{-1})\,. & (6') \end{cases}$$

Recall the alternate description of $X_{p,q}$ as surgery on $X \# Y_{p,q}$, where X is a ribbon knot exterior with $\pi_1 X = G$ and $\pi_2 X = \mathbb{Z}G/(1+x^{-1}-t^{-1})$ (II, §2) . Equations (5), (5'), (6), (6') provide an $\bar{\alpha}$-isomorphism $\bar{\beta}\colon \pi_2 X \to \pi_2 X$. Since the k-invariant of X vanishes ($\mathrm{cd}G = 2$) , we can find a homotopy equivalence $H: X \longrightarrow X$ with $H_* = \overline{\alpha}$. Since $\overline{\alpha}(x) = x$, we may assume that H preserves a neighborhood $D^3 \times S^1$ of x, in which we take connected sum with $Y_{p,q}$ and do surgery, so that H extends to a homotopy equivalence $\tilde{H}: X_{p,q} \longrightarrow X_{p,q}$. Replacing f by $f \circ \tilde{H}^{-1}$ enables us to assume that $\alpha \in \mathrm{Aut}\,\pi$ projects to $\overline{\alpha} = \mathrm{id} \in \mathrm{Aut}\,G$.

We now continue our quest for the right $\overline{\theta}$. Recall equation (3): $\overline{c}(x-1) = N\overline{u} - \overline{d}N$. Let $\ell = \varepsilon(\overline{d}) = \varepsilon(u) \in \mathbb{Z}$. The element $\overline{d} - \ell$ belongs to $I\pi$. By fact (iv) above, there are $\mu, \nu, \xi \in \mathbb{Z}\pi$ such that

$$\overline{d} = \mu(t-1) + \nu(x-1) + \xi(a-1) + \ell . \qquad (7)$$

Combining equations (3) and (7) yields

$$\overline{c}(x-1) = N(\overline{u}-\ell) - (\mu(t-1) + \nu(x-1))N , \qquad (8)$$

or, after left-multiplying both sides by (a-1):

$$(a-1)(\overline{c}+\nu N)(x-1) = -(a-1)\mu(t-1)N . \qquad (9)$$

Right-multiplying by (a-1) gives

$$0 = (a-1)(\overline{c}+\nu N)(x-1)(a-1) = (a-1)(\overline{c}+\nu N)(1-a)(1+a^{-1}x) .$$

By (i), $(a-1)(\overline{c}+\nu N)(1-a) = 0$, which implies $(a-1)(\overline{c}+\nu N) =$

$c_1 N$, for some $c_1 \in \mathbb{Z}\pi$. This combines with (9) to give

$$c_1(x-1)N = -(a-1)\mu(t-1)N \ .$$

Consequently, there is $\xi_1 \in \mathbb{Z}\pi$ such that

$$c_1(x-1) + (a-1)\mu(t-1) = \xi_1(a-1) \ .$$

The above equation provides a relation among the generators of $I\pi$. Looking at the coefficient of $(t-1)$ in this relation, and recalling the presentation of $I\pi$ given in (iv) , we find $\bar{\mu} \in \mathbb{Z}\pi$ such that

$$(a-1)\mu = \bar{\mu}(1-x^2) \ . \qquad (10)$$

Left-multiplying by N gives $N\bar{\mu}(1-x^2) = 0$. Hence $N\bar{\mu} = 0$, which implies $\bar{\mu} = (a-1)\mu_1$. Equation (10) becomes $(a-1)(\mu-\mu_1(1-x^2)) = 0$, and so

$$\mu = N\mu_2 + \mu_1(1-x^2) \ .$$

Combining this with (8) yields

$$\begin{aligned} \bar{c}(x-1) &= N(\bar{u}-\ell) - (N\mu_2(t-1) + \mu_1(1-x^2)(t-1) + \nu(x-1))N \\ &= N(\bar{u}-\ell-\mu_2(t-1)N) - (\mu_1(1+x-t) + \nu)N(x-1) \ , \end{aligned}$$

which, if we set $y = \mu_1(1+x-t) + \nu$, gives

$$(a-1)(\bar{c} + yN)(x-1) = 0 ,$$

or

$$\bar{c} + yN = Nc .$$

Now return to equation (3):

$$N\bar{u} = \bar{c}(x-1) + \bar{d}N = (Nc-yN)(x-1) + \bar{d}N = Nc(x-1) + d_1N ,$$

where

$$d_1 = -y(x-1) + \bar{d} = -y(x-1) + (N\mu_2+\mu_1(1-x^2))(t-1) +$$

$$\nu(x-1) + \xi(a-1) + \ell$$

$$= N\mu_2(t-1) + (-y+\mu_1(1+x-t)+\nu)(x-1) + \xi(a-1) + \ell$$

$$= N\mu_2(t-1) + \xi(a-1) + \ell .$$

Therefore $N\bar{u} = Nc(x-1) + N\mu_2(t-1)N + \ell N$. Setting $d = \mu_2(t-1)$, we get $N\bar{u} = N(c(x-1) + dN + \ell)$. Equation (1b) now becomes

$$N(1+x^{-1}-\alpha(t^{-1}))u = N(c(x-1) + dN + \ell)(1+x^{-1}-t^{-1}) . \tag{11}$$

We just proved that θ lifts to the α-map $\overline{\theta} = \begin{pmatrix} \overline{a} & \overline{b} \\ Nc & Nd+\ell \end{pmatrix} : (\mathbb{Z}\pi)^2 \longrightarrow (\mathbb{Z}\pi)^2$:

$$\begin{tikzcd}[column sep=large]
\mathbb{Z}\pi \arrow[r, "{(-N,x-1)}"] & \mathbb{Z}\pi\oplus\mathbb{Z}\pi \arrow[r, "{\begin{pmatrix}(x-1) & (1+x^{-1}-t^{-1}) \\ N & (1+x^{-1}-t^{-1})\end{pmatrix}}"] \arrow[d, "{\overline{\theta} = \begin{pmatrix} \overline{a} & \overline{b} \\ Nc & Nd+\ell \end{pmatrix}}"] & \mathbb{Z}\pi \arrow[r] \arrow[d, "{\theta = (u)}"] & \pi_2 \arrow[r] \arrow[d, "\beta"] & 0 \\
\mathbb{Z}\pi \arrow[r] & \mathbb{Z}\pi\oplus\mathbb{Z}\pi \arrow[r] & \mathbb{Z}\pi \arrow[r] & \pi_2 \arrow[r] & 0 \text{ .}
\end{tikzcd}$$

In order to analyze the action of β on $H^3(\pi_1,\pi_2)$, we want to lift $\overline{\theta}$ to $\tilde{\theta} = (v) : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$. Unfortunately, the above complex fails to be exact at $\mathbb{Z}\pi \oplus \mathbb{Z}\pi$. But it is easy to see that $\ker \begin{pmatrix}(x-1) & (1+x^{-1}-t^{-1}) \\ N & (1+x^{-1}-t^{-1})\end{pmatrix}$ is generated by $\mathrm{Im}(-N,x-1)$ and $0 \oplus \mathrm{Im}(a-1)$ (c.f. the proof of Lemma 2.2). Hence there are $v, w_1, w_2 \in \mathbb{Z}\pi$ such that

$$\begin{cases} -vN = -N\overline{a} + (x-1)Nc + w_1(a-1) & \text{(12a)} \\[2ex] v(x-1) = -N\overline{b} + (x-1)(Nd+\ell) + w_2(a-1) & \text{(12b)} \end{cases}$$

Equation (12b) can be rewritten as

$$(v-\ell)(x-1) = N(-\overline{b} + (x-1)d) + w_2(a-1) \ ;$$

or, if we set $-\overline{b} + (x-1)d = \mu(t-1) + \nu(x-1) + \xi(a-1)$,

$$-N\mu(t-1) + (v-\ell-N\nu)(x-1) - (N\xi + w_2)(a-1) = 0 .$$

The presentation of $I\pi$ given in (iv) provides elements $\bar{\mu}, \bar{\nu}$ such that

$$-N\mu = \bar{\mu}(1-x^2) \tag{13a}$$

$$v-\ell-N\nu = \bar{\mu}(t-x-1) + \bar{\nu}(1-a^{-1}) . \tag{13b}$$

Equation (13a) gives $(a-1)\bar{\mu}(1-x^2) = 0$, or $(a-1)\bar{\mu} = 0$. Hence $\bar{\mu} = N\mu_1$, and equation (13b) becomes:

$$v-\ell = N(\nu+\mu_1(t-x-1)) + \bar{\nu}(1-a^{-1}) ,$$

or

$$v = \ell + Nv_1 + v_2(a-1) . \tag{14}$$

A map $\cdot Nc : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ induces well defined maps $\cdot Nc: \mathbb{Z}(\pi/\mathbb{Z}_p) \longrightarrow \mathbb{Z}\pi$ and $\cdot Nc: \mathbb{Z}(\pi/\mathbb{Z}_p) \longrightarrow \mathbb{Z}(\pi/\mathbb{Z}_p)$. The map $\cdot(a-1): \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ induces the zero map $\mathbb{Z}(\pi/\mathbb{Z}_p) \xrightarrow{0} \mathbb{Z}(\pi/\mathbb{Z}_p)$. Therefore β lifts to:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}(\pi/\mathbb{Z}_p) \arrow[r, "{(-N,x-1)}"] \arrow[d, "\tilde{\theta}=(Nv_1+\ell)"] & \mathbb{Z}\pi\oplus\mathbb{Z}(\pi/\mathbb{Z}_p) \arrow[rr, "{\begin{pmatrix}(x-1)(1+x^{-1}-t^{-1}) \\ N\ (1+x^{-1}-t^{-1})\end{pmatrix}}"] \arrow[d, "\bar{\theta}=\begin{pmatrix}\bar{a} & \bar{b} \\ Nc & Nd+\ell\end{pmatrix}"] & & \mathbb{Z}\pi \arrow[r] \arrow[d] & \pi_2 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathbb{Z}(\pi/\mathbb{Z}_p) \arrow[r] & \mathbb{Z}\pi\oplus\mathbb{Z}(\pi/\mathbb{Z}_p) \arrow[rr] \arrow[rr, phantom, shift left=6ex, "\theta=(u)"] & & \mathbb{Z}\pi \arrow[r] & \pi_2 \arrow[r] & 0
\end{tikzcd} .$$

Recall from §6 that $H^3(\pi_1,\pi_2) \cong H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}_p)) \cong$

$H^5(H,\mathbb{Z}(H/\mathbb{Z}p)) \cong H^4(H,\mathbb{Z}) \cong \mathbb{Z}p$. Hence β_* acts on $H^3(\pi_1,\pi_2)$ via $\cdot(Nv_1+\ell)$: $H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}p)) \longrightarrow H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}p)_\alpha)$. From the naturality of the above isomorphisms comes the commuting diagram

$$\begin{array}{ccc} H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}p)) & \xrightarrow{N} & H^5(\pi,\mathbb{Z}(\pi/\mathbb{Z}_p)_\alpha) \\ \downarrow\wr & & \downarrow\wr \\ H^4(H,\mathbb{Z}) & \xrightarrow{N} & H^4(H,\mathbb{Z}) \\ \downarrow\wr & & \downarrow\wr \\ \mathbb{Z}p & \xrightarrow{0} & \mathbb{Z}p \end{array} ,$$

and so the term Nv_1 does not contribute to the action of β_* . Therefore β_*: $H^3(\pi_1,\pi_2) \longrightarrow H^3(\pi_1,(\pi_2)_\alpha)$ is the map $\cdot\ell$: $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p$. All that is left to show is that $\ell = \pm 1$ (mod p) .

We now go back and modify u until we find a suitable lift θ of β . Recall that u satisfies equation (11) $N(1+x^{-1}-\alpha(t^{-1}))u = N\ (c(x-1)+d\ N+\ell)(1+x^{-1}-t^{-1})$ and that we can replace u by $u + \mu(x-1)(1+x^{-1}-t^{-1}) + \nu\ N(1+x^{-1}-t^{-1})$ without changing β . This replaces c by $c+(1+x^{-1}-\alpha(t^{-1}))\mu$ and d by $d + (1+x^{-1} - \alpha(t^{-1}))\nu$. Project (11) to $\mathbb{Z}G$ via λ: $\mathbb{Z}\pi \longrightarrow \mathbb{Z}G$ and recall that $\bar{\alpha} = id_G$. We get

$$(1+x^{-1}-t^{-1})\lambda(u) = (\lambda(c)(x-1) + p\lambda(d) + \ell)(1+x^{-1}-t^{-1}) \qquad (15)$$

<u>Lemma 7.2</u>: There is a lift $\theta = (u): \mathbb{Z}\pi \to \mathbb{Z}\pi$ of β such that $\lambda(u) \in \mathbb{Z}$ and $\lambda(u) = \ell \pmod{p}$.

Assuming the lemma, we finish the proof. Let u and u' be lifts of β and β^{-1} as in Lemma 7.2. Recall equation (6): $\lambda(u)\,\lambda(u') = w(1+x^{-1}-t^{-1})$, with $w \in \mathbb{Z}G$. The abelianization map $G \to \mathbb{Z}$ extends to a ring homomorphism $\mathbb{Z}G \to \mathbb{Z}\mathbb{Z}$. Projecting equation (6) to $\mathbb{Z}\mathbb{Z}$, we find

$$uu' = w(2-t^{-1}) + 1 ,$$

where u and u' are integers. Writing

$$w = \sum_{k\in\mathbb{Z}} n_k t^k \in \mathbb{Z}\mathbb{Z}, \text{ we get } \quad uu' = \Sigma(2n_k - n_{k+1})t^k + 1 ,$$

which gives

$$\begin{cases} uu' = 2n_0 - n_1 + 1 \\ 0 = 2n_k - n_{k+1} , \quad \text{for } k \neq 0 . \end{cases}$$

Since only finitely many n_k are nonzero, we easily see that $n_k = 0$, for all k. Hence $uu' = 1$, which gives $u = \pm 1$. Therefore $\beta_* = \ell = \pm 1 \pmod{p}$, completing the proof of the theorem.

$\square$

<u>Proof of Lemma 7.2:</u> From now on, all computations will take place in $\mathbb{Z}G$. For simplicity, we will write $\lambda(u) = u$, etc,

so that equation (15) reads $(1+x^{-1}-t^{-1})u = (c(x-1)+pd+\ell)\cdot(1+x^{-1}-t^{-1})$. We are allowed to replace u by $u+\mu(x-1)\cdot(1+x^{-1}-t^{-1})+p\,\nu(1+x^{-1}-t^{-1})$, where $\mu,\nu \in \mathbb{Z}G$, replacing in the process c by $c+(1+x^{-1}-t^{-1})\mu$ and d by $d+(1+x^{-1}-t^{-1})\nu$.

Notice that $G = G' \rtimes \mathbb{Z}$, where $G' = \mathbb{Z}[1/2]$ is a torsion-free abelian group generated by $\{t^{-j}x\,t^{j}\ ;\ j \geq 0\}$. We write a typical element $g \in G$ as $g = t^{i}h$, with $h = \prod_{i=1}^{r} t^{-j_i} x^{k_i} t^{j_i} = t^{-j}x^{k}t^{j} \in G'$, where j is always chosen to be as small as possible. Notice also that $(1+x^{-1}-t^{-1})t^{i}h = t^{i}(h+t^{-i}x^{-1}t^{i}h) - t^{i-1}h$. Since we can add multiples of $(1+x^{-1}-t^{-1})$ to c and d, we may replace equation (15) by

$$(1+x^{-1}-t^{-1})\ u = (\bar{u}\cdot t^{r} + \ell)\ (1+x^{-1}-t^{-1})\ , \qquad (16)$$

where $\bar{u} \in \mathbb{Z}G'$, $\varepsilon(\bar{u}) = 0 \pmod p$, and $r > 1$.

We now determine what sort of $u \in \mathbb{Z}G$, $\bar{u} \in \mathbb{Z}G'$ can satisfy (16). Write $u = \sum_{g\in G} n_g g$, $\bar{u} = \sum_{h\in G'} \bar{n}_h h$. Equation (16) yields

$$n_g + n_{xg} - n_{tg} = \begin{cases} -\ell & ;\ g = t^{-1} \\ \ell & ;\ g = 1 \text{ or } x^{-1} \\ \bar{n}_{gt^{-r}} + \bar{n}_{gxt^{-r}} + \bar{n}_{gt^{-r+1}} & ;\ g = ht^{r-1} \text{ or } ht^{r} \\ 0 & ;\ \text{otherwise}, \end{cases}$$

where $\bar{n}_g = 0$ if $g \notin G'$.

For $g = t^i h$, $i \leq -2$, we have $n_{t^i h} + n_{xt^i h} - n_{t^{i+1}h} = 0$.

Since only a finite number of the n_g are nonzero, there is an integer L such that $n_{t^i h} = 0$ for $|i| > L$. Hence $n_{t^i h} = 0$, for $i \leq -1$.

For $i \geq r+1$, we again have $n_{t^i h} + n_{xt^i h} - n_{t^{i+1}h} = 0$. Let j be the largest integer $\geq r+1$ such that there is an $h \in G'$ with $n_{t^j h} \neq 0$. Then $n_{t^j h} + n_{xt^j h} = 0$, and so

$$n_{t^j h} = -n_{t^j t^{-j} x t^j h} = \ldots = (-1)^s \, n_{t^j t^{-j} x^s t^j h} = \ldots \quad .$$

This is an infinite sequence of equalities, since x has infinite order. Hence $n_{t^j h} = 0$, which proves $n_{t^i h} = 0$, for $i \geq r+1$.

For $g = t^{-1}h$, we have $n_{t^{-1}h} + n_{xt^{-1}h} - n_h = \begin{cases} -\ell & ; h=1 \\ 0 & ; h \neq 1 \end{cases}$.

Hence $n_1 = \ell$ and $n_h = 0$, for $h \neq 1$. For $g = h$, we have $n_h + n_{xh} - n_{th} = \begin{cases} \ell & ; h=1, x^{-1} \\ 0 & ; h \neq 1, x^{-1} \end{cases}$, which gives $n_{th} = 0$.

For $1 \leq i \leq r-2$, we have $n_{t^i h} + n_{xt^i h} - n_{t^{i+1}h} = 0$, which implies $n_{t^i h} = 0$, for $2 \leq i \leq r-1$.

Finally, from

$$n_{ht^{r-1}} + n_{xht^{r-1}} - n_{tht^{r-1}} = \bar{n}_{ht^{r-1}t^{-r}} + \bar{n}_{ht^{r-1}xt^{-r}} - \bar{n}_{ht^{r-1}t^{-r+1}}$$

$$n_{ht^{r}} + n_{xht^{r}} - n_{tht^{r}} = \bar{n}_{ht^{r}t^{-r}} + \bar{n}_{ht^{r}xt^{-r}} - \bar{n}_{ht^{r}t^{-r+1}} ,$$

we get

$$n_{tht^{-1}t^{r}} = \bar{n}_{h} \tag{17}$$

$$n_{ht^{r}} + n_{xht^{r}} = \bar{n}_{h} + \bar{n}_{hx^{2^{r}}} , \tag{18}$$

which combine to give

$$n_{ht^{r}} + n_{xht^{r}} = n_{tht^{-1}t^{r}} + n_{thx^{2^{r}}t^{-1}t^{r}} . \tag{19}$$

Replacing h by $t^{-1}ht$, we find

$$n_{ht^{r}} = n_{t^{-1}htt^{r}} + n_{xt^{-1}htt^{r}} - n_{x^{2^{r+1}}ht^{r}} . \tag{20}$$

Let j be the largest integer >0 such that there is an $h = t^{-j}x^{k}t^{j} \in G'$ with $n_{ht^{r}} \neq 0$. For such h , equation (20) gives

$$n_{ht^{r}} = -n_{x^{2^{r+1}}ht^{r}} = \ldots = (-1)^{s}\, n_{x^{2^{r+s}}ht^{r}} = \ldots ,$$

which implies $n_{ht^{r}} = 0$. Hence $n_{ht^{r}} = 0$, for $h \neq x^{k}$.

Let j be the largest positive integer such that $n_{x^j t^r} \neq 0$. Equation (19) gives

$$n_{x^j t^r} = -n_{x^{j+1} t^r} + n_{x^{2j} t^r} + n_{x^{2^{r+1}+2j} t^r} = 0 \ .$$

Hence $n_{x^k t^r} = 0$, for $k > 0$. The conclusion of these computations is

$$u = \ell + (\sum_{k \leq 0} n_{x^k t^r} x^k) t^r \ . \tag{21}$$

To see what $\bar{u}$ can be, we go back to equation (17), which implies $\bar{n}_h = 0$, for $h = t^{-i} x^k t^i$, $i \geq 2$ or $h = x^k$, $k > 0$. Combining (17) and (18) gives

$$\bar{n}_h + \bar{n}_{hx^{2^r}} = \bar{n}_{t^{-1}ht} + \bar{n}_{t^{-1}xht} \ .$$

Setting $h = t^{-1} x^k t$, k odd, in the above equation yields $\bar{n}_{t^{-1} x^k t} + \bar{n}_{t^{-1} x^k tx^{2^r}} = 0$. This gives

$$\bar{n}_{t^{-1} x^k t} = - \bar{n}_{t^{-1} x^k tx^{2^r}} = \ldots = (-1)^{s-1} \bar{n}_{t^{-1} x^k tx^{2^{r+s}}} = \ldots,$$

which implies $\bar{n}_{t^{-1} x^k t} = 0$. The upshot is

$$\bar{u} = \sum_{k \leq 0} \bar{n}_{x^k} x^k \ . \tag{22}$$

If we set $u_1 = \sum_{k\le 0} n_{x^k t^r} x^k$, equations (16), (21) and (22) combine to give

$$(1+x^{-1}-t^{-1})\ u_1 = \bar{u}\ (1+x^{-2^r}\ -t^{-1}) \ ,$$

or $t\bar{u}t^{-1} = u_1$ and $(1+x^{-1})\ u_1 = \bar{u}(1+x^{-2^r})$.

<u>Lemma 7.3</u>: Given a polynomial $P(x) \in \mathbb{Z}[x]$ satisfying $P(x^2)(1+x) = (1+x^{2^r})P(x)$, then $P(x) = m\cdot(\sum_{k=0}^{2^r-1} x^k)$.

<u>Proof</u>: Let $P(x) = \sum_{k=0}^{i} m_k x^k$. Comparing degrees yields $i = 2^r-1$. Comparing coefficients proves the lemma. $\square$

The lemma, with $P(x) = \sum_{k\ge 0} \bar{n}_{x^k} x^k$, implies $\bar{u} = m\ (\sum_{k=-2^r+1}^{0} x^k)$. In particular, $\epsilon(\bar{u}) = m\cdot 2^r$. But $\epsilon(\bar{u}) = 0 \pmod p$ and p is odd, hence $m = pm_1$. We now have

$$u = \ell + u_1 t^r = \ell + m \cdot \sum_{k=-2^r+1}^{0} x^{2k} t^r$$

$$= \ell + m\ (\sum_{k=-2^{r-1}+1}^{0} x^{2k})(1+x^{-2^r})t^r = \ell + p(\underbrace{m_1 \cdot \sum_{k=-2^{r-1}+1}^{0} x^{2k} t^r}_{\nu_1})(1+x^{-1})$$

$$= \ell + p\nu_1(1+x^{-1}-t^{-1}) + m\cdot \sum_{k=-2^{r-1}+1}^{0} x^{2k} t^{r-1} \ .$$

Induction on r produces an element $\nu \in \mathbb{Z}G$ such that

$$u = \ell + p\nu(1+x^{-1}-t^{-1}) + m ,$$

thus proving lemma 7.2. $\square$

Why does this proof work for these examples, but not for the 2-twist spun knots? That is, why is it that $X_{p,q} \not\simeq X_{p,q'}$, but $B^2_{p,q} \simeq B^2_{p,q'}$? We can see this algebraically as follows. Recall that $\pi_1 B^2_{p,q} = H$, and $\pi_2 = \pi_2 B^2_{p,q}$ is given by the exact sequence $\mathbb{Z}H \oplus \mathbb{Z}H \xrightarrow{\binom{x-1}{N}} \mathbb{Z}H \longrightarrow \pi_2 \longrightarrow 0$. Take $\alpha = \mathrm{id}$; β is given by

$$\begin{array}{ccccccc}
\mathbb{Z}H \oplus \mathbb{Z}H & \xrightarrow{\binom{x-1}{N}} & \mathbb{Z}H & \longrightarrow & \pi_2 & \longrightarrow & 0 \\
\Big\downarrow \bar{\theta}=\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} & & \Big\downarrow u & & \Big\downarrow \beta & & \\
\mathbb{Z}H \oplus \mathbb{Z}H & \longrightarrow & \mathbb{Z}H & \longrightarrow & \pi_2 & \longrightarrow & 0
\end{array} .$$

This imposes the conditions $\begin{cases} (x-1)u = (x-1)\bar{a} + N\bar{b} \\ N\,u = (x-1)\bar{c} + N\bar{d} \end{cases}$.

Since β is an isomorphism, we also want a u' satisfying analogous conditions and $uu' = (u_1(x-1) + v_1 N) + 1$. These conditions are not hard to satisfy. For example, if $p = 5$, $q = 1$, $q' = 2$, we need a u with $\beta_* = \epsilon(u) = 2$. We may pick $u = a + a'$, $u' = 1 + a + a^{-1}$. Then $\bar{\theta} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ and $uu' = N + 1$. With the more complicated module structure on

$\pi_2(X_{p,q})$, these u's are no longer available.

One might also ask, why not project $G \to D_3 = (t,x \mid txt^{-1} = x^2,\ t^2 = 1)$, and work in $\mathbb{Z}D_3$, as in [45]. Equations (1a), (1b), (2), (2') project to

$$\begin{cases} (x-1)(1+x^{-1}-t^{-1})u = (\bar{a}(x-1) + p\bar{b})(1+x^{-1}-t^{-1}) \\ p(1+x^{-1}-t^{-1})u = (\bar{c}(x-1) + p\bar{d})(1+x^{-1}-t^{-1}) \\ uu' = (u_0(x-1) + pu_1)(1+x^{-1}-t^{-1}) + 1 \\ u'u = (u_0'(x-1) + pu_1')(1+x^{-1}-t^{-1}) + 1 . \end{cases}$$

Unfortunately, one can produce examples which satisfy these equations and have the required augmentations. For example, if $p = 5$, pick $u = 2 + tx$. We get

$$\begin{tikzcd} \mathbb{Z}D_3 \oplus \mathbb{Z}D_3 \arrow[r, "{\begin{pmatrix} (x-1)(1+x^{-1}-t^{-1}) \\ 5(1+x^{-1}-t^{-1}) \end{pmatrix}}"] \arrow[d, "{\bar{\theta} = \begin{pmatrix} 3 & 0 \\ 0 & 2+tx^{-1} \end{pmatrix}}"] & \mathbb{Z}D_3 \arrow[r] \arrow[d, "u=2+tx"] & \mathbb{Z}D_3/(*) \arrow[r] \arrow[d, "\beta"] & 0 \\ \mathbb{Z}D_3 \oplus \mathbb{Z}D_3 \arrow[r] & \mathbb{Z}D_3 \arrow[r] & \mathbb{Z}D_3/(*) \arrow[r] & 0 \end{tikzcd}$$

Pick also $u' = 2tx + 1$. We have

$$uu' = (2+tx)(2tx+1) = 5tx + 4 = (x+tx^{-1}) \cdot 5(1+x^{-1}-t^{-1})-1$$

So it really seems necessary to work in $\mathbb{Z}G$ as we did in order to prove the theorem.

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