COHOMOLOGY JUMP LOCI OF 3-MANIFOLDS

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Abstract. The cohomology jump loci of a space X are of two basic types: the characteristic varieties, defined in terms of homology with coefficients in rank one local systems, and the resonance varieties, constructed from information encoded in either the cohomology ring, or an algebraic model for X. We explore here the geometry of these varieties and the delicate interplay between them in the context of closed, orientable 3-dimensional manifolds and link complements. The classical multivariable Alexander polynomial plays an important role in this analysis. As an application, we derive some consequences regarding the formality and the existence of finite-dimensional models for such 3-manifolds.

Contents

1. Introduction

1.1. **Cohomology jump loci.** Let X be a finite, connected CW-complex and let $\pi = \pi_1(X)$ be its fundamental group. The *characteristic varieties* $\mathcal{V}^i_k(X)$ are the Zariski closed subsets of the algebraic group $\mathrm{Hom}(\pi, \mathbb{C}^*)$ consisting of those characters $\rho\colon \pi \to \mathbb{C}^*$ for which the

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i-th homology group of X with coefficients in the rank 1 local system defined by ρ has dimension at least k; in particular, the trivial character $\mathbf{1}$ belongs to $\mathscr{V}^i_k(X)$ precisely when the i-th Betti number $b_i(X)$ is at least k.

Now let $H^\bullet = H^\bullet(X,\mathbb{C})$ be the cohomology algebra of X. For each $a \in H^1$, we may form a cochain complex, (H,a), with differentials $\delta_a\colon H^i \to H^{i+1}$ given by left-multiplication by a. The *resonance varieties* $\mathscr{R}^i_k(X)$, then, are the subvarieties of the affine space H^1 consisting of those classes a for which the i-th cohomology of (H,a) has dimension at least k.

Finally, suppose we are given an algebraic model for X, that is, a commutative differential graded algebra (A,d) connected by a zig-zag of quasi-isomorphisms to the Sullivan algebra of polynomial forms on X. Assuming A is connected and of finite type, we may form a cochain complex (A,δ_a) as above, with differentials now given by $\delta_a(u) = au + \mathrm{d}\,u$, and we may define the resonance varieties $\mathscr{R}^i_k(A) \subseteq H^1(A)$ analogously. (When if X is formal, that is, the cohomology algebra $H^\bullet(X,\mathbb{C})$ with $\mathrm{d}=0$ is a model for X, we recover the previous definition of resonance.)

All these notions admit 'partial' versions: e.g., for a fixed $q \geqslant 1$, we may speak of a q-finite q-model (A,d) for X, in which case the sets $\mathscr{R}^i_k(A)$ are Zariski closed for all $i \leqslant q$. For more details on all this, we refer to [9, 10, 12, 26, 44] and references therein.

For $q = 1$, the aforementioned properties of the space X can be interpreted purely in terms of the Malcev Lie algebra of it fundamental group, $\mathfrak{m}(\pi)$. For instance, as shown in [38], X admits a 1-finite 1-model if and only if $\mathfrak{m}(\pi)$ is the lower central series completion of a finitely presented Lie algebra L. More stringently, as shown in the foundational work of Quillen [40] and Sullivan [48], X is 1-formal if and only if L can be chosen to be a quadratic Lie algebra.

1.2. **The Tangent Cone formula.** A crucial tool in both the theory and the applications of cohomology jump loci is a formula relating the behavior around the origin of the characteristic and resonance varieties of a space.

Given a subvariety $W \subseteq (\mathbb{C}^*)^n$, we consider two types of approximations around the trivial character. One is the usual tangent cone, $\mathrm{TC}_{\mathbf{1}}(W) \subseteq \mathbb{C}^n$, while the other is the exponential tangent cone, $\tau_{\mathbf{1}}(W)$, which consists of those $z \in \mathbb{C}^n$ for which $\exp(\lambda z) \in W$, for all $\lambda \in \mathbb{C}$. As shown by Dimca–Papadima–Suciu in [12], $\tau_{\mathbf{1}}(W)$ is a finite union of rationally defined linear subspaces, all contained in $\mathrm{TC}_{\mathbf{1}}(W)$.

Now let X be a space as above. Combining the previous observation together with a result of Libgober [24] yields a chain of inclusions,

$$\tau_{\mathbf{1}}(\mathscr{V}^i_k(X)) \subseteq \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(X)) \subseteq \mathscr{R}^i_k(X), \tag{1.1}$$

in arbitrary degree i and depth k. As we shall see, each of these inclusions may be strict. Nevertheless, if X admits a q-finite q-model A, it follows from work of Dimca–Papadima [10] and Budur–Wang [5] that the following "Tangent Cone formula" holds:

$$\tau_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathscr{R}^i_k(A) \tag{1.2}$$

for all $i \leqslant q$ and $k \geqslant 0$. In particular, if X is q-formal, then, in the same range,

$$\tau_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathscr{R}^i_k(X), \tag{1.3}$$

a result originally proved in [12] for $i = 1$.

1.3. **Cohomology jump loci of closed 3-manifolds.** Most of the applications of these results have centered on the case when X admits a finite-dimensional model, which happens for instance if X is a smooth, quasi-projective variety (in particular, the complement of a hyperplane arrangement), or a compact Kähler manifold, or a Sasakian manifold, or a nilmanifold, or a classifying space for a right-angled Artin group.

We focus here instead on the cohomology jump loci of 3-dimensional manifolds, which in general fail to possess finite-dimensional models. Let M be a compact, connected 3-manifold; we shall assume for simplicity that M is orientable and $\partial M = \emptyset$, although we shall also treat in §10 the case when M is a link complement. Set $n = b_1(M)$. Sending each element of $\pi = \pi_1(M)$ to its inverse induces an automorphism of the character group of π, which in turn restricts to isomorphisms $\mathscr{V}^i_k(M) \cong \mathscr{V}^{3-i}_k(M)$. Thus, in order to compute the characteristic varieties of M, it is enough to determine the jump loci $\mathscr{V}^1_k(M)$ for $1 \leqslant k \leqslant n$.

Work of McMullen [31] and Turaev [53] implies that, at least away from the origin $\mathbf{1}$, the intersection of $\mathscr{V}^1_1(M)$ with the identity component of the character group coincides with $V(\Delta_M)$, the hypersurface defined by the Alexander polynomial $\Delta_M \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. It follows that $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M))$ is either $\{\mathbf{0}\}$, or $\mathbb{C}^n$, or the subvariety of $\mathbb{C}^n$ defined defined by the initial form of the polynomial $\Delta_M|_{t_i - 1 = x_i} \in \mathbb{Z}[x_1, \dots, x_n]$.

Now fix an orientation $[M] \in H_3(M, \mathbb{Z})$; then the cup product on M determines an alternating 3-form μ_M on $H^1(M, \mathbb{Z})$, given by $a \wedge b \wedge c \mapsto \langle a \cup b \cup c, [M] \rangle$. Let $\mathrm{Pf}(\mu_M) \in \mathbb{Z}[x_1, \dots, x_n]$ be the Pfaffian of μ_M, as defined in [53]. As shown in [45], except for the trivial cases when $n \leqslant 1$, the first resonance variety of M is given by

$$\mathscr{R}^1_1(M) = \begin{cases} H^1(M, \mathbb{C}) & \text{if } n \text{ is even,} \\ V(\mathrm{Pf}(\mu_M)) & \text{if } n = 2g+1 \geqslant 3 \text{ and } \mu_M \text{ is generic.} \end{cases} \tag{1.4}$$

Here, we say that μ_M is generic (in the sense of [3]) if there is an element $c \in H^1(M, \mathbb{C})$ such that the 2-form on $H^1(M, \mathbb{C})$ given by $a \wedge b \mapsto \mu_A(a \wedge b \wedge c)$ has rank $2g$.

The higher depth resonance varieties also exhibit a nice pattern, revealed in [45]: $\mathscr{R}^1_{2k}(M) = \mathscr{R}^1_{2k+1}(M)$ if n is even, and $\mathscr{R}^1_{2k-1}(M) = \mathscr{R}^1_{2k}(M)$ if n is odd; moreover, if $n \geqslant 3$ and μ_M has maximal rank, then $\mathscr{R}^1_{n-2}(M) = \mathscr{R}^1_{n-1}(M) = \mathscr{R}^1_n(M) = \{\mathbf{0}\}$.

1.4. **A Tangent Cone theorem for closed 3-manifolds.** As is well-known, 3-manifolds may be non-formal, due to the presence of non-vanishing Massey products in their cohomology. Thus, we do not expect the Tangent Cone formula (1.3) to hold in this context.

Nevertheless, something very special happens in degree 1 and depth 1. The next result (proved in Theorem 7.3), delineates exactly the class of closed, orientable 3-manifolds M for which the second half of the Tangent Cone formula holds, except in the case when $n = b_1(M)$ is odd and at least 3 and μ_M is *not* generic, which remains open.

Theorem 1.1. *With notation as above,*

(1) *If $n \leqslant 1$, or n is odd, $n \geqslant 3$, and μ_M is generic, then* $\operatorname{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathscr{R}^1_1(M)$.
(2) *If n is even and $n \geqslant 2$, then* $\operatorname{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathscr{R}^1_1(M)$ *if and only if* $\Delta_M = 0$.

This result, together with those mentioned in §1.1–1.2, have definite implications regarding the kind of algebraic models a closed, orientable 3-manifolds M has, or, the kind of presentations the Malcev Lie algebra of $\pi = \pi_1(M)$ admits. For instance, if n is even, $n \geqslant 2$, and $\Delta_M \neq 0$, then M is not 1-formal and so $\mathfrak{m}(\pi)$ admits no quadratic presentation. In Example 7.7, we exhibit a 3-manifold M with $b_1(M) = 2$ for which the first half of the Tangent Cone formula fails, thus showing that M actually has no 1-finite 1-model, or, equivalently, $\mathfrak{m}(\pi)$ admits no finite presentation.

1.5. **Connected sums and graph manifolds.** As is well-known, every closed, orientable 3-manifold decomposes as the connected sum of finitely many irreducible 3-manifolds. We give in Theorem 8.1 an explicit formula that expresses the cohomology jump loci of the connected sum of two closed, orientable, smooth m-manifolds ($m \geqslant 3$) in terms of the jump loci of the summands. Since every 3-manifold is smooth, this reduces the computation of the cohomology jump loci of arbitrary closed, orientable 3-manifolds to that of irreducible ones. In particular, if $M = M_1 \# M_2$, and both summands have non-zero first Betti number, then $\mathscr{V}^1_1(M) = H^1(M,\mathbb{C}^*)$ and $\mathscr{R}^1_1(M) = H^1(M,\mathbb{C})$, and so the full Tangent Cone formula holds for M.

Every irreducible closed, orientable 3-manifold M admits a Jaco–Shalen–Johannson (JSJ) decomposition along incompressible tori; M is a graph-manifold is each of the pieces is Seifert fibered. We discuss in §9 three classes of graph-manifolds where the cohomology jump loci can be described in a fairly detailed fashion: (1) closed, orientable Seifert manifolds with orientable base; (2) graph-manifolds whose closed-up Seifert pieces are of type (1) and whose underlying graph is a tree; and (3) boundary manifolds of complex projective line arrangements.

1.6. **Links in the 3-sphere.** In the final section we explore the extent to which the Tangent Cone formula applies to link complements. Given a link $L = \{L_1, \dots, L_n\}$ in S^3, we let X denote its complement. Then $\mathscr{V}^1_1(X) = V(\Delta_L) \cup \{\mathbf{1}\}$, where $\Delta_L \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the (multivariable) Alexander polynomial of the link. Moreover, $\mathscr{R}^1_1(X)$ is the vanishing locus of the codimension 1 minors of the linearized Alexander matrix, whose entries are certain linear forms in the variables $x_1, \dots, x_n$, with coefficients solely depending on the linking numbers $\ell_{i,j} = \operatorname{lk}(L_i, L_j)$.

For 2-component links, we obtain a complete answer regarding the validity of the full Tangent Cone formula (in depth 1), and the formality of the link complement. In Theorem 10.3 we show the following: the complement X is formal if and only if $\tau_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \operatorname{TC}_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \mathscr{R}^1_1(X)$, and this happens precisely when the linking number of the two components is non-zero. We conclude with several examples of links with 3 or more components for which the second equality holds yet the first one does not, thereby showing that such link complements admit no 1-finite 1-models.

2. Resonance varieties

2.1. **Commutative differential graded algebras.** Let $\Bbbk$ be a field of characteristic 0, and let $A = (A^{\bullet}, \mathrm{d})$ be a commutative, differential graded algebra (for short, a CDGA) over $\Bbbk$. That is, A is a non-negatively graded $\Bbbk$-vector space, endowed with a multiplication map $\cdot\colon A^i \otimes_{\Bbbk} A^j \to A^{i+j}$ satisfying $a \cdot b = (-1)^{ij} b \cdot a$, and a differential $\mathrm{d}\colon A^i \to A^{i+1}$ satisfying $\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i a \cdot \mathrm{d}(b)$, for all $a \in A^i$ and $b \in A^j$. The cohomology of the underlying cochain complex, $H^{\bullet}(A)$, inherits the structure of a commutative, graded algebra (CGA); we will let $b_i(A) = \dim_{\Bbbk} H^i(A)$ be its Betti numbers.

A morphism between two CDGAS, $\varphi\colon A \to B$, is both an algebra map and a cochain map. Consequently, φ induces a morphism $\varphi^*\colon H^{\bullet}(A) \to H^{\bullet}(B)$ between the respective cohomology algebras. We say that φ is a quasi-isomorphism if φ^* is an isomorphism. Likewise, we say φ is a q-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees up to q and a monomorphism in degree $q+1$.

Two CDGAS A and B are *weakly equivalent* (or just *q-equivalent*) if there is a finite zig-zag of quasi-isomorphisms (or q-isomorphisms) connecting A to B,

$$A \longrightarrow A_1 \longleftarrow A_2 \longrightarrow \cdots \longleftarrow A_n \longrightarrow B, \tag{2.1}$$

with arrows going either way. In this case, we write $A \simeq B$ (or $A \simeq_q B$). A CDGA (A, d) is said to be *formal* (or just *q-formal*) if it is weakly equivalent (or just q-equivalent) to its cohomology algebra, $H^{\bullet}(A)$, endowed with the zero differential.

2.2. **Resonance varieties.** Assume now that our CDGA A is connected, i.e., $A^0 = \Bbbk$, generated by the unit 1. Since $\mathrm{d}(1) = 0$, we may identify the vector space $H^1(A)$ with $Z^1(A) = \ker(\mathrm{d})$. For each element a of this space, we turn A into a cochain complex,

$$(A^{\bullet}, \delta_a)\colon\ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots, \tag{2.2}$$

with differentials given by $\delta_a^i(u) = a \cdot u + \mathrm{d}(u)$, for all $u \in A^i$. (The fact that $\delta_a^{i+1} \circ \delta_a^i = 0$ for all $i \geqslant 0$ easily follows from the definitions.) Computing the homology of these chain complexes for various values of the parameter a, and keeping track of the dimensions of the resulting $\Bbbk$-vector spaces yields the sets

$$\mathscr{R}^i_k(A) = \{a \in H^1(A) \mid \dim_{\Bbbk} H^i(A^{\bullet}, \delta_a) \geqslant k\}. \tag{2.3}$$

Suppose now that A is *q-finite*, for some $q \geqslant 1$, that is, the Betti numbers $b_i = b_i(A)$ are finite for all $i \leqslant q$. Clearly, $H^1(A)$ is also a finite-dimensional $\Bbbk$-vector space. Moreover, as we shall see in §2.4, the sets $\mathscr{R}^i_k(A)$ are algebraic subsets of the ambient affine space $H^1(A)$, for all $i \leqslant q$. We call these sets the *resonance varieties* of A, in degree $i \geqslant 0$ and depth $k \geqslant 0$. For each $0 \leqslant i \leqslant q$, we obtain a descending filtration,

$$H^1(A) = \mathscr{R}^i_0(A) \supseteq \mathscr{R}^i_1(A) \supseteq \cdots \supseteq \mathscr{R}^i_{b_i+1}(A) = \emptyset. \tag{2.4}$$

Clearly, $H^i(A^\bullet, \delta_0) = H^i(A)$; thus, the point $\mathbf{0} \in H^1(A)$ belongs to $\mathscr{R}^i_k(A)$ if and only if $b_i \geqslant k$. In particular, since A is connected, we have that $\mathscr{R}^0_1(A) = \{\mathbf{0}\}$.

In general, the resonance varieties of a CDGA may not be invariant under scalar multiplication, see [10, 26, 44]. Nevertheless, when the differential of A is zero (that is, A is simply a CGA), the varieties $\mathscr{R}^i_k(A)$ are homogeneous subsets of $H^1(A) = A^1$. When $i = 1$, these subsets admit a particularly simple description. First note that the differential δ^0_a takes the generator $1 \in A^0 = \Bbbk$ to $a \in A^1$. Thus, a non-zero element $a \in A^1$ belongs to $\mathscr{R}^1_k(A)$ if and only if there exist elements $u_1, \dots, u_k \in A^1$ such that the set $\{a, u_1, \dots, u_k\}$ is linearly independent and $au_1 = \cdots = au_k = 0$ in A^2. In particular, if $b_1 = 0$ then $\mathscr{R}^1_1(A) = \emptyset$, and if $b_1 = 1$ then $\mathscr{R}^1_1(A) = \{\mathbf{0}\}$.

2.3. **Fitting ideals.** Our next goal is to explain why the resonance varieties of a locally finite CDGA are Zariski closed sets, and how to find defining equations for these varieties. We start with some basic notions from commutative algebra, following Eisenbud [16]. Let S be a commutative ring with unit. If φ is a matrix with entries in S, we let $I_k(\varphi)$ be the ideal of S generated by all minors of size k of φ. We then have a descending chain of ideals, $S = I_0(\varphi) \supseteq I_1(\varphi) \supseteq \cdots$.

Now suppose S is Noetherian. Then every finitely generated S-module Q admits a finite presentation, say $S^p \xrightarrow{\varphi} S^q \to Q \to 0$. We can arrange that $p \geqslant q$, by adding zero columns to the matrix φ if necessary. We then define the *k-th elementary ideal* (or, *Fitting ideal*) of Q as $E_k(Q) = I_{q-k}(\varphi)$. As is well-known, this ideal depends only on the module Q, and not on the choice of presentation matrix φ, whence the notation.

The Fitting ideals form an ascending chain, $E_0(Q) \subseteq E_1(Q) \subseteq \cdots \subseteq S$. Furthermore, $E_0(Q) \subseteq \operatorname{ann}(Q)$ and $(\operatorname{ann}(Q))^q \subseteq E_0(Q)$, while $\operatorname{ann}(Q) \cdot E_k(Q) \subseteq E_{k-1}(Q)$, for all $k > 0$. Consequently, if we denote by $V(\mathfrak{a}) \subset \operatorname{Spec}(S)$ the zero-locus of an ideal $\mathfrak{a}$, then $V(E_0(Q)) = V(\operatorname{ann}(Q))$.

2.4. **Equations for the resonance varieties.** Once again, let (A, d) be a connected $\Bbbk$-CDGA with $\dim_{\Bbbk} A^1 < \infty$. Pick a basis $\{e_1, \dots, e_n\}$ for the $\Bbbk$-vector space $H^1(A)$; to avoid trivialities, we shall assume that $n = b_1(A)$ is positive. Let $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for the dual vector space $H_1(A) = (H^1(A))^*$. Upon identifying the symmetric algebra $\operatorname{Sym}(H_1(A))$ with the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$, we obtain a cochain complex of finitely generated, free S-modules,

$$(A \otimes S, \delta)\colon \ \cdots \longrightarrow A^i \otimes_{\Bbbk} S \xrightarrow{\delta^i_A} A^{i+1} \otimes_{\Bbbk} S \xrightarrow{\delta^{i+1}_A} A^{i+2} \otimes_{\Bbbk} S \longrightarrow \cdots, \tag{2.5}$$

with differentials given by $\delta^i_A(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j + \mathrm{d}\, u \otimes s$, for $u \in A^i$ and $s \in S$. It is readily verified that the evaluation of this cochain complex at an element $a \in H^1(A)$ coincides with the cochain complex (A, δ_a) from (2.2).

Suppose that A is q-finite, for some $q \geqslant 1$. It is easy to see then that the sets $\mathscr{R}^i_k(A)$ with $i < q$ are Zariski closed. Indeed, an element $a \in A^1$ belongs to $\mathscr{R}^i_k(A)$ if and only if

$$\operatorname{rank} \delta^{i+1}_a + \operatorname{rank} \delta^i_a \leqslant c_i - k. \tag{2.6}$$

where $c_i = \dim_{\Bbbk} A^i$. Hence, $\mathscr{R}^i_k(A)$ is the zero-set of the ideal generated by all minors of size $c_i - k + 1$ of the block-matrix $\delta^{i+1}_A \oplus \delta^i_A$. It turns out that the sets $\mathscr{R}^q_k(A)$ are also Zariski closed even when $\dim_{\Bbbk} A^{q+1} = \infty$, see [10, 4].

The degree 1 resonance varieties $\mathscr{R}^1_k(A)$ admit an even simpler description: away from $\mathbf{0}$, they are the vanishing loci of the codimension k minors of δ^1_A. More precisely,

$$\mathscr{R}^1_k(A) = \begin{cases} V(I_{n-k}(\delta^1_A)) & \text{if } 0 < k < n, \\ \{\mathbf{0}\} & \text{if } k = n. \end{cases} \tag{2.7}$$

3. Characteristic varieties and the Alexander polynomial

3.1. **Characteristic varieties.** We say that a space X is *q-finite* (for some integer $q \geqslant 1$) if it has the homotopy type of a connected CW-complex with finite q-skeleton. We will denote by $\pi = \pi_1(X, x_0)$ the fundamental group of such a space, based at a 0-cell x_0. Clearly, if the space X is 1-finite, the group π is finitely generated, and if X is 2-finite, π admits a finite presentation.

So let X be a 1-finite space, and let $\operatorname{Char}(X) = \operatorname{Hom}(\pi, \mathbb{C}^*)$ be the group of complex-valued, multiplicative characters of π, whose identity $\mathbf{1}$ corresponds to the trivial representation. This is a complex algebraic group, which may be identified with $H^1(X, \mathbb{C}^*)$. The identity component, $\operatorname{Char}(X)^0$, is a an algebraic torus of dimension $n = b_1(X)$; the other connected components are translates of this torus by characters indexed by the torsion subgroup of $\pi_{\mathrm{ab}} = H_1(X, \mathbb{Z})$.

For each character $\rho \colon \pi \to \mathbb{C}^*$, let $\mathbb{C}_\rho$ be the corresponding rank 1 local system on X. The *characteristic varieties* of X (in degree i and depth k) are the jump loci for homology with coefficients in such local systems,

$$\mathscr{V}^i_k(X) = \{\rho \in \operatorname{Char}(X) \mid \dim H_i(X, \mathbb{C}_\rho) \geqslant k\}. \tag{3.1}$$

In more detail, let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover, with group of deck transformations π_{ab}. Upon lifting the cell structure of X to this cover, we obtain a chain complex of $\mathbb{Z}[\pi_{\mathrm{ab}}]$-modules,

$$\cdots \longrightarrow C_{i+1}(X^{\mathrm{ab}}, \mathbb{Z}) \xrightarrow{\partial^{\mathrm{ab}}_{i+1}} C_i(X^{\mathrm{ab}}, \mathbb{Z}) \xrightarrow{\partial^{\mathrm{ab}}_{i}} C_{i-1}(X^{\mathrm{ab}}, \mathbb{Z}) \longrightarrow \cdots . \tag{3.2}$$

Tensoring this chain complex with the $\mathbb{Z}[\pi_{\mathrm{ab}}]$-module $\mathbb{C}_\rho$, we obtain a chain complex of $\mathbb{C}$-vector spaces,

$$\cdots \to C_{i+1}(X, \mathbb{C}_\rho) \xrightarrow{\partial^{\mathrm{ab}}_{i+1}(\rho)} C_i(X, \mathbb{C}_\rho) \xrightarrow{\partial^{\mathrm{ab}}_{i}(\rho)} C_{i-1}(X, \mathbb{C}_\rho) \to \cdots , \tag{3.3}$$

where the evaluation of ∂^{ab}_i at ρ is obtained by applying the ring morphism $\mathbb{C}[\pi] \to \mathbb{C}$, $g \mapsto \rho(g)$ to each entry. Taking homology in degree i of this chain complex, we obtain the twisted homology groups $H_i(X, \mathbb{C}_\rho)$ which appear in definition (3.1).

If X is a q-finite space, the sets $\mathscr{V}^i_k(X)$ are Zariski closed subsets of the algebraic group $\operatorname{Char}(X)$, for all $i \leqslant q$ and all $k \geqslant 0$. If $i < q$, this is again easy to see. Indeed, let

$R = \mathbb{C}[\pi_{ab}]$ be the coordinate ring of the algebraic group $\mathrm{Hom}(\pi, \mathbb{C}^*) = \mathrm{Hom}(\pi_{ab}, \mathbb{C}^*)$. By definition, a character $\rho \in \mathrm{Char}(X)$ belongs to $\mathcal{V}^i_k(X)$ if and only if

$$\text{rank}\, \partial^{ab}_{i+1}(\rho) + \text{rank}\, \partial^{ab}_i(\rho) \leqslant c_i - k, \tag{3.4}$$

where $c_i = c_i(X)$ is the number of i-cells of X. Hence, $\mathcal{V}^i_k(X)$ is the zero-set of the ideal of minors of size $c_i - k + 1$ of the block-matrix $\partial^{ab}_{i+1} \oplus \partial^{ab}_i$. The case $i = q$ is covered in [35, Lemma 2.1] and [37, Proposition 4.1].

Clearly, $\mathcal{V}^i_0(X) = \mathrm{Char}(X)$. Moreover, $\mathbf{1} \in \mathcal{V}^i_k(X)$ if and only if the i-th Betti number $b_i(X)$ is at least k. In degree 0, we have that $\mathcal{V}^0_1(X) = \{\mathbf{1}\}$ and $\mathcal{V}^0_k(X) = \emptyset$ for $k > 1$. In degree 1, the sets $\mathcal{V}^1_k(X)$ depend only on the fundamental group $\pi = \pi_1(X, x_0)$, and, in fact, only on its maximal metabelian quotient, π/π''; thus, we shall sometimes write these sets as $\mathcal{V}^1_k(\pi) \subseteq \mathrm{Char}(\pi)$, and refer to them as the characteristic varieties of π.

If $b_1(\pi) = 0$, then $\mathrm{Char}(\pi)$ is a finite set of torsion characters in bijection with π_{ab}; although $1 \notin \mathcal{V}^1_1(\pi)$, other characters may belong to $\mathcal{V}^1_1(\pi)$. For instance, if $\pi = \mathbb{Z}_2 * \mathbb{Z}_2$, then $\mathrm{Char}(\pi) = \{(\pm 1, \pm 1)\}$, while $\mathcal{V}^1_1(\pi) = \{(-1, -1)\}$.

3.2. **Alexander varieties.** There is an alternative, very useful interpretation of the degree one characteristic varieties, first noted by Hironaka in [23]. Namely, let $B_X = H_1(X^{ab}, \mathbb{Z})$ be the *Alexander invariant* of a 2-finite space X, viewed as a $\mathbb{Z}[\pi_{ab}]$-module, and let $\mathcal{W}^1_k(X) = V(E_{k-1}(B_X \otimes \mathbb{C}))$ be the zero sets of the elementary ideals of the complexification of this module. Then, at least away from the trivial representation, the degree 1 characteristic varieties of X coincide with the *Alexander varieties*,

$$\mathcal{V}^1_k(X) \backslash \{\mathbf{1}\} = \mathcal{W}^1_k(X) \backslash \{\mathbf{1}\}. \tag{3.5}$$

Indeed, if $\rho\colon \pi \to \mathbb{C}^*$ is a non-trivial character, then, by the universal coefficients theorem, $H_1(X, \mathbb{C}_\rho)$ has dimension at least k if and only if $(B_X \otimes \mathbb{C}) \otimes_{\mathbb{C}[\pi_{ab}]} \mathbb{C}_\rho$ has dimension at least k; in turn, this condition is equivalent to $\rho \in V(E_{k-1}(B_X \otimes \mathbb{C}))$.

More generally, one may define the Alexander varieties $\mathcal{W}^i_k(X)$ as the zero sets of the ideals $E_{k-1}(H_i(X^{ab}, \mathbb{C}))$. Provided X is q-finite, a formula analogous to (3.5) holds for all $i \leqslant q$, but only in depth $k = 1$, see [35, Corollary 3.7].

If X is 2-finite, the degree 1 characteristic varieties can be computed algorithmically, starting from a finite presentation of the group $\pi = \pi_1(X)$. If $\pi = \langle x_1, \dots, x_m \mid r_1, \dots, r_s \rangle$ is such a presentation, then $\partial^{ab}_2 \colon \mathbb{Z}[\pi_{ab}]^s \to \mathbb{Z}[\pi_{ab}]^m$, the second boundary map in the chain complex (3.2), coincides with the Alexander matrix $\left(\partial_j r_i\right)^{ab}$ of abelianized Fox derivatives of the relators. An argument as above shows that $\mathcal{V}^1_k(\pi)$ coincides, at least away from $\mathbf{1}$, with the zero locus of the ideal of codimension k minors of ∂^{ab}_2; that is,

$$\mathcal{V}^1_k(\pi) \backslash \{\mathbf{1}\} = V(E_k(\mathrm{coker}\, \partial^{ab}_2)) \backslash \{\mathbf{1}\}. \tag{3.6}$$

The characteristic varieties of a space or a group can be arbitrarily complicated. For instance, let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be an integral Laurent polynomial. Then, as shown in [47], there is a finitely presented group π with $\pi_{ab} = \mathbb{Z}^n$ such that $\mathcal{V}^1_1(\pi) = V(f) \cup \{\mathbf{1}\}$. More generally, let Z be a an algebraic subset of $(\mathbb{C}^*)^n$, defined over $\mathbb{Z}$, and let j be a positive integer. Then, as shown in [55], there is a finite, connected CW-complex X with $\mathrm{Char}(X) = (\mathbb{C}^*)^n$ such that $\mathcal{V}^i_1(X) = \{\mathbf{1}\}$ for $i < j$ and $\mathcal{V}^j_1(X) = Z \cup \{\mathbf{1}\}$.

3.3. **The Alexander polynomials of a space.** Let X be a 2-finite space, with fundamental group $\pi = \pi_1(X)$. We shall let $H = \pi_{\mathrm{ab}}/\operatorname{Tors}(\pi_{\mathrm{ab}})$ be the maximal torsion-free abelian quotient of π. It is readily seen that the group ring $\mathbb{Z}[H]$ is a commutative Noetherian ring and a unique factorization domain.

Let $q\colon X^H \to X$ be the regular cover corresponding to the projection $\pi \twoheadrightarrow H$, i.e., the maximal torsion-free abelian cover of X. Fixing a basepoint $x_0 \in X$, the *Alexander module* of X is defined as the relative homology group $A_X = H_1(X^H, q^{-1}(x_0), \mathbb{Z})$, viewed as a $\mathbb{Z}[H]$-module. For each integer $k \geqslant 0$, the *k-th Alexander ideal* is the determinantal ideal $E_k(A_X)$, while the *k-th Alexander polynomial* is $\Delta^k_X = \gcd(E_k(A_X))$, the greatest common divisor of the elements in the ideal $E_k(A_X) \subseteq \mathbb{Z}[H]$.

Fixing a basis for $H \cong \mathbb{Z}^n$, we may identify the group ring $\mathbb{Z}[H]$ with the ring of Laurent polynomials in $t_1^{\pm 1}, \dots, t_n^{\pm 1}$. The Laurent polynomials $\Delta^k_X \in \mathbb{Z}[H]$ are well-defined up to multiplication by units in this ring, i.e., monomials of the form $\pm t_1^{a_1} \cdots t_n^{a_n}$ (the equivalence relation is written as $\doteq$).

Of particular importance is the polynomial $\Delta_X := \Delta^0_X$, simply called the *Alexander polynomial* of X. As shown in [53, Lemma II.5.5], if $b_1(X) \geqslant 2$ this ideal is contained in $\Delta_X \cdot I_H$, where $I_H = \ker(\varepsilon\colon \mathbb{Z}[H] \to \mathbb{Z})$ is the augmentation ideal.

3.4. **The zero sets of the Alexander polynomials.** Henceforth, we identify the identity component of the character torus, $\mathrm{Char}(X)^0$, with the algebraic torus $\operatorname{Hom}(H, \mathbb{C}^*) = (\mathbb{C}^*)^n$, where recall H is the torsion-free part of $H_1(X, \mathbb{Z})$ and $n = b_1(X)$. The Laurent polynomials in n variables are precisely the regular functions on this algebraic torus. As such, the Alexander polynomials Δ^k_X define algebraic hypersurfaces,

$$V(\Delta^k_X) = \big\{\rho \in \mathrm{Char}(X)^0 \mid \Delta^k_X(\rho) = 0\big\}. \tag{3.7}$$

Write $\mathcal{Z}^1_k(X) = \mathcal{V}^1_k(X) \cap \mathrm{Char}(X)^0$, and let $\check{\mathcal{Z}}^1_k(X)$ be the union of all codimension-one irreducible components of $\mathcal{Z}^1_k(X)$. The next lemma details the relationships between the hypersurfaces defined by the Alexander polynomials of X and the degree 1 characteristic varieties of X.

Lemma 3.1 ([11, 22])**.** *For each $k \geqslant 1$, the following hold.*

1. *The polynomial Δ^{k-1}_X is identically 0 if and only if $\mathcal{Z}^1_k(X) = \mathrm{Char}(X)^0$, in which case $\check{\mathcal{Z}}^1_k(X) = \emptyset$.*
2. *Suppose that $b_1(X) \neq 0$ and $\Delta^{k-1}_X \neq 0$. Then $\check{\mathcal{Z}}^1_k(X) = V(\Delta^{k-1}_X)$ if $b_1(X) \geqslant 2$, and $\check{\mathcal{Z}}^1_k(X) = V(\Delta^{k-1}_X) \sqcup \{\mathbf{1}\}$ if $b_1(X) = 1$.*
3. *Suppose that $b_1(X) \geqslant 2$. Then $\Delta^{k-1}_X \doteq 1$ if and only if $\check{\mathcal{Z}}^1_k(X) = \emptyset$.*

In particular, if $b_1(X) = 0$, then $\Delta_X = 0$ and $\mathcal{Z}^1_1(X) = \mathrm{Char}(X)^0 = \{1\}$.

In a special type of situation (singled out in [11]), the relationship between the first characteristic variety and the Alexander polynomial is even tighter.

Proposition 3.2. *Suppose $I_H^s \cdot (\Delta_X) \subseteq E_1(A_X)$, for some $s \geqslant 0$. Then*

$$\mathscr{Z}_1^1(X) = V(\Delta_X) \cup \{\mathbf{1}\}.$$

In particular, if $H_1(X,\mathbb{Z})$ is torsion-free and the assumption of Proposition 3.2 holds, then $\mathscr{V}_1^1(X)$ itself coincides with $V(\Delta_X)$, at least away from $\mathbf{1}$; if, moreover, $\Delta_X(\mathbf{1}) = 0$, then $\mathscr{V}_1^1(X) = V(\Delta_X)$.

4. Algebraic models and the tangent cone theorem

4.1. **Tangent cones.** We start by reviewing two constructions which provide approximations to a subvariety W of a complex algebraic torus $(\mathbb{C}^*)^n$. The first one is the classical tangent cone, while the second one is the exponential tangent cone, a construction introduced in [12] and further studied in [43, 10, 47].

Let I be an ideal in the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ such that $W = V(I)$. Picking a finite generating set for I, and multiplying these generators with suitable monomials if necessary, we see that W may also be defined by the ideal $I \cap R$ in the polynomial ring $R = \mathbb{C}[t_1, \dots, t_n]$. Let J be the ideal in the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ generated by the polynomials $g(x_1, \dots, x_n) = f(x_1 + 1, \dots, x_n + 1)$, for all $f \in I \cap R$.

The *tangent cone* of W at $\mathbf{1} \in (\mathbb{C}^*)^n$ is the algebraic subset $\operatorname{TC}_{\mathbf{1}}(W) \subseteq \mathbb{C}^n$ defined by the ideal $\operatorname{in}(J) \subset S$ generated by the initial forms of all non-zero elements from J. The set $\operatorname{TC}_{\mathbf{1}}(W)$ is a homogeneous subvariety of $\mathbb{C}^n$, which depends only on the analytic germ of W at the identity. In particular, $\operatorname{TC}_{\mathbf{1}}(W) \neq \emptyset$ if and only if $\mathbf{1} \in W$.

Let $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ be the exponential map, given in coordinates by $x_i \mapsto e^{x_i}$. The *exponential tangent cone* at $\mathbf{1}$ to a subvariety $W \subseteq (\mathbb{C}^*)^n$ is the set

$$\tau_{\mathbf{1}}(W) = \{x \in \mathbb{C}^n \mid \exp(\lambda x) \in W, \text{ for all } \lambda \in \mathbb{C}\}. \tag{4.1}$$

It is readily seen that $\tau_{\mathbf{1}}$ commutes with finite unions and arbitrary intersections. Furthermore, $\tau_{\mathbf{1}}(W)$ only depends on $W_{(\mathbf{1})}$, the analytic germ of W at the identity; in particular, $\tau_{\mathbf{1}}(W) \neq \emptyset$ if and only if $\mathbf{1} \in W$. The main property of this construction is encapsulated in the following lemma.

Lemma 4.1 ([12, 43, 47])**.** *The exponential tangent cone $\tau_{\mathbf{1}}(W)$ of a subvariety $W \subseteq (\mathbb{C}^*)^n$ is a finite union of rationally defined linear subspaces of the affine space $\mathbb{C}^n$.*

For instance, if W is an algebraic subtorus of $(\mathbb{C}^*)^n$, then $\tau_{\mathbf{1}}(W)$ equals $\operatorname{TC}_{\mathbf{1}}(W)$, and both coincide with $T_{\mathbf{1}}(W)$, the tangent space to W at the identity $\mathbf{1}$. More generally, there is always an inclusion between the two types of tangent cones associated to an algebraic subset $W \subseteq (\mathbb{C}^*)^n$, namely,

$$\tau_{\mathbf{1}}(W) \subseteq \operatorname{TC}_{\mathbf{1}}(W). \tag{4.2}$$

As we shall see, though, this inclusion is far from being an equality for arbitrary W. For instance, the tangent cone $\operatorname{TC}_{\mathbf{1}}(W)$ may be a non-linear, irreducible subvariety of $\mathbb{C}^n$, or $\operatorname{TC}_{\mathbf{1}}(W)$ may be a linear space containing the exponential tangent cone $\tau_{\mathbf{1}}(W)$ as a union of proper linear subspaces.

4.2. **The Exponential Ax–Lindemann theorem.** In [5], Budur and Wang establish the following version of the Exponential Ax–Lindemann theorem [2], which proves to be very useful in this context.

Theorem 4.2 ([5]). *Let $V \subseteq \mathbb{C}^n$ and $W \subseteq (\mathbb{C}^*)^n$ be irreducible algebraic subvarieties.*

(1) *Suppose $\dim V = \dim W$ and $\exp(V) \subseteq W$. Then V is a translate of a linear subspace, and W is a translate of an algebraic subtorus.*
(2) *Suppose the exponential map $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ induces a local analytic isomorphism $V_{(\mathbf{0})} \to W_{(\mathbf{1})}$. Then $W_{(\mathbf{1})}$ is the germ of an algebraic subtorus.*

A standard dimension argument shows the following: if W and W' are irreducible algebraic subvarieties of $(\mathbb{C}^*)^n$ which contain $\mathbf{1}$ and whose germs at $\mathbf{1}$ are locally analytically isomorphic, then $W \cong W'$. Using this fact, we obtain the following corollary to part (2) of the above theorem.

Corollary 4.3. *Let $V \subseteq \mathbb{C}^n$ and $W \subseteq (\mathbb{C}^*)^n$ be irreducible algebraic subvarieties. Suppose the exponential map $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$ induces a local analytic isomorphism $V_{(\mathbf{0})} \cong W_{(\mathbf{1})}$. Then W is an algebraic subtorus and V is a rationally defined linear subspace.*

4.3. **Tangent cones and jump loci.** Let X be a q-finite space. Its cohomology algebra, $H^{\bullet}(X,\mathbb{C})$, is then q-finite; thus, the resonance varieties $\mathscr{R}^i_k(X) := \mathscr{R}^i_k(H^{\bullet}(X,\mathbb{C}))$ are homogeneous algebraic subsets of the affine space $H^1(X,\mathbb{C})$, for all $i \leqslant q$ and $k \geqslant 0$.

The following basic relationship between the characteristic and resonance varieties was established by Libgober in [24] in the case when X is a finite CW-complex and i is arbitrary; a similar proof works in the generality that we work in here (see [44, 9] for an even more general setup).

Theorem 4.4 ([24]). *Suppose X is a q-finite space. Then, for all $i \leqslant q$ and $k \geqslant 0$,*

$$\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(X)) \subseteq \mathscr{R}^i_k(X). \tag{4.3}$$

Putting together these inclusions with those from (4.2), we obtain the following corollary.

Corollary 4.5. *Suppose X is a q-finite space. Then, for all $i \leqslant q$ and $k \geqslant 0$,*

$$\tau_{\mathbf{1}}(\mathscr{V}^i_k(X)) \subseteq \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(X)) \subseteq \mathscr{R}^i_k(X). \tag{4.4}$$

Note that we may replace in Corollary 4.5 the characteristic varieties $\mathscr{V}^i_k(X)$ by the subvarieties $\mathscr{Z}^i_k(X) = \mathscr{V}^i_k(X) \cap \mathrm{Char}^0(X)$. Also note that, if $\mathscr{R}^i_k(X)$ is empty or equal to $\{\mathbf{0}\}$, then all of the above inclusions become equalities. In particular, $\tau_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \mathscr{R}^1_1(X)$ if $b_1(X) \leqslant 1$. In general, though, each of the inclusions from (4.4)—or both—can be strict, as examples to follow will show.

A particular case of the above corollary is worth mentioning separately.

Corollary 4.6. *Let π be a finitely generated group. Then, for all $k \geqslant 0$,*

$$\tau_{\mathbf{1}}(\mathscr{V}^1_k(\pi)) \subseteq \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_k(\pi)) \subseteq \mathscr{R}^1_k(\pi).$$

4.4. **Algebraic models for spaces and groups.** Given a space X, we let $A^{\bullet}_{\mathrm{PL}}(X)$ be the commutative differential graded $\mathbb{Q}$-algebra of rational polynomial forms, as defined by Sullivan in [49] (see [19, 20, 21] for a detailed exposition). There is then a natural isomorphism $H^{\bullet}(A_{\mathrm{PL}}(X)) \cong H^{\bullet}(X,\mathbb{Q})$ under which the respective induced homomorphisms in cohomology correspond.

As before, let $\Bbbk$ be a field of characteristic 0, and q a positive integer. We say that a $\Bbbk$-CDGA (A,d) is a *model* (or just a *q-model*) over $\Bbbk$ for X if A is weakly equivalent (or just q-equivalent) to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$. For instance, if X is a smooth manifold, then $\Omega^{\bullet}_{\mathrm{dR}}(X)$, the de Rham algebra of smooth forms on X, is a model of X over $\mathbb{R}$. By considering a classifying space $X = K(\pi,1)$ for a group π, we may speak about q-models for groups.

A continuous map $f\colon X \to Y$ is said to be a *q-rational homotopy equivalence* if the induced homomorphism $f^*\colon H^i(Y,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ is an isomorphism for $i \leqslant q$ and a monomorphism for $i = q+1$. Such a map induces a q-equivalence $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$. Therefore, whether a space X admits a q-finite q-model depends only on its q-rational homotopy type, in particular, on its q-homotopy type. Consequently, a path-connected space X admits a 1-finite 1-model if and only if the fundamental group $\pi = \pi_1(X)$ admits one. The existence of such a model puts rather stringent constraints on the group π. One such constraint is given in [38, Theorem 1.5].

Following Quillen [40], let us define the Malcev Lie algebra of π, denoted $\mathfrak{m}(\pi)$, as the complete, filtered Lie algebra of primitive elements in the I-adic completion of the Hopf algebra $\mathbb{Q}[\pi]$, where $I = \ker(\varepsilon\colon \mathbb{Q}[\pi] \to \mathbb{Q})$ is the augmentation ideal. That is to say, $\mathfrak{m}(\pi) = \operatorname{Prim}(\widehat{\mathbb{Q}[\pi]})$, where $\widehat{\mathbb{Q}[\pi]} = \varprojlim_r \mathbb{Q}[\pi]/I^r$.

Theorem 4.7 ([38]). *A finitely generated group π admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(\pi)$ is the lower central series (LCS) completion of a finitely presented Lie algebra.*

The above condition means that $\mathfrak{m}(\pi) = \widehat{L}$, for some finitely presented Lie algebra L, where $\widehat{L} = \varprojlim_r L/\Gamma_r L$, with the LCS series $\{\Gamma_r L\}_{r\geqslant 1}$ defined inductively by $\Gamma_1 L = L$ and $\Gamma_r L = [L, \Gamma_{r-1} L]$ for $r > 1$.

4.5. **Algebraic models and cohomology jump loci.** Work of Dimca and Papadima [10], generalizing previous work from [12], establishes a tight connection between the geometry of the characteristic varieties of a space and that of resonance varieties of a model for it, around the origins of the respective ambient spaces, provided certain finiteness conditions hold.

More precisely, let X be a path-connected space with $b_1(X) < \infty$, and consider the analytic map $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ induced by the coefficient homomorphism $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$. Let (A,d) be a CDGA model for X, defined over $\mathbb{C}$. Upon identifying $H^1(A) \cong H^1(X,\mathbb{C})$, we obtain an analytic map $H^1(A) \to H^1(X,\mathbb{C}^*)$, which takes $\mathbf{0}$ to $\mathbf{1}$.

Theorem 4.8 ([10]). *Let X be a q-finite space, and suppose X admits a q-finite, q-model A, for some $q \geqslant 1$. Then, the aforementioned map, $H^1(A) \to H^1(X,\mathbb{C}^*)$, induces a local* *analytic isomorphism $H^1(A)_{\mathbf{0}} \to H^1(X,\mathbb{C}^*)_{\mathbf{1}}$, which identifies the germ at $\mathbf{0}$ of $\mathscr{R}^i_k(A)$ with the germ at $\mathbf{1}$ of $\mathscr{V}^i_k(X)$, for all $i \leqslant q$ and all $k \geqslant 0$.*

Recent work of Budur and Wang [5] builds on this theorem, providing a structural result on the geometry of the characteristic varieties of spaces satisfying the hypothesis of the above theorem. Putting together Theorem 4.8 and Corollary 4.3 yields their result, in the slightly stronger form given in [38].

Theorem 4.9 ([5])**.** *Suppose X is a q-finite space which admits a q-finite q-model. Then all the irreducible components of $\mathscr{V}^i_k(X)$ passing through $\mathbf{1}$ are algebraic subtori of $H^1(X,\mathbb{C}^*)$, for all $i \leqslant q$ and $k \geqslant 0$.*

As an immediate corollary of the previous two theorems, we obtain the following "Tangent Cone formula."

Theorem 4.10. *Suppose X is a q-finite space which admits a q-finite q-model A. Then, for all $i \leqslant q$ and $k \geqslant 0$,*

$$\tau_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathscr{R}^i_k(A). \tag{4.5}$$

This theorem, together with Theorem 4.7, yields the following corollary.

Corollary 4.11. *Suppose π is a finitely generated group whose Malcev Lie algebra is the LCS completion of a finitely presented Lie algebra. Then $\tau_{\mathbf{1}}(\mathscr{V}^1_k(\pi)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_k(\pi))$, for all $k \geqslant 0$.*

In other words, if the first half of the Tangent Cone formula fails in degree 1, i.e., if $\tau_{\mathbf{1}}(\mathscr{V}^1_k(\pi)) \subsetneq \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_k(\pi))$ for some $k > 0$, then $\mathfrak{m}(\pi) \not\cong \widehat{L}$, for any finitely presented Lie algebra L. This will happen automatically if the variety $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_k(\pi))$ has an irreducible component which is not a rationally defined linear subspace of $H^1(\pi,\mathbb{C})$.

4.6. **Formality.** A path-connected space X is said to be *formal* (over a field $\Bbbk$ of characteristic 0) if Sullivan's algebra $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$ is formal; in other words, if the cohomology algebra, $H^{\bullet}(X,\Bbbk)$, endowed with the zero differential, is weakly equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$. Likewise, a space X is merely *q-formal* (for some $q \geqslant 1$) if $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$ has this property. These formality and partial formality notions are independent of the field $\Bbbk$, as long as its characteristic is 0. Furthermore, if X is a q-formal CW-complex of dimension at most $q+1$, then X is formal, cf. [25].

Evidently, every q-finite, q-formal space X admits a q-finite q-model, namely, $A = H^{\bullet}(X,\Bbbk)$ with $d = 0$. Examples of formal spaces include suspensions, rational cohomology tori, surfaces, compact connected Lie groups, as well as their classifying spaces. On the other hand, the only nilmanifolds which are formal are tori. Formality is preserved under wedges and products of spaces, and connected sums of manifolds.

It is readily seen that the 1-formality property of a space X depends only on its fundamental group, $\pi = \pi_1(X)$. Alternatively, a finitely generated group π is 1-formal if and only if its Malcev Lie algebra $\mathfrak{m}(\pi)$ is isomorphic to the LCS completion of a finitely generated, quadratic Lie algebra L. Examples of 1-formal groups include free groups and free abelian groups of finite rank, surface groups, and groups with first Betti number equal to 0 or 1. The 1-formality property is preserved under finite free products and direct products of (finitely generated) groups. We refer to [25, 34, 38, 46] for more details and references regarding all these notions.

4.7. **Formality and cohomology jump loci.** The main connection between the formality property of a space and the geometry of its cohomology jump loci is provided by the next result. This result, which was first proved in degree $i = 1$ in [12], and in arbitrary degree in [10], is now an immediate consequence of Theorem 4.10.

Corollary 4.12. *Let X be a q-finite, q-formal space. Then, for all $i \leqslant q$ and $k \geqslant 0$,*

$$\tau_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(X)) = \mathscr{R}^i_k(X). \tag{4.6}$$

In particular, if π is a finitely generated, 1-formal group, then, for all $k \geqslant 0$,

$$\tau_{\mathbf{1}}(\mathscr{V}^1_k(\pi)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_k(\pi)) = \mathscr{R}^1_k(\pi). \tag{4.7}$$

As an application of Corollary 4.12, we have the following characterization of the irreducible components of the cohomology jump loci in the formal setting.

Corollary 4.13. *Suppose X is a q-finite, q-formal space. Then, for all $i \leqslant q$ and $k \geqslant 0$, the following hold.*

1. *All irreducible components of the resonance varieties $\mathscr{R}^i_k(X)$ are rationally defined linear subspaces of $H^1(X, \mathbb{C})$.*
2. *All irreducible components of the characteristic varieties $\mathscr{V}^i_k(X)$ which contain the origin are algebraic subtori of $\mathrm{Char}(X)^0$, of the form $\exp(L)$, where L runs through the linear subspaces comprising $\mathscr{R}^i_k(X)$.*

5. Resonance varieties of 3-manifolds

We now switch our focus from the general theory of cohomology jump loci to some of the applications of this theory in low-dimensional topology. We start by describing the resonance varieties attached to the cohomology ring of a closed, orientable, 3-dimensional manifold, based on the approach from [45].

5.1. **The intersection form of a 3-manifold.** Let M be a compact, connected 3-manifold without boundary. For short, we shall refer to M as being a *closed* 3-manifold. Throughout, we will also assume that M is orientable.

Fix an orientation class $[M] \in H_3(M, \mathbb{Z}) \cong \mathbb{Z}$. With this choice, the cup product on M determines an alternating 3-form μ_M on $H^1(M, \mathbb{Z})$, given by

$$\mu_M(a \wedge b \wedge c) = \langle a \cup b \cup c, [M] \rangle, \tag{5.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the Kronecker pairing. In turn, the cup-product map $\bigwedge^2 H^1(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$ is determined by the intersection form μ_M via $\langle a \cup b, \gamma\rangle = \mu_M(a \wedge b \wedge c)$, where c is the Poincaré dual of $\gamma \in H_2(M,\mathbb{Z})$.

Now fix a basis $\{e_1,\dots,e_n\}$ for $H^1(M,\mathbb{Z})$, and choose $\{e_1^\vee,\dots,e_n^\vee\}$ as basis for the torsion-free part of $H^2(M,\mathbb{Z})$, where $e_i^\vee$ denotes the Kronecker dual of the Poincaré dual of e_i. Write

$$\mu_M = \sum_{1\leqslant i<j<k\leqslant n} \mu_{ijk} e_i e_j e_k, \tag{5.2}$$

where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$. Using formula (5.1), we find that $e_i e_j = \sum_{k=1}^n \mu_{ijk} e_k^\vee$.

As shown by Sullivan [48], for every finitely generated, torsion-free abelian group H and every 3-form $\mu \in \bigwedge^3 H^*$, there is a closed, oriented 3-manifold M with $H^1(M,\mathbb{Z}) = H$ and cup-product form $\mu_M = \mu$. Such a 3-manifold can be constructed by a process known as "Borromean surgery" or "T^3-surgery", see for instance [6, Corollary 3.5] or [28, Theorem 6.1]. More precisely, if $n = \operatorname{rank} H$, such a manifold M may be defined as 0-framed surgery on a link in S^3 obtained from the trivial n-component link by replacing a collection of trivial 3-string braids by a collection of 3-string braid whose closure is the Borromean rings.

Of course, there are many closed 3-manifolds that realize a given intersection 3-form μ. For instance, if M is such a manifold, then the connected sum of M with any rational homology 3-sphere will also realize μ. As another example, if M is the link in S^5 of an isolated singularity of a complex algebraic surface, then $\mu_M = 0$ [48]; more generally, if M bounds a compact, orientable 4-manifold W such that the cup-product pairing on $H^2(W,M)$ is non-degenerate, then $\mu_M = 0$, see [27, Proposition 13].

Remark 5.1. Two closed, oriented 3-manifolds, M_1 and M_2, are said to be *homology cobordant* if there is a compact oriented 4-manifold W such that $\partial W = M_1 \sqcup -M_2$ and the inclusion-induced maps $H_n(M_i,\mathbb{Z}) \to H_n(W,\mathbb{Z})$ are isomorphisms for $i = 1,2$ and all n. It is readily seen that homology cobordism is an equivalence relation. Moreover, if M_1 and M_2 are homology cobordant, then their cohomology rings are isomorphic.

5.2. **Resonance varieties of 3-manifolds.** Let A be a graded, graded-commutative algebra over a field $\Bbbk$ such that A is connected and all the Betti numbers $b_i(A) = \dim_{\Bbbk} A^i$ are finite. We say that A is a *Poincaré duality algebra* of dimension m (for short, a PD_m algebra) if there exists a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular. In this case, ε is an isomorphism, $A^i = 0$ for $i > m$, and $b_i(A) = b_{m-i}(A)$.

Now suppose M is a compact, connected, oriented m-dimensional manifold; then, by Poincaré duality, the cohomology algebra $H^\bullet(M,\Bbbk)$ is a PD_m algebra over $\Bbbk$, with the homomorphism $\varepsilon\colon H^m(M,\Bbbk) = H_m(M,\mathbb{Z}) \otimes \Bbbk \to \Bbbk$ being determined by the orientation class $[M] \in H_m(M,\mathbb{Z})$ by setting $\varepsilon([M] \otimes 1) = 1$.

We now restrict to the case $m = 3$, so that M is a closed, oriented 3-manifold. Then the cohomology algebra $A = H^\bullet(M,\mathbb{C})$ is a Poincaré duality $\mathbb{C}$-algebra of dimension 3. Two such PD_3 algebras are isomorphic if and only if the corresponding 3-forms are isomorphic, see [45].

Let $S = \mathrm{Sym}(A_1)$ be the symmetric algebra on $A_1 = H_1(M,\mathbb{C})$, which we will identify as before with the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$. In our situation, the chain complex from (2.5) has the form

$$A^0 \otimes_{\mathbb{C}} S \xrightarrow{\delta^0_A} A^1 \otimes_{\mathbb{C}} S \xrightarrow{\delta^1_A} A^2 \otimes_{\mathbb{C}} S \xrightarrow{\delta^2_A} A^3 \otimes_{\mathbb{C}} S\,, \tag{5.3}$$

where the S-linear differentials are given by $\delta^q_A(u) = \sum_{j=1}^n e_j u \otimes x_j$ for $u \in A^q$. In our chosen basis, the matrix of δ^0 is $\begin{pmatrix} x_1 & \cdots & x_n\end{pmatrix}$, the matrix of δ^2_A is the transpose of δ^0, while the matrix of δ^1_A is the $n \times n$ skew-symmetric matrix of linear forms in the variables of S, with entries given by

$$\delta^1_A(e_i) = \sum_{j=1}^n \sum_{k=1}^n \mu_{jik} e_k^{\vee} \otimes x_j\,. \tag{5.4}$$

We wish to describe the resonance varieties $\mathscr{R}^i_k(M) = \mathscr{R}^i_k(A)$. To avoid trivialities, we will assume for the rest of this section that $n \geqslant 3$, since, otherwise $\mathscr{R}^i_k(M) \subseteq \{\mathbf{0}\}$. Furthermore, we may assume that $i = 1$; indeed, as shown in [45, Proposition 6.1],

$$\mathscr{R}^2_k(M) = \mathscr{R}^1_k(M) \quad \text{for } 1 \leqslant k \leqslant n, \tag{5.5}$$

while $\mathscr{R}^i_0(M) = H^1(M,\mathbb{C})$, $\mathscr{R}^3_1(M) = \mathscr{R}^0_1(M) = \{\mathbf{0}\}$, and $\mathscr{R}^i_k(M) = \emptyset$, otherwise. Now, by (2.7), the resonance variety $\mathscr{R}^1_k(M)$ is the vanishing locus of the ideal of codimension k minors of the matrix $\delta_M := \delta^1_A$; that is,

$$\mathscr{R}^1_k(M) = V(I_{n-k}(\delta_M)). \tag{5.6}$$

The rank of a 3-form $\mu\colon \bigwedge^3 U \to \mathbb{C}$ on a finite-dimensional $\mathbb{C}$-vector space U is the minimum dimension of a subspace $W \subset U$ such that μ factors through $\bigwedge^3 W$.

Proposition 5.2 ([45]). *If $n \geqslant 3$ and μ_M has rank $n = b_1(M)$, then*

$$\mathscr{R}^1_{n-2}(M) = \mathscr{R}^1_{n-1}(M) = \mathscr{R}^1_n(M) = \{\mathbf{0}\}.$$

5.3. **Pfaffians and resonance.** For a skew-symmetric matrix θ, we shall denote by $\mathrm{Pf}_{2r}(\theta)$ the ideal of $2r \times 2r$ Pfaffians of θ.

Proposition 5.3 ([45]). *The following hold:*

$$\begin{aligned} &\mathscr{R}^1_{2k}(M) = \mathscr{R}^1_{2k+1}(M) = V(\mathrm{Pf}_{n-2k}(\delta_M)), && \text{if } n \text{ is even},\\ &\mathscr{R}^1_{2k-1}(M) = \mathscr{R}^1_{2k}(M) = V(\mathrm{Pf}_{n-2k+1}(\delta_M)), && \text{if } n \text{ is odd}. \end{aligned} \tag{5.7}$$

The skew-symmetric matrix δ_M is singular, since the vector $(x_1,\dots,x_n)$ is in its kernel. Hence, both its determinant $\det(\delta_M)$ and its Pfaffian $\mathrm{pf}(\delta_M)$ vanish. In [53, Ch. III, Lemmas 1.2 and 1.3.1], Turaev shows how to remedy this situation, so as to obtain well-defined determinant and Pfaffian polynomials for the 3-form μ_M. Let $\delta_M(i;j)$ be the sub-matrix obtained from δ_M by deleting the i-th row and j-th column.

Lemma 5.4 ([53]). *Suppose $n \geqslant 3$. There is then a polynomial* $\mathrm{Det}(\mu) \in S$ *such that* $\det \delta_M(i;j) = (-1)^{i+j} x_i x_j \,\mathrm{Det}(\mu)$. *Moreover, if n is even, then* $\mathrm{Det}(\mu) = 0$*, while if n is odd, then* $\mathrm{Det}(\mu) = \mathrm{Pf}(\mu)^2$*, where* $\mathrm{pf}(\delta_M(i;i)) = (-1)^{i+1} x_i \,\mathrm{Pf}(\mu)$.

5.4. **The top resonance variety.** We will need in the sequel the notion of 'generic' alternating 3-form, introduced and studied by Berceanu and Papadima in [3]. For our purposes, it will be enough to consider the case when $n = 2g+1$, for some $g \geqslant 1$. We say that a 3-form μ_A is *generic* if there is an element $c \in A^1$ such that the 2-form $\gamma_c \in A_1 \wedge A_1$ defined by

$$\gamma_c(a \wedge b) = \mu_A(a \wedge b \wedge c) \quad \text{for } a, b \in A^1 \tag{5.8}$$

has rank $2g$, that is, $\gamma_c^g \neq 0$ in $\bigwedge^{2g} A_1$. Equivalently, in a suitable basis for A^1, we may write $\mu_A = \sum_{i=1}^g a_i b_i c + \sum q_{ijk} z_i z_j z_k$, where each z_i belongs to the span of $a_1, b_1, \dots, a_g, b_g$ in A^1, and the coefficients q_{ijk} are in $\mathbb{C}$.

Example 5.5. Let $M = \Sigma_g \times S^1$ be the product of a circle with a closed, orientable surface of genus $g \geqslant 1$. If $a_1, b_1, \dots, a_g, b_g$ is the standard symplectic basis for $H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$, and c generates $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$, then $\mu_M = \sum_{i=1}^g a_i b_i c$, and so μ_M is generic. A routine computation shows that $\mathrm{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Furthermore, $\mathscr{R}^1_k(M) = \{x_{2g+1} = 0\}$ for $1 \leqslant k \leqslant 2g-2$ and $\mathscr{R}^1_{2g-1}(M) = \{\mathbf{0}\}$. ◇

More generally, if M is homology cobordant to $S^1 \times \Sigma_g$ for some $g \geqslant 1$, then $\mu_M = \mu_{S^1 \times \Sigma_g}$ is generic. For $n = 3$ and $n = 5$, there is a single irreducible 3-form of rank n (up to isomorphism), and that form is of the type just discussed. In rank $n = 7$, there are 5 irreducible 3-forms: two generic, $\mu = e_7(e_1e_4 + e_2e_5 + e_3e_6)$ and $\mu' = \mu + e_4e_5e_6$, and three non-generic.

Using [45, Theorem 8.6], we obtain the following description of the top resonance variety of a closed, orientable 3-manifold.

Theorem 5.6. *Let M be a closed, orientable 3-manifold. Set $n = b_1(M)$ and let μ_M be the associated 3-form. Then*

$$\mathscr{R}^1_1(M) = \begin{cases} \emptyset & \text{if } n = 0; \\ \{0\} & \text{if } n = 1 \text{ or } n = 3 \text{ and } \mu_M \text{ has rank } 3; \\ V(\mathrm{Pf}(\mu_M)) & \text{if } n \text{ is odd, } n > 3, \text{ and } \mu_M \text{ is generic}; \\ H^1(M, \mathbb{C}) & \text{otherwise}. \end{cases} \tag{5.9}$$

Remark 5.7. The case when $b_1(M)$ is even and positive is worth dwelling upon. In this case, the equality $\mathscr{R}^1(M) = H^1(M, \mathbb{C})$ was first proved in [13], where it was used to show that the only 3-manifold groups which are also Kähler groups are the finite subgroups of $\mathrm{O}(4)$. Another application of this equality was given in [36]: if M is a closed, orientable 3-manifold such that $b_1(M)$ is even and M fibers over the circle, then M is not 1-formal. ◇

6. Alexander polynomials and characteristic varieties of 3-manifolds

In this section, we collect some facts regarding the Alexander polynomials and the characteristic varieties of closed, orientable, 3-dimensional manifolds.

6.1. **Poincaré duality and characteristic varieties.** Let M be a smooth, closed, orientable manifold of dimension m. By Morse theory, M admits a finite cell decomposition; consequently, its fundamental group, $\pi = \pi_1(M)$ admits a finite presentation. The involution $g \mapsto g^{-1}$ taking each element of π to its inverse induces an algebraic automorphism of $\operatorname{Hom}(\pi, \mathbb{C}^*)$, taking a character ρ to the character $\bar{\rho}$ given by $\bar{\rho}(g) = \rho(g^{-1})$.

Proposition 6.1. *The above automorphism of* $\operatorname{Hom}(\pi, \mathbb{C}^*)$ *restricts to isomorphisms*

$$\mathcal{V}^i_k(M) \cong \mathcal{V}^{m-i}_k(M),$$

for all $i \geqslant 0$ and $k \geqslant 0$.

Proof. Poincaré duality with local coefficients (see e.g. [54, §2]) yields isomorphisms $H^i(M, \mathbb{C}_\rho) \cong H_{m-i}(M, \mathbb{C}_{\bar{\rho}})$. The claim follows. ◻

A well-known theorem of E. Moise insures that every 3-manifold has a smooth structure. Thus, the above proposition together with the discussion from §3.1 yield the following corollary.

Corollary 6.2. *Let M be a closed, orientable 3-manifold. Then $\mathcal{V}^i_0(M) = H^1(M, \mathbb{C}^*)$ for all $i \geqslant 0$, $\mathcal{V}^3_1(M) = \mathcal{V}^0_1(M) = \{\mathbf{1}\}$,*

$$\mathcal{V}^2_k(M) = \mathcal{V}^1_k(M) \quad \text{for } 1 \leqslant k \leqslant b_1(M),$$

and otherwise $\mathcal{V}^i_k(M) = \emptyset$.

Thus, in order to compute the characteristic varieties of a 3-manifold M as above, it is enough to determine the sets $\mathcal{V}^1_k(M)$ for $1 \leqslant k \leqslant b_1(M)$.

6.2. **The Alexander polynomial of a closed 3-manifold.** As before, let H be the quotient of $H_1(M, \mathbb{Z})$ by its torsion subgroup. We will identify the group ring $\mathbb{Z}[H]$ with the ring of Laurent polynomials $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, where $n = b_1(M)$. For a Laurent polynomial λ, we will denote by $\bar{\lambda}$ its image under the involution $t_i \mapsto t_i^{-1}$, and say that λ is *symmetric* if $\lambda \doteq \bar{\lambda}$.

Assume now that M is 3-dimensional, and let $\Delta_M \in \mathbb{Z}[H]$ be its Alexander polynomial. Recall that Δ_M is only defined up to units, i.e., up to multiplication by monomials $\pm t_1^{a_1} \cdots t_n^{a_n}$ with $a_i \in \mathbb{Z}$. Work of Milnor [32] and Turaev [51, 53] shows that Δ_M is symmetric.

Conversely, if $H = \mathbb{Z}$ or $\mathbb{Z}^2$, then every symmetric Laurent polynomial $\lambda \in \mathbb{Z}[H]$ can be realized as the Alexander polynomial of a closed, orientable 3-manifold M with $H_1(M, \mathbb{Z}) = H$; see [53, VII.5.3]. Furthermore, every symmetric Laurent polynomial λ in $n \leqslant 3$ variables such that $\lambda(\mathbf{1}) \neq 0$ can be realized as the Alexander polynomial of a closed, orientable 3-manifold M with $b_1(M) = n$; see [1].

On the other hand, for $n \geqslant 4$, the situation is quite different.

Theorem 6.3 ([53]). *If M is a closed, orientable 3-manifold M with $b_1(M) \geqslant 4$, then $\Delta_M(\mathbf{1}) = 0$.*

The theorem follows at once from [53, §II, Corollaries 2.2 and 5.2.1]. As an application, we deduce that $\Delta_M \neq 1$ if $b_1(M) \geqslant 4$, a result which is also proved in [1, Theorem 8] by different means.

6.3. **Characteristic varieties and the Alexander polynomial.** Now consider the maximal torsion-free abelian cover $M^H \to M$, and let $A_M = H_1(M^H, \mathbb{Z})$, viewed as a $\mathbb{Z}[H]$-module as in §3.3. Recall that the determinantal ideal $E_1(A_M)$ is always contained in the ideal $\Delta_M \cdot I_H$, where $I_H = \ker(\varepsilon\colon \mathbb{Z}[H] \to \mathbb{Z})$ is the augmentation ideal, provided $b_1(M) \geqslant 2$. In [31, Theorem 5.1], McMullen established a closer relationship between these ideals, in the case when M is a closed, orientable 3-manifold M (see also Turaev [53, Theorem II.1.2]).

Theorem 6.4 ([31]). *Let $n = b_1(M)$. Then*

$$E_1(A_M) = \begin{cases} (\Delta_M) & \text{if } n \leqslant 1, \\ I_H^2 \cdot (\Delta_M) & \text{if } n \geqslant 2. \end{cases} \tag{6.1}$$

Recall now that $\mathscr{W}^1_1(M)$ denotes the intersection of the characteristic variety $\mathscr{V}^1_1(M)$ with the identity component of the character group, $\mathrm{Char}^0(M) = (\mathbb{C}^*)^n$.

Proposition 6.5. *Let M be a closed, orientable, 3-dimensional manifold. Then*

$$\mathscr{W}^1_1(M) = V(\Delta_M) \cup \{\mathbf{1}\}. \tag{6.2}$$

Moreover, if $b_1(M) \geqslant 4$, then $\mathscr{W}^1_1(M) = V(\Delta_M)$.

Proof. The first equality follows at once from Proposition 3.2 and Theorem 6.4. If $b_1(M) \geqslant 4$, the second equality follows from the first one and Theorem 6.3. ∎

Remark 6.6. If the group $H_1(M, \mathbb{Z})$ has non-trivial torsion, the inclusion $\mathscr{W}^1_1(M) \subseteq \mathscr{V}^1_1(M)$ may very well be strict. A rich source of examples illustrating this phenomenon is provided by Seifert fibered manifolds (see Example 9.1 below).

Corollary 6.7. *Let M be a closed, orientable, 3-dimensional manifold, and set $W = \mathscr{V}^1_1(M)$.*

1. *If $\Delta_M(\mathbf{1}) \neq 0$, then $\tau_{\mathbf{1}}(W) = \mathrm{TC}_{\mathbf{1}}(W) = \{\mathbf{1}\}$.*
2. *If $\Delta_M(\mathbf{1}) = 0$, yet $\Delta_M \neq 0$, then $\tau_{\mathbf{1}}(W) = \tau_{\mathbf{1}}(V(\Delta_M))$ and $\mathrm{TC}_{\mathbf{1}}(W) = \mathrm{TC}_{\mathbf{1}}(V(\Delta_M))$.*
3. *If $\Delta_M = 0$, then $\tau_{\mathbf{1}}(W) = \mathrm{TC}_{\mathbf{1}}(W) = H^1(M, \mathbb{C})$.*

Moreover, if $b_1(M) \geqslant 4$, then case (1) does not occur.

Proof. Recall from 4.1 that both $\tau_{\mathbf{1}}(W)$ and $\mathrm{TC}_{\mathbf{1}}(W)$ depend only on the analytic germ of W around the identity $\mathbf{1} \in \mathrm{Char}(M)^0$. Thus, in computing these tangent cones at $\mathbf{1}$, we may replace W by $W \cap \mathrm{Char}(M)^0 = \mathscr{W}^1_1(M)$. All the claims now follow directly from Proposition 6.5. ∎

7. A Tangent Cone theorem for 3-manifolds

For closed, orientable, 3-dimensional manifolds, the Tangent Cone theorem takes a rather surprisingly concrete form, which we proceed to describe in this section.

7.1. A 3-dimensional Tangent Cone theorem.

We start by isolating a class of closed 3-manifolds for which the full Tangent Cone formula (4.6) holds in degree $i = 1$ and depth $k = 1$.

Lemma 7.1. *Let M be a closed, orientable, 3-dimensional manifold such that $\Delta_M = 0$. Then $\tau_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathscr{R}^1_1(M) = H^1(M,\mathbb{C})$.*

Proof. By case (3) of Corollary 6.7, we have that $\tau_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = H^1(M,\mathbb{C})$. On the other hand, by Corollary 4.5, we always have $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) \subseteq \mathscr{R}^1_1(M)$, while $\mathscr{R}^1_1(M) \subseteq H^1(M,\mathbb{C})$ by definition. The claim follows. $\square$

Example 7.2. Let $M = \#_1^n S^1 \times S^2$. Then clearly $\mu_M = 0$ and $\Delta_M = 0$; in particular, Lemma 7.1 applies. In fact, M is formal, and so the Tangent Cone formula holds in all degrees and depths.

The next result shows that the second half of the Tangent Cone formula holds for a large class of closed 3-manifolds with odd first Betti number (regardless of whether these manifolds are 1-formal or not), yet fails for most 3-manifolds with even first Betti number.

Theorem 7.3. *Let M be a closed, orientable 3-manifold, and set $n = b_1(M)$.*

1. *If $n \leqslant 1$, or n is odd, $n \geqslant 3$, and μ_M is generic, then $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathscr{R}^1_1(M)$.*
2. *If n is even, $n \geqslant 2$, then $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathscr{R}^1_1(M)$ if and only if $\Delta_M = 0$.*

Proof. (1) If $n \leqslant 1$, the Tangent Cone formula always holds. If $n = 3$, then our genericity assumption implies that $\mu_M = e_1e_2e_3$ in a suitable basis for $H^1(M,\mathbb{C})$. It follows that $\mathscr{R}^1_1(M) = \{\mathbf{0}\}$, and so $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \{\mathbf{0}\}$, too, by Theorem 4.4.

So let assume that n is odd and $n > 3$. In this case, Proposition 6.5 insures that $\mathscr{Z}^1_1(M) = V(\Delta_M)$. Moreover, as noted previously, $\mathscr{V}^1_1(M)$ coincides with $\mathscr{Z}^1_1(M)$ around the identity, and so the two varieties share the same tangent cone at $\mathbf{1}$.

Now, as explained in §4.1, $\mathrm{TC}_{\mathbf{1}}(V(\Delta_M))$ is the variety defined by the homogeneous polynomial $\mathrm{in}(\widetilde{\Delta}_M)$, where $\widetilde{\Delta}_M(x_1,\dots,x_n) = \Delta_M(x_1+1,\dots,x_n+1)$. Putting things together, we conclude that

$$\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = V(\mathrm{in}(\widetilde{\Delta}_M)). \tag{7.1}$$

On the other hand, as shown by Turaev in [53, Theorem III.2.2], for $n \geqslant 3$ and n odd, we have that

$$\mathrm{in}(\widetilde{\Delta}_M) = \mathrm{Det}(\mu_M). \tag{7.2}$$

We also know from Lemma 5.4 that $\mathrm{Det}(\mu) = \mathrm{Pf}(\mu)^2$; hence, $V(\mathrm{Det}(\mu_M) = V(\mathrm{Pf}(\mu_M))$. Finally, since n is odd, $n > 3$, and μ_M is generic, Theorem 5.6 implies that $V(\mathrm{Pf}(\mu_M)) =$

$\mathscr{R}^1_1(M)$. Combining the aforementioned equalities, we conclude that

$$\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = V(\mathrm{in}(\widetilde{\Delta}_M)) = V(\mathrm{Det}(\mu_M) = V(\mathrm{Pf}(\mu_M)) = \mathscr{R}^1_1(M).$$

(2) Now suppose that n is even and $n \geqslant 2$. By Theorem 5.6, we have that $\mathscr{R}^1_1(M) = \mathbb{C}^n$. On the other hand, by Corollary 6.7, the following alternative holds: if $\Delta_M = 0$, then $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M))$ also equals $\mathbb{C}^n$; otherwise $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M))$ is a proper subvariety of $\mathbb{C}^n$. This completes the proof. □

7.2. **Algebraic models for 3-manifolds.** As an application of the techniques developed so far, we derive a partial characterization of the formality and finiteness properties for rational models of 3-manifolds.

Theorem 7.4. *Let M be a closed, orientable, 3-dimensional manifold, and set $n = b_1(M)$.*

(1) *If $n \leqslant 1$, then M is formal, and has the rational homotopy type of S^3 or $S^1 \times S^2$.*
(2) *If n is even, $n \geqslant 2$, and $\Delta_M \neq 0$, then M is not 1-formal.*
(3) *If $\Delta_M \neq 0$, yet $\Delta_M(\mathbf{1}) = 0$ and $\mathrm{TC}_{\mathbf{1}}(V(\Delta_M))$ is not a finite union of rationally defined linear subspaces, then M admits no 1-finite 1-model.*

Proof. (1) As mentioned previously, any connected CW-complex X with finite 2-skeleton and with $b_1(X) \leqslant 1$ is 1-formal. On the other hand, if M is a closed, orientable 3-manifold, then 1-formality is equivalent to formality, see [18]. Thus, if $b_1(M) = 0$ or 1, then M is formal, and so, as noted in [34], M must be rationally homotopy equivalent to either S^3 or $S^1 \times S^2$.

(2) Now suppose $b_1(M)$ is even and positive, and $\Delta_M \neq 0$. Then, by part (2) of Theorem 7.3, we have that $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) \neq \mathscr{R}^1_1(M)$. Thus, by Corollary 4.12, M is not 1-formal.

(3) Finally, if $\Delta_M \neq 0$ and $\Delta_M(\mathbf{1}) = 0$, then, by Corollary 6.7, $\tau_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \tau_{\mathbf{1}}(V(\Delta_M))$ and $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathrm{TC}_{\mathbf{1}}(V(\Delta_M))$. On the other hand, if not all the irreducible components of $\mathrm{TC}_{\mathbf{1}}(V(\Delta_M))$ are linear subspaces defined over $\mathbb{Q}$, then, by Lemma 4.1, $\tau_{\mathbf{1}}(V(\Delta_M)) \neq \mathrm{TC}_{\mathbf{1}}(V(\Delta_M))$. Therefore, if both assumptions are satisfied, $\tau_{\mathbf{1}}(\mathscr{V}^1_1(M))$ is a proper subset of $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M))$, and so, by Theorem 4.10, M cannot have a 1-finite 1-model. □

Now let $\pi = \pi_1(M)$ be the fundamental group of M, and let $\mathfrak{m} = \mathfrak{m}(\pi)$ be its Malcev Lie algebra. In the three cases treated in Theorem 7.4, the following hold:

(1) $\mathfrak{m} = 0$ (if $n = 0$) or $\mathfrak{m} = \mathbb{Q}$ (if $n = 1$).
(2) $\mathfrak{m}$ is not the LCS completion of a finitely generated, quadratic Lie algebra.
(3) $\mathfrak{m}$ is not the LCS completion of a finitely presented Lie algebra.

7.3. **Discussion and examples.** If $b_1(M) = 2$, then all three possibilities laid out in Corollary 6.7 do occur.

Example 7.5. Let $M = S^1 \times S^2 \# S^1 \times S^2$; then $\Delta_M = 0$, and so $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathscr{R}^1_1(M) = \mathbb{C}^2$. Clearly, the manifold M is formal. ◇

Example 7.6. Let M be the Heisenberg 3-dimensional nilmanifold; then $\Delta_M = 1$ and $\mu_M = 0$, and so $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(M)) = \{\mathbf{0}\}$, whereas $\mathcal{R}^1_1(M) = \mathbb{C}^2$. The manifold M admits a finite model, namely, $A = \bigwedge(a,b,c)$ with $\mathrm{d}\,a = \mathrm{d}\,b = 0$ and $\mathrm{d}\,c = ab$, but M is not 1-formal. ◇

Example 7.7. Consider the symmetric Laurent polynomial $\lambda = (t_1+t_2)(t_1t_2+1)-4t_1t_2$. By the discussion from §6.2, there is a closed, orientable 3-manifold M with $H_1(M,\mathbb{Z}) = \mathbb{Z}^2$ and $\Delta_M = \lambda$. It is readily seen that $\tau_{\mathbf{1}}(\mathcal{V}^1_1(M)) = \{\mathbf{0}\}$, which is a proper subset of $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(M)) = \{x_1^2 + x_2^2 = 0\}$. Note that the latter variety decomposes as the union of two lines defined over $\mathbb{C}$, but not over $\mathbb{Q}$; hence, M admits no 1-finite 1-model. ◇

Now consider the case when $n = b_1(M)$ is odd and at least 3, and μ_M is not generic, a case which is not covered by Theorem 7.3. In this situation, $\mathcal{R}^1_1(M) = H^1(M,\mathbb{C})$, by Theorem 5.6, while the equality $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$ may or may not hold. For instance, if M is the connected sum of n copies of $S^1 \times S^2$, then $\mu_M = 0$ is not generic, yet the aforementioned equality holds (see Corollary 8.3 below for a more general instance of this phenomenon). On the other hand, as we shall see in Example 9.7, there are 3-manifolds M with $n = 15, 21, 45, 55, 91, \dots$ for which μ_M is not generic, while $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(M))$ is a proper subset of $\mathcal{R}^1_1(M)$.

8. Connected sums

Let $M = M_1 \# M_2$ be the connected sum of two closed, orientable manifolds of dimension $m \geqslant 3$. By the van Kampen theorem, the fundamental group of M splits as a free product, $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$, from which we get a direct product decomposition of the corresponding character tori,

$$\mathrm{Char}(\pi_1(M)) = \mathrm{Char}(\pi_1(M_1)) \times \mathrm{Char}(\pi_1(M_2)). \tag{8.1}$$

Likewise, we have that $H^1(M,\mathbb{C}) = H^1(M_1,\mathbb{C}) \times H^1(M_2,\mathbb{C})$. The next result describes the behavior of the cohomology jump loci under these decompositions.

Theorem 8.1. *Let $M = M_1 \# M_2$ be the connected sum of two closed, orientable, smooth m-manifolds, $m \geqslant 3$. Then, for $i = 1$ or $m-1$ and for all $k \geqslant 0$,*

$$\mathcal{V}^i_k(M) = \bigcup_{r+s=k-1} \mathcal{V}^i_r(M_1) \times \mathcal{V}^i_s(M_2), \qquad \mathcal{R}^i_k(M) = \bigcup_{r+s=k-1} \mathcal{R}^i_r(M_1) \times \mathcal{R}^i_s(M_2),$$

while, for $1 < i < m$,

$$\mathcal{V}^i_k(M) = \bigcup_{r+s=k} \mathcal{V}^i_r(M_1) \times \mathcal{V}^i_s(M_2), \qquad \mathcal{R}^i_k(M) = \bigcup_{r+s=k} \mathcal{R}^i_r(M_1) \times \mathcal{R}^i_s(M_2).$$

Proof. The claims involving resonance varieties are proved in [45, Proposition 5.4]. A completely similar proof works for the characteristic varieties. □

Staying with the same notation, we obtain the following corollary regarding the compatibility of the Tangent Cone formula (at least of its second half) with respect to connected sums.

Corollary 8.2. *Suppose that* $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_s(M_j)) = \mathscr{R}^i_s(M_j)$ *for* $j = 1,2$, *in some fixed degree* $0 < i < m$, *and in depths* $s < k$ *if* $i = 1$ *or* $m-1$, *or* $s \leqslant k$ *otherwise. Then* $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^i_k(M_1 \# M_2)) = \mathscr{R}^i_k(M_1 \# M_2)$.

In degree $i = 1$ and depth $k = 1$, Theorem 8.1 yields another corollary, the conclusions of which can also be deduced from [12, Lemma 9.8] and [33, Lemma 5.2], respectively.

Corollary 8.3. *Let* $M = M_1 \# M_2$ *be the connected sum of two closed, orientable, smooth* m*-manifolds* ($m \geqslant 3$) *with* $b_1(M_1)$ *and* $b_1(M_2)$ *both non-zero. Then* $\mathscr{V}^1_1(M) = H^1(M,\mathbb{C}^*)$ *and* $\mathscr{R}^1_1(M) = H^1(M,\mathbb{C})$.

In particular, the full Tangent Cone formula in this degree and depth, $\tau_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(M)) = \mathscr{R}^1_1(M)$, holds for manifolds which admit a connected sum decomposition as above. Combining this corollary with Lemma 3.1, part (1), we obtain the following—presumably well-known—application.

Corollary 8.4. *Let* $M = M_1 \# M_2$ *be the connected sum of two closed, smooth* m*-manifolds* ($m \geqslant 3$) *with non-zero first Betti number. Then* $\Delta_M = 0$.

A classical theorem of J. Milnor insures that every closed, orientable 3-manifold decomposes as the connected sum of finitely many irreducible 3-manifolds. Since every 3-manifold is smooth, Theorem 8.1 reduces the computation of the cohomology jump loci of arbitrary closed, orientable 3-manifolds to that of irreducible ones.

9. Graph manifolds

In this section we study in more detail the cohomology jump loci and the formality properties of certain classes of graph manifolds. We start with a look at the Seifert fibered spaces, which are the basic building blocks for such manifolds.

9.1. **Seifert manifolds.**

A compact 3-manifold is a Seifert fibered space if and only if it is foliated by circles. One can think of such a manifold M as a bundle in the category of orbifolds, in which the circles of the foliation are the fibers, and the base space of the orbifold bundle is the quotient space of M obtained by identifying each circle to a point. We refer to [42] as a general reference for the subject.

For our purposes here, we will only consider closed, orientable Seifert manifolds with orientable base. Every such manifold M admits an effective circle action, with orbit space a Riemann surface Σ_g, and finitely many exceptional orbits, encoded in pairs of coprime integers $(\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)$ with $\alpha_j \geqslant 2$. The fundamental group $\pi = \pi_1(M)$ admits a presentation of the form

$$\begin{aligned}\pi = \big\langle x_1,y_1,\dots,x_g,y_g,z_1,\dots,z_s,h \mid\ & h \text{ central}, \\ & [x_1,y_1]\cdots[x_g,y_g]z_1\cdots z_s = h^b,\ \ z_1^{\alpha_1}h^{\beta_1} = \cdots = z_s^{\alpha_s}h^{\beta_s} = 1\big\rangle,\end{aligned} \tag{9.1}$$

where the integer b encodes the obstruction to trivializing the bundle $p\colon M \to \Sigma_g$ outside tubular neighborhoods of the exceptional orbits.

Let $e = -\big(b + \sum_{i=1}^{s} \beta_i/\alpha_i\big)$ be the Euler number of the orbifold bundle. If $g = 0$, then $b_1(M) = 0$ or 1, according to whether $e \neq 0$ or 0; therefore, by Theorem 7.4(1), the manifold M is formal. So let us assume that $g > 0$. Then M admits a finite-dimensional model, $A = \big(H^{\bullet}(\Sigma;\mathbb{Q}) \otimes_{\mathbb{Q}} \bigwedge(c), \mathrm{d}\,\big)$, where $\deg c = 1$ and the differential d is defined as follows: $\mathrm{d} = 0$ on $H^{\bullet}(\Sigma;\mathbb{Q})$, while $\mathrm{d}\,c = 0$ if $e = 0$ and $\mathrm{d}\,c = \omega$, where $\omega \in H^2(\Sigma;\mathbb{Q})$ is the orientation class, otherwise. As shown in [39, 46], the Malcev Lie algebra $\mathfrak{m}(\pi)$ is the LCS completion of graded algebra with relations in degrees 2 and 3; furthermore, π is 1-formal if and only if $e = 0$.

The simplest Seifert manifold with $e = 0$ is the product $M = \Sigma_g \times S^1$ ($g \geqslant 1$) from Example 5.5; in this case, $\mathcal{V}^1_k(M) = \{t \in (\mathbb{C}^*)^{2g+1} \mid t_{2g+1} = 1\}$ for $1 \leqslant k \leqslant 2g-2$ and $\mathcal{V}^1_{2g-1}(M) = \{\mathbf{1}\}$. On the other hand, if $e \neq 0$, then, as shown in [39], the morphism $p^*\colon H^1(\Sigma_g,\mathbb{C}^*) \to H^1(M,\mathbb{C}^*)$ induced by the orbit map $p\colon M \to \Sigma_g$ defines an isomorphism of analytic germs, $\mathcal{V}^1_k(\Sigma_g)_{(\mathbf{1})} \cong \mathcal{V}^1_k(M)_{(\mathbf{1})}$, for each $k \geqslant 0$.

On the other hand, if $H_1(M,\mathbb{Z})$ has torsion, then the corresponding connected components of $H^1(M,\mathbb{C}^*)$ may contain irreducible components of $\mathcal{V}^1_1(M)$ which do not pass through $\mathbf{1}$. Here is a concrete such example, extracted from [11, 47].

Example 9.1. Consider the Brieskorn manifold $M = \Sigma(2,4,8)$. Then $H_1(M,\mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}_4$, and so $\mathrm{Char}(M) = (\mathbb{C}^*)^2 \times \{\pm 1, \pm i\}$. Direct computation shows that $\Delta_M = 1$, and so $\mathcal{Z}^1_1(M) = \{\mathbf{1}\}$, whereas $\mathcal{V}^1_1(M) = \{\mathbf{1}\} \cup (\mathbb{C}^*)^2 \times \{-1\}$. $\diamond$

9.2. **Tree graph-manifolds.** Every irreducible closed, orientable 3-manifold M admits a JSJ decomposition along incompressible tori. That is to say, there is a finite collection of subtori T with product neighborhood $N(T)$, such that each connected component of $M \backslash N(T)$ is irreducible. A closed, orientable 3-manifold M is a *graph-manifold* if its JSJ decomposition consists only of Seifert fibered pieces. Associated to such a manifold there is a graph $\Gamma = (V, E)$ with a vertex v for each component M_v of $M \backslash N(T)$, and with an edge $e = \{v, w\}$ whenever M_v and M_w are glued along a torus T_e from T.

In [14], Doig and Horn provide an algorithm for computing the rational cohomology ring of a closed, orientable graph manifold M. For instance, if M is a *tree graph-manifold* (that is, the underlying graph Γ is a tree), and all closed-up base surfaces Σ_v are orientable, then

$$H^{\bullet}(M;\mathbb{Q}) \cong \mathop{\#}_{v\in V} H^{\bullet}(\Sigma_v \times S^1;\mathbb{Q}). \tag{9.2}$$

Thus, if we let g_v be the genus of Σ_v, the intersection form of M can be written, in a suitable basis for $H^1(M,\mathbb{Z})$, as

$$\mu_M = \sum_{v\in V} \sum_{i=1}^{g_v} a_{v,i} b_{v,i} c_v\,. \tag{9.3}$$

Proposition 9.2. *Let M be a tree graph-manifold with orientable base surfaces. Then the resonance varieties $\mathscr{R}^i_k(M)$ are either empty, or equal to $H^1(M,\mathbb{C})$, or are finite unions of coordinate subspaces in $H^1(M,\mathbb{C})$.*

Proof. By the computation from Example 5.5, we know that the claim is true when $M = \Sigma_g \times S^1$. The general case follows at once from (9.2) and Theorem 8.1. ◻

Example 9.3. The main result of [14] is Theorem 6.1, which states that not every closed 3-manifold is homology cobordant to a tree graph manifold. The proof reduces to showing that the intersection forms $\mu = e_1e_2e_3 + e_1e_5e_6 + e_2e_4e_5$ and $\mu' = e_1e_2e_3 + e_4e_5e_6$ are not equivalent, up to a change of basis in $\mathrm{GL}(6,\mathbb{Q})$. This is done in [14, Theorem 6.4] by a rather long argument; here is a much shorter proof of this fact.

A computation recorded in [45] shows that $\mathscr{R}^1_2(\mu) = \{x_1 = x_2 = x_5 = 0\}$; on the other hand, by Theorem 8.1, $\mathscr{R}^1_2(\mu') = \{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}$. Thus, the respective resonance varieties are not isomorphic, and hence the two intersection forms are not equivalent (over $\mathbb{Q}$). ◇

9.3. **Boundary manifolds of line arrangements.** Let $\mathscr{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of projective lines in $\mathbb{CP}^2$. We associate to $\mathscr{A}$ a graph $\Gamma = (V,E)$, with vertex set $V = \mathscr{A} \cup \mathscr{P}$, where $\mathscr{P}$ are the points $P_J = \bigcap_{j\in J} \ell_j$ where three or more lines intersect. The graph Γ has an edge from ℓ_i to ℓ_j if those lines are transverse, and an edge from a multiple point P to each line ℓ_i on which it lies.

Now let $M = M(\mathscr{A})$ be the boundary of a regular neighborhood of $\mathscr{A}$. Then M is a closed, orientable graph manifold, with underlying graph Γ; the vertex manifolds M_v are of the form $S^1 \times S_v$, where S_v is the 2-sphere with $\deg(v)$ open disks removed, and all the gluing maps are flips, i.e., diffeomorphisms of the boundary tori given by the matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

For instance, if $\mathscr{A}$ is a pencil of lines defined by $\{z_1^{n+1} - z_2^{n+1} = 0\}$, then $M = \#^n S^1 \times S^2$, whereas if $\mathscr{A}$ is a near-pencil defined by $\{z_0(z_1^n - z_2^n) = 0\}$, then $M = S^1 \times \Sigma_{n-1}$.

The group $H_1(M,\mathbb{Z})$ is free abelian, of rank equal to $n + b_1(\Gamma)$. We fix a basis for $H^1(M,\mathbb{Z})$, consisting of classes e_i dual to the meridians of the lines $\ell_1, \dots, \ell_n$, as well as classes $f_{i,j}$ dual to the cycles in the graph. The latter classes are indexed by the set B of pairs (i, j) with $i < j$ for which either $\ell_i \pitchfork \ell_j$, or $i = \min J$ and $j \in J \backslash \{i\}$, where $P_J \in \mathscr{P}$. As shown in [8], the intersection 3-form of M may then be written as

$$\mu_M = \sum_{(i,j)\in B} e_{I(i,j)} e_j f_{i,j}\,, \tag{9.4}$$

where $I(i, j) = \{k \in [n] \mid \ell_i \cap \ell_j \in \ell_k\}$ and $e_J = \sum_{k\in J} e_k$.

Theorem 9.4 ([7, 8])**.** *If $n \geqslant 2$ and $\mathscr{A}$ is not a near-pencil, then $\mathscr{R}^1_1(M) = H^1(M,\mathbb{C})$.*

In depth $k > 1$, though, the resonance varieties $\mathscr{R}^1_k(M)$ may have non-linear irreducible components.

The next result expresses the Alexander polynomial and the first characteristic variety of the boundary manifold $M = M(\mathscr{A})$ in terms of the underlying graph $\Gamma = (V,E)$.

Theorem 9.5 ([8]). *If $\mathscr{A}$ is not a pencil, then*

(1) $\Delta_M = \prod_{v\in V}(t_v - 1)^{\deg(v)-2} \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, *where* $t_v = \prod_{i\in v} t_i$ *and* $t_0 \cdots t_n = 1$.

(2) $\mathscr{V}_1^1(M) = \bigcup_{v\in V\,:\,\deg(v)\geqslant 3} \{t_v - 1 = 0\}$.

Putting now together Corollary 4.12 with the above two theorems easily implies the next result.

Corollary 9.6 ([8]). *For the boundary manifold M of a line arrangement $\mathscr{A}$ the following conditions are equivalent:*

(1) *M is formal.*
(2) *M is 1-formal.*
(3) $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}_1^1(M)) = \mathscr{R}_1^1(M)$.
(4) *$\mathscr{A}$ is either a pencil or a near-pencil.*

Example 9.7. Let $\mathscr{A}$ be an arrangement of $n+1 \geqslant 4$ lines in general position in $\mathbb{CP}^2$. Then $\mu_M = \sum_{1\leqslant i<j\leqslant n} e_i e_j f_{i,j}$ and $\mathscr{R}_1^1(M) = H^1(M,\mathbb{C})$ properly contains the tangent cone at $\mathbf{1}$ to $\mathscr{V}_1^1(M) = \{\Delta_M = 0\}$, where $\Delta_M = [(t_1 - 1)\cdots(t_n - 1)(t_1\cdots t_n - 1)]^{n-2}$.

Note that $b_1(M) = \binom{n+1}{2}$, which is an odd integer if $n \equiv 1$ or $2 \bmod 4$. In this case, the fact that $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}_1^1(M)) \subsetneqq \mathscr{R}_1^1(M)$ together with Theorem 7.3(1) imply that μ_M is not generic. $\diamond$

10. Links in the 3-sphere

Finally, we analyze the Tangent Cone theorem in the setting of knots and links, where the Alexander polynomial originated from.

10.1. **Cohomology ring and resonance varieties.** A link in S^3 is a finite collection, $L = \{L_1, \dots, L_n\}$, of disjoint, smoothly embedded circles in the 3-sphere. Let $M = S^3 \setminus \bigcup_{i=1}^n N(L_i)$ be the link exterior, i.e., the complement of an open tubular neighborhood of L. Then M is a compact, connected, orientable 3-manifold, with boundary ∂M consisting of n disjoint tori. Furthermore, M is homotopy equivalent to the link complement, $X = S^3 \setminus \bigcup_{i=1}^n L_i$.

Picking orientations on the link components yields a preferred basis for $H_1(X,\mathbb{Z}) = \mathbb{Z}^n$ consisting of oriented meridians; let $\{e_1, \dots, e_n\}$ be the Kronecker dual basis for $H^1(X,\mathbb{Z})$. For each $i \neq j$, choose arcs in X connecting L_i to L_j, and let $b_{i,j} \in H^2(X,\mathbb{Z})$ be their Poincaré–Lefschetz duals. Furthermore, let $\ell_{i,j} = \mathrm{lk}(L_i, L_j)$ be the linking number of those two components (as is well-known, $\ell_{i,j} = \ell_{j,i}$). The cohomology ring $H^{\bullet}(X,\mathbb{Z})$, then, is the quotient of the exterior algebra on generators e_i and $b_{i,j}$, truncated in degrees 3 and higher, modulo the ideal generated by the relations

$$e_i e_j = \ell_{i,j} b_{i,j} \text{ and } b_{i,j} + b_{j,k} + b_{k,i} = 0. \tag{10.1}$$

In particular, we may choose $\{b_{1,n}, \dots, b_{n-1,n}\}$ as a basis for $H^2(X,\mathbb{Z}) = \mathbb{Z}^{n-1}$.

Set $A = H^\bullet(X,\mathbb{C})$ and $S = \mathbb{C}[x_1,\dots,x_n]$, and consider the chain complex from (2.5). The S-linear map $\delta^1\colon A^1\otimes S \to A^2\otimes S$ is given by $\delta^1(e_i) = -\sum_{j=1}^n \ell_{i,j} b_{i,j}\otimes x_j$. Rewriting in the chosen basis for A^2, we find that the transpose matrix, $\partial_2\colon S^{n-1}\to S^n$, has entries

$$(\partial_2)_{i,j} = \ell_{i,j}x_i - \delta_{i,j}\left(\sum_{k=1}^n \ell_{i,k}x_k\right). \tag{10.2}$$

The degree 1 resonance varieties of the link complement, then, are the vanishing loci of the codimension k minors of this matrix: $\mathscr{R}^1_k(X) = V(E_k(\partial_2)) \subseteq \mathbb{C}^n$.

Example 10.1. If all the linking numbers are equal to ± 1, then the cohomology ring is the exterior algebra on $e_1,\dots,e_n$ modulo the relations $\ell_{i,j}e_ie_j + \ell_{j,k}e_je_k + \ell_{k,i}e_ke_i = 0$. In the special case when all $\ell_{i,j}$ are equal to 1, we conclude that $\mathscr{R}^1_1(X) = \{\mathbf{0}\}$ if $n = 2$ and $\mathscr{R}^1_1(X) = \{\sum_{i=1}^n x_i = 0\}$ if $n > 2$. On the other hand, if some $\ell_{i,j} = -1$, then the variety $\mathscr{R}^1_1(X)$ can be quite complicated, as shown in several examples from [29, §6]. $\diamond$

10.2. **Characteristic varieties.** Let $\pi = \pi_1(X)$ be the fundamental group of a link complement. Using the preferred meridian basis for $H_1(X,\mathbb{Z}) = \mathbb{Z}^n$, we may identify the group ring $\mathbb{Z}[\mathbb{Z}^n]$ with the ring of Laurent polynomials $\mathbb{Z}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$, and view the Alexander polynomial of the link, $\Delta_L = \Delta_X$, as an element in this ring. Likewise, we may also identify the character group $\mathrm{Char}(X)$ with the algebraic torus $(\mathbb{C}^*)^n$. The depth 1 characteristic variety of the complement, $\mathscr{V}^1_1(X) = \mathscr{Z}^1_1(X)$, is a subvariety of $(\mathbb{C}^*)^n$ determined by the Alexander polynomial, as follows.

First suppose that L is a knot, that is, a 1-component link. Then the polynomial $\Delta_L \in \mathbb{Z}[t^{\pm 1}]$ satisfies $\Delta_L(1) = \pm 1$ and $\Delta_L(t^{-1}) \doteq \Delta_L(t)$. In fact, every Laurent polynomial satisfying these two conditions occurs as the Alexander polynomial of a knot. By definition, the Alexander variety $\mathscr{W}^1_1(X) \subset \mathbb{C}^*$ is the set of roots of Δ_L; in particular, $1 \notin \mathscr{W}^1_1(X)$. On the other hand, $\mathscr{V}^1_1(X)$ consists of all those roots, together with 1.

Now suppose that the link L has at least two components. Work of Eisenbud and Neumann [17] shows that the first Alexander ideal, $E_1(A_X)$, is equal to $I\cdot(\Delta_L)$, where I is the augmentation ideal of $\mathbb{Z}[\mathbb{Z}^n]$. Hence, by Proposition 3.2,

$$\mathscr{V}^1_1(X) = \{z \in (\mathbb{C}^*)^n \mid \Delta_L(z) = 0\} \cup \{\mathbf{1}\}. \tag{10.3}$$

As before, set $\widetilde{\Delta}_L(z_1,\dots,z_n) = \Delta_L(z_1+1,\dots,z_n+1)$. The tangent cone to the characteristic variety is then given by the following formula:

$$\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \begin{cases} V(\mathrm{in}(\widetilde{\Delta}_L)) & \text{if } \Delta_L(\mathbf{1}) = 0,\\ \{\mathbf{0}\} & \text{otherwise.}\end{cases} \tag{10.4}$$

10.3. **Formality.** The link complement X has the homotopy type of a 2-complex; thus, X is formal if and only if it is 1-formal. For a geometrically defined class of links (which includes the Hopf links of arbitrarily many components) formality holds.

Example 10.2. Suppose L is an *algebraic* link, that is, $\bigcup_{i=1}^n L_i$ is the intersection of a complex plane algebraic curve having an isolated singularity at a point p with a small 3-sphere centered at p. Then, as shown in [15, Theorem 4.2], the complement X is a formal space.

In general, though, link complements are far from being formal, and, in fact, may even fail to admit a 1-finite 1-model. Their non-formality has been traditionally detected by higher-order Massey products. As we shall see below, the resonance varieties and the Tangent Cone theorem provide an efficient, alternative way to ascertain the non-formality or the non-existence of finite models for link complements.

10.4. **Two-component links.** To start with, let $L = \{L_1, L_2\}$ be a 2-component link, and set $\ell = \operatorname{lk}(L_1, L_2)$. The Alexander polynomial $\Delta_L(t_1, t_2)$, then, satisfies the following formula due to Torres [50]:

$$\Delta_L(t, 1) = \frac{t^\ell - 1}{t - 1}\Delta_{L_1}(t), \tag{10.5}$$

and analogously for $\Delta_L(1, t)$. Using now the fact that the Alexander polynomial of a knot evaluates to ± 1 at 1, we see that

$$\Delta_L(1, 1) = \pm\ell. \tag{10.6}$$

Theorem 10.3. *Let $L = \{L_1, L_2\}$ be a 2-component link, with complement X. The following statements are equivalent:*

(1) *The space X is formal.*
(2) *The space X is 1-formal.*
(3) *The tangent cone formula $\tau_{\mathbf{1}}(\mathscr{V}_1^1(X)) = \operatorname{TC}_{\mathbf{1}}(\mathscr{V}_1^1(X)) = \mathscr{R}_1^1(X)$ holds.*
(4) *The linking number $\ell = \operatorname{lk}(L_1, L_2)$ is non-zero.*

Proof. As explained previously, implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold for arbitrary finite, connected CW-complexes.

To prove (3) $\Rightarrow$ (4), suppose that $\ell = 0$. Then of course $\mathscr{R}_1^1(X) = \mathbb{C}^2$. On the other hand, the variety $V(\Delta_L)$ is an algebraic curve in $(\mathbb{C}^*)^2$. By (10.6), this curve passes through $\mathbf{1}$, and thus, by (10.3), it coincides with $\mathscr{V}_1^1(X)$. By (10.4), the tangent cone to this variety is the algebraic curve in $\mathbb{C}^2$ defined by the ideal $\operatorname{in}(\widetilde{\Delta}_L)$; in particular, $\operatorname{TC}_{\mathbf{1}}(\mathscr{V}_1^1(X))$ is properly contained in $\mathscr{R}_1^1(X)$.

Finally, to prove (4) $\Rightarrow$ (1), suppose that $\ell \neq 0$. Then $H^{\boldsymbol{\cdot}}(X, \mathbb{Q}) \cong H^*(T^2, \mathbb{Q})$, and so X is formal. $\square$

We illustrate this theorem and related phenomena with several examples. The links which appear in these examples are numbered c_r^n, where c is the crossing number, n is the number of components, and r is the index from Rolfsen's tables [41].

Example 10.4. Let L be the 4_1^2 link. This is a 2-component link with linking number 2 and Alexander polynomial $\Delta_L = t_1 + t_2$. By Theorem 10.3, the complement X is formal, and the tangent cone formula (4.6) holds. Nevertheless, the variety $\mathcal{V}^1_1(X)$ has an irreducible component (the translated subtorus $t_1t_2^{-1} = -1$), which is not detected by the resonance variety $\mathcal{R}^1_1(X) = \{\mathbf{0}\}$. ◇

Example 10.5. Let L be the 6^2_3 link. This is a 2-component link with linking number 2 and Alexander polynomial $\Delta_L = t_1t_2 - 2(t_1 + t_2) + 1$. Again, the complement X is formal, yet $\mathcal{V}^1_1(X)$ has an irreducible component (not passing through $\mathbf{1}$) which this time is not a translated algebraic subtorus of $(\mathbb{C}^*)^2$. ◇

Example 10.6. Let L be the 5^2_1 link, also known as the Whitehead link. This 2-component link has linking number 0; thus, $\mathcal{R}^1_1(X) = \mathbb{C}^2$ and, by Theorem 10.3, X is not formal. On the other hand, $\Delta_L = (t_1 - 1)(t_2 - 1)$, and so $\mathcal{V}^1_1(X) \subset (\mathbb{C}^*)^2$ consists of the two coordinate subtori, $\{t_1 = 1\}$ and $\{t_2 = 1\}$. Consequently $\tau_{\mathbf{1}}(\mathcal{V}^1_1(X)) = \mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(X)) = \{x_1 = 0\} \cup \{x_2 = 0\}$, and so this computation leaves open the question whether X admits a 1-finite 1-model. ◇

10.5. **Links of many components.** We conclude this section with a discussion of links having 3 or more components. In the first example, $\mathcal{R}^1_1(X)$ is linear, yet it strictly contains $\tau_{\mathbf{1}}(\mathcal{V}^1_1(X))$.

Example 10.7. Let L be the 6^3_1 link. Then $\mathcal{V}^1_1(X)$ is the subvariety of $(\mathbb{C}^*)^3$ defined by the polynomial $t_1t_2 + t_1t_3 + t_2t_3 - t_1 - t_2 - t_3$. A quick computation shows that $\tau_1(\mathcal{V}^1_1(X))$ is a union of three lines in $\mathbb{C}^3$, namely, $\{x_1 = x_2 + x_3 = 0\}$, $\{x_2 = x_1 + x_3 = 0\}$, and $\{x_3 = x_1 + x_2 = 0\}$. On the other hand, $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(X)) = \mathcal{R}^1_1(X)$ is the plane defined by the equation $x_1 + x_2 + x_3 = 0$. By Theorem 4.10, X admits no 1-finite 1-model. ◇

In the next example, the resonance variety $\mathcal{R}^1_1(X)$ is non-linear.

Example 10.8. Let L be the 8^4_2 link. Then $\mathcal{V}^1_1(X) \subset (\mathbb{C}^*)^4$ is the zero locus of the polynomial $t_1t_2t_3t_4 - t_1t_2t_4 - t_1t_3t_4 + t_1t_3 + t_2t_4 - t_2 - t_3 + 1$. It follows that $\tau_{\mathbf{1}}(\mathcal{V}^1_1(X))$ is a union of eight planes in $\mathbb{C}^4$,

$$\begin{aligned}
&\{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\} \cup \{x_1 = x_3 + x_4 = 0\} \cup \\
&\{x_1 + x_2 = x_4 = 0\} \cup \{x_1 - x_2 = x_3 = 0\} \cup \{x_2 = x_3 - x_4 = 0\} \cup \\
&\{x_1 - x_2 + 2x_3 = x_2 - x_3 + x_4 = 0\} \cup \{x_1 - x_2 + x_3 = 2x_2 - x_3 + x_4 = 0\}.
\end{aligned}$$

On the other hand, $\mathrm{TC}_{\mathbf{1}}(\mathcal{V}^1_1(X)) = \mathcal{R}^1_1(X)$ is an irreducible quadric, given by the equation $(x_1 + x_2)x_3 = (x_1 - x_2)x_4$. Hence, once again, X admits no 1-finite 1-model. ◇

An interesting class of examples, studied in detail in [29, 30], consists of the singularity links of arrangements of transverse planes in $\mathbb{R}^4$. By intersecting such an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ with a 3-sphere about $\mathbf{0}$, we obtain a link L of n great circles in S^3. The link complement X is aspherical, and its fundamental group is a semidirect product of free groups, $\pi = F_{n-1} \rtimes \mathbb{Z}$. If $\mathcal{A}$ is defined by complex equations, then L is the Hopf link, and X is formal; in general, though, things are much more complicated.

Example 10.9. Consider the arrangement $\mathscr{A} = \mathscr{A}(31425)$ defined in complex coordinates by the function $Q(z,w) = z(z-w)(z-2w)(2z+3w-5\overline{w})(2z-w-5\overline{w})$. The cohomology jump loci of the corresponding link complement X were computed in [29, Example 6.5] and [30, Example 10.2]. As noted in [12, Example 8.2], we have that $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_2(X)) \subsetneqq \mathscr{R}^1_2(X)$, and so X is not 1-formal. In fact, $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \mathscr{R}^1_1(X)$, yet $\tau_{\mathbf{1}}(\mathscr{V}^1_1(X)) \subsetneqq \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(X))$, and so X admits no 1-finite 1-model.

These examples (and many others) lead to the following problem regarding the range of applicability of the tangent cone formula in the setting of classical links.

Problem 10.10. Given a link complement X, determine which (if any) of the following equalities is true.

(1) $\tau_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(X))$.
(2) $\mathrm{TC}_{\mathbf{1}}(\mathscr{V}^1_1(X)) = \mathscr{R}^1_1(X)$.

Does the complement need to admit a 1-finite 1-model for the first equality to hold? Does it need to be formal for both equalities to hold?

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