Polyhedral products

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I Construction

- Input:
  - K, a simplicial complex on $\{1,\dots,n\}$
  - (X,A), a pair of spaces
- Output: $$Z_K(X,A) := \bigcup_{\sigma\in K} (X,A)^\sigma \subset X^{\times n}$$

$$(X,A)^\sigma = \{x\in X^{\times n} : x_i \in A \text{ if } i\notin\sigma\}$$

— the "polyhedral product" of (X,A) interpolates between

$$Z_\emptyset(X,A) = Z_K(A,A) = A^{\times n}$$

and

$$Z_{\Delta^{n-1}}(X,A) = Z_K(X,X) = X^{\times n}$$

- Particular case:

$$Z_K(X) := Z_K(X,*) \qquad * = \text{basepoint}$$

eg:

- $Z_{\{n \text{ points}\}}(X) = \overset{n}{\vee} X$ wedge
- $Z_{\partial(\Delta^{n-1})}(X) = T^n(X)$ fat wedge

II Properties

(a) $L \subset K$ subcomplex $\rightsquigarrow Z_L(X,A) \subset Z_K(X,A)$

(b) (X,A) (finite) CW-pair $\rightsquigarrow Z_K(X,A)$ (finite) CW-complex

(c) $Z_{K*L}(X,A) = Z_K(X,A) \times Z_L(X,A)$

(d) $f:(X,A)\longrightarrow(Y,B)$ continuous $\rightsquigarrow$

$$Z^f: \begin{tikzcd} Z_K(X,A) \arrow[r] \arrow[d, phantom, "\cap"] & Z_K(Y,B) \arrow[d, phantom, "\cap"] \\ X^{\times n} \arrow[r, "f^{\times n}"] & Y^{\times n} \end{tikzcd}$$

hence: $(X,A)\simeq(Y,B) \Longrightarrow Z_K(X,A)\simeq Z_K(Y,B)$

(e) If X comm. topological monoid & A submonoid then [Strickland]: (eg: $(X,A)=(D^2,S^1)$)

$f:K\longrightarrow L$ simplicial $\rightsquigarrow$ $Z_f: Z_K(X,A)\longrightarrow Z_L(X,A)$

III Fibrations

__Theorem__ (Denham–S. PAMQ 07)

Let
$$\begin{tikzcd} F \arrow[r] & E \arrow[r] & B \\ F' \arrow[r] \arrow[u, phantom, "\cup"] & E' \arrow[r] \arrow[u, phantom, "\cup"] & B' \arrow[u, phantom, "\cup"] \end{tikzcd}$$
fibrations (or G-bundles) with $F=F'$ or $B=B'$

Then $Z_K(F,F')\longrightarrow Z_K(E,E')\longrightarrow Z_K(B,B')$

is also a fibration (or G-bundle)

__Example__

$$\begin{tikzcd} G \arrow[r] & EG \arrow[r] & BG \\ G \arrow[r] \arrow[u, phantom, "\cup"] & G \arrow[r] \arrow[u, phantom, "\cup"] & * \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\overset{\rightsquigarrow}{\text{thm}}$ $G^{\times n}\longrightarrow Z_K(EG,G)\longrightarrow Z_K(BG)$

Furthermore:

__Lemma__ The homotopy fiber of $Z_K(BG)\hookrightarrow BG^{\times n}$ is $Z_K(EG,G)$.

IV Manifolds

Prop (Denham–S. in preparation)

Let $(M, \partial M)$ be a d-dimensional compact manifold and K a PL-triangulation of S^m on n vertices [K is PL-homeo to $\partial\Delta^{m+1}$]

Then: $Z_K(M, \partial M)$ is a compact manifold of dimension $= (d-1)n + m + 1$

Also, a similar result for PL-triangulations of D^{m+1}

Examples: $\cdot\ M_K = Z_K([0,1], \{0,1\})$ has $\dim = m+1$

$\cdot\ Z_K = Z_K(D^2, S^1)$ has $\dim = n+m+1$

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Theorem (Bosio–Meersseman, Acta 06)

Let K be a polytopal triangulation of S^m [K = boundary complex of a convex polytope of $\dim = m+1$]

Then:

- Z_K if $n+m+1$ even
- $Z_K \times S^1$ if $n+m+1$ odd

admits a complex structure.

- This generalizes classical constructions of Hopf ($S^1 \times S^{2n+1}$) Calabi–Eckmann ($S^k \times S^\ell$, k, ℓ odd), and Lopez de Medrano–Verjovsky.

- These compact, complex manifolds are almost never Kähler (as B–M note). In fact, we show that, quite often, they are not even formal.

V Toric complexes

$T_K := Z_K(S^1)$ a subcomplex of $T^n = (S^1)^{\times n}$ (with product cell structure from $S^1 = e^0 \cup e^1$)

- Let $C_*(T_K)$ be the cellular chain complex
  Since $T_K \subset T^n$, all $\partial_k : C_k \to C_{k-1}$ vanish
  Since k cells of $T_K \longleftrightarrow (k-1)$ simplices of K,

$$H_k(T_K) = C_{k-1}(K) \cong \mathbb{Z}^{d_k(K)}$$

where $d_k(K) = \#\{\sigma \in K : \underset{\dim\sigma+1}{|\sigma|} = k\}$

- Cohomology ring [Kim–Roush 1980]:

$$H^*(T_K) = \Lambda(x_1,\dots,x_n) \big/ \text{ideal}\langle x_{i_1}\cdots x_{i_r} : \{i_1,\dots,i_r\} \notin K\rangle$$

i.e.,

$$H^*(T_K, \Bbbk) = \Bbbk\langle K\rangle = E/J_K$$

– the exterior Stanley–Reisner ring

In particular, if

$K = \Delta(\Gamma)$ $\qquad \Gamma = K^{(1)}$ 1-skeleton of K
$\qquad\qquad\qquad\quad \Delta(\Gamma)$ flag complex of Γ

then J_K is a quadratic monomial ideal, and so
$H^*(T_{\Delta(\Gamma)}; \Bbbk)$ is a Koszul algebra [Fröberg 1975]

- Fundamental group

$$\pi_1(T_K) = G_\Gamma := \langle v \in V_\Gamma : vw = wv \text{ if } \{v,w\} \in E_\Gamma\rangle$$

right-angled Artin group of Γ

K	T_K	G_Γ
•	S^1	$\mathbb{Z}$
• •	$S^1 \vee S^1$	F_2
•–•	$S^1 \times S^1$	$\mathbb{Z}^2$

K	T_K	G_Γ
• • •	$S^1 \vee S^1 \vee S^1$	F_3
•–• •	$T^2 \vee S^1$	$\mathbb{Z}^2 * \mathbb{Z}$
•–•–•	$S^1 \times (S^1 \vee S^1)$	$\mathbb{Z} \times F_2$

• Classifying space for G_Γ is also a toric complex:

$$K(G_\Gamma, 1) = T_{\Delta(\Gamma)}$$

[Meier–VanWyck, Charney–Davis based on a lemma of Gromov]

In general, though, T_K is not aspherical.

Theorem (Papadima–S. 08) Let Δ be the flag complex of K. Set

$$p := \sup\{k : d_i(\Delta) = d_i(K) \text{ for } i \le k\}$$

Then:

$$T_K \text{ aspherical} \iff K \text{ flag complex} \iff p = \infty$$

Moreover, if $p < \infty$, then:

- $\pi_2(T_K) = \cdots = \pi_{p-1}(T_K) = 0$
- $\pi_p(T_K)$ has (explicit) finite presentation as $\mathbb{Z}G_\Gamma$-module
- $\pi_p(T_K)_{G_\Gamma} = \mathbb{Z}^{d_{p+1}(\Delta) - d_{p+1}(K)} \neq 0$

[The implication $T_K \simeq K(G_\Gamma, 1) \Rightarrow K = \Delta(\Gamma)$ recovers a recent result of Leary and Saadetoğlu]

• Formality

Theorem [Notbohm–Ray AGT 05] $T_K = Z_K(S^1)$ formal, $\forall K$

[A different proof is given in [PS MathAnn 06] in the case when $K = \Delta(\Gamma)$:

- G_Γ is 1-formal [Kapovich–Millson IHES 98]
- $\mathbb{Q}\langle\Delta(\Gamma)\rangle$ Koszul
- [$T_{\Delta(\Gamma)}$ aspherical]

$\Rightarrow T_{\Delta(\Gamma)}$ formal

use [Papadima–Yuzvinsky JPAA 99]]

6)

Lie algebras associated to RAAG's

G finitely generated group $\rightsquigarrow$ graded Lie algebras:

- $gr(G) = \bigoplus_{k\geq 1} G_k/G_{k+1}$ $\qquad G_1 = G, \dots, G_{k+1} = [G, G_k], \dots$
  $\leftarrow$ associated graded Lie algebra
- $gr(G/G'')$ $\leftarrow$ Chen Lie algebra
- $h_G = \mathrm{Lie}(H_1 G)/\mathrm{ideal}(H_2 G \xrightarrow{\Delta} H_1 G \wedge H_1 G)$
  $\uparrow$ holonomy Lie algebra

In general:
- $h_G \twoheadrightarrow gr(G)$ with iso in deg ≤ 3
- $h_G \otimes \mathbb{Q} \xrightarrow{\cong} gr(G) \otimes \mathbb{Q}$ if G is 1-formal [Sullivan]
- $gr(G) \twoheadrightarrow gr(G/G'')$ with iso in deg ≤ 3
- $h_G/h_G'' \twoheadrightarrow gr(G/G'')$
- $h_G \otimes \mathbb{Q}/h_G'' \otimes \mathbb{Q} \xrightarrow{\cong} gr(G/G'') \otimes \mathbb{Q}$ if G is 1-formal [Papadima-S. IMRN 04]

Now let $G = G_\Gamma$ ($\Gamma = (V,E)$). Then:

Theorem (PS – Math Ann 06)

- $gr(G) \cong h_G = \mathrm{Lie}(V) / \langle [v,w] = 0 \text{ if } \{v,w\} \in E \rangle$
  and is torsion-free
- The ranks $\phi_k = \mathrm{rank}\, gr_k(G)$ are given by

$$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k} = P_\Gamma(-t)$$

where $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma)\, t^k$ is clique polynomial

[recovers and sharpens results of Duchamp & Krob – 1992]

7)

Theorem (PS 06) For $G = G_\Gamma$:

- $gr(G/G'') \cong \mathfrak{h}_G/\mathfrak{h}_G''$ and is torsion-free
- The Chen ranks $\theta_k = \operatorname{rank} gr_k(G/G'')$ given by

$$\sum_{k=2}^{\infty} \theta_k(G)t^k = Q_\Gamma\left(t/_{1-t}\right)$$

where $Q_\Gamma(t) = \sum_{j\geq 2} c_j(\Gamma)t^j$ is the "cut polynomial"

$$c_j(\Gamma) = \sum_{\substack{W\subset V \\ |W|=j}} \tilde{b}_0(\Gamma_W)$$

Γ_W: induced subgraph

Proof uses

Thm (Fröberg – Löfwall, HHA 02)

A graded algebra$/\Bbbk$ with:
- A^1 finite-dim
- $a\cdot a = 0$, $\forall a\in A^1$
- $E := \Lambda A^1 \twoheadrightarrow A$ (A gen. in deg 1)

Then:

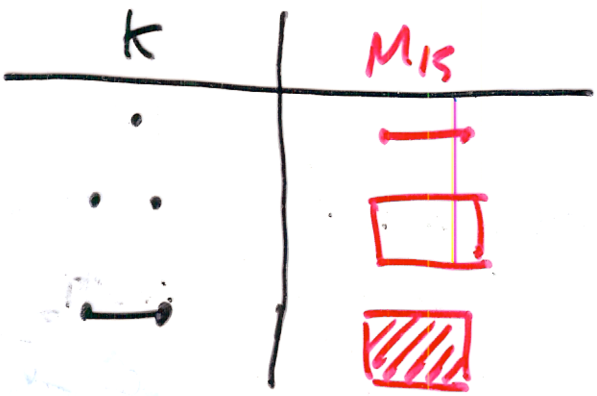
$$\left(\mathfrak{h}_A'/\mathfrak{h}_A''\right)_k \cong \operatorname{Tor}^E_{k-1}(A,\Bbbk)_k \quad , \forall k\geq 2$$

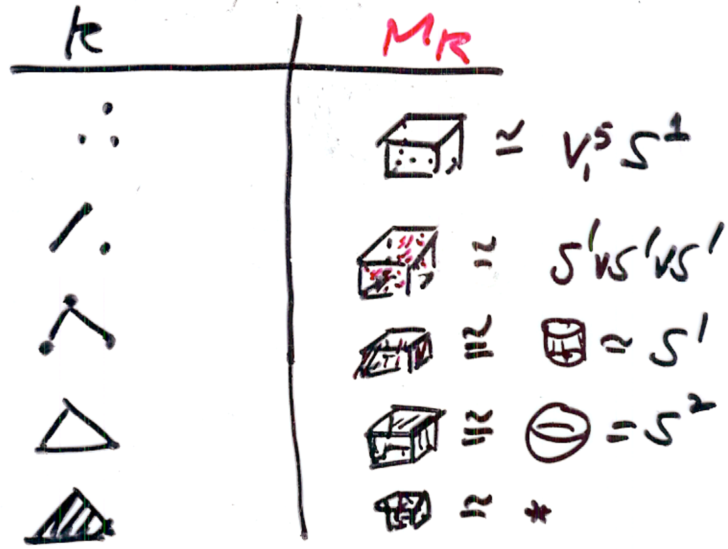
Mirror complexes

$M_K = Z_K(I, \partial I)$ — where $I = [0,1]$, $\partial I = \{0,1\}$

— a subcomplex of n-cube I^n

[at each vertex of I^n, select faces corresponding to $K \subset \Delta^{n-1}$]





- K = n points $\rightsquigarrow$ $M_K = (I^n)^{(1)} \simeq \bigvee_{1+2^{n-1}(n-2)} S^1$
- $K = K_n$ $\rightsquigarrow$ $M_K = (I^n)^{(2)}$
- $K = \partial\Delta^{n-1}$ $\rightsquigarrow$ $M_K = \partial I^n \cong S^{n-1}$
- K = n-gon $\xrightarrow[\text{[Coxeter 1936]}]{}$ M_K = closed, orientable surface of $g = 1 + 2^{n-3}(n-4)$
- K triangulation of S^m $\rightsquigarrow$ M_K is an $(m+1)$-dim manifold

eg:
- $K \cong S^2$ $\rightsquigarrow$ M_K 3-dim mfd
- $K \cong S^3$, $K^{(1)} = K_n$ $\rightsquigarrow$ M_K simply-connected 4-dim mfd

<u>Problem</u>: Classify these manifolds, compute their top invariants directly from K.

Fundamental group

$\Gamma = K^{(1)}$ graph

$$W_\Gamma = \langle v \in V_\Gamma : \underset{v^2=1}{vw = wv} \quad \text{if } \{v,w\} \in E_\Gamma \rangle$$

(right-angled) Coxeter group

$$1 \longrightarrow W_\Gamma' \longrightarrow W_\Gamma \xrightarrow{ab} \mathbb{Z}_2^n \longrightarrow 1$$

commutator subgroup

Then:

$$\pi_1(M_K) = W_\Gamma'$$

Cohomology ring

$$H^*(M_K; \mathbb{Z}_2) = \operatorname{Tor}^S(S/I_K, \mathbb{Z}_2)$$

where $S = \mathbb{Z}_2[x_1, \ldots, x_n]$, $\deg(x_i) = 1$

I_K = Stanley-Reisner ideal

Problem Compute $H^*(M_K; \mathbb{Z})$

Moment-angle complexes

$$Z_K := Z_K(D^2, S^1)$$

a subcomplex of polydisk $(D^2)^{\times n} \cong D^{2n}$, with product cell structure from $D^2 = e^0 \cup e^1 \cup e^2$

e.g.:

- $K = \partial\Delta^{n-1} \rightsquigarrow Z_K = S^{2n-1}$
- $K = \{n \text{ points}\} \rightsquigarrow Z_K = \bigvee_{k=2}^{n} (k-1)\binom{n}{k} S^{k+1}$
- K a "shifted complex" $\rightsquigarrow Z_K \simeq$ wedge of spheres
  [Grbić & Theriault, Top 07]
- K an n-gon $\rightsquigarrow Z_K = \#_{j=1}^{n-3}\, j\binom{n-2}{j+1} S^{j+2} \times S^{n-j}$
  [McGavran, TAMS 79]

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Closely related is the Davis-Januszkiewicz space [DJ, Duke 91]

$$DJ_K = Z_K(BS^1) \subset (\mathbb{CP}^\infty)^{\times n}$$

- Fibrations:

$$\begin{array}{ccccccc}
(S^1)^{\times n} & \longrightarrow & Z_K(ES^1, S^1) & \longrightarrow & Z_K(BS^1) & \hookrightarrow & B(S^1)^{\times n} \\
\| & & \wr\wr & & \| & & \| \\
T^n & \longrightarrow & Z_K & \longrightarrow & DJ_K & \hookrightarrow & (\mathbb{CP}^\infty)^{\times n}
\end{array}$$

Clearly:
- $\pi_1(DJ_K) = 0$
- $\pi_1(Z_K) = \pi_2(Z_K) = 0$
- $\pi_i(Z_K) \cong \pi_i(DJ_K)$ for $i > 2$

- Cohomology ring:

- $H^*(DJ_K, \mathbb{Z}) = S/I_K$ — $S = \mathbb{Z}[x_1, \dots, x_n]$, $\deg x_i = 2$; I_K = Stanley-Reisner ideal of K ← [DJ]

- $H^*(Z_K, \mathbb{Z}) = \mathrm{Tor}^S(S/I_K; \mathbb{Z})$ ← [Buchstaber-Panov 00]

$\xrightarrow{\text{Hochster}}$ $H^s(Z_K) = \bigoplus_{w \subset [n],\, |w| = p} \tilde{H}^{s-p-1}(K_w)$

Homotopy groups

Let $\phi_r := \operatorname{rank} \pi_r(Z_K) = \dim_{\Bbbk} \pi_r(Z_K) \otimes \Bbbk$ $\quad$ ($\operatorname{char} \Bbbk = 0$)

Thm [Denham–S 07]:

$$\prod_{r=1}^{\infty} \frac{(1+t^{2r-1})^{\phi_{2r}}}{(1-t^{2r})^{\phi_{2r+1}}} = (1+t)^{-n} \sum_{p,q} \dim_{\Bbbk} \operatorname{Tor}_p^{S/I_K}(\Bbbk,\Bbbk)_q\, t^q$$

given by Berglund's formula

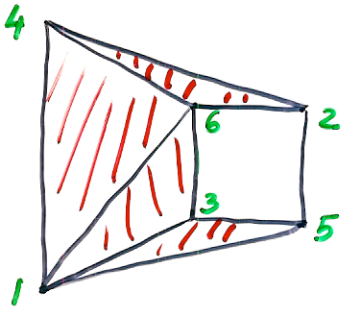
Formality

DJ_K always formal [Notbohm–Ray]

Z_K is not formal in general [Baskakov 03] [DS 07]

Simplest example:

K



$H^3(Z_K) = \mathbb{Z}^3$, generated by

$\alpha \leftrightarrow 12$
$\beta \leftrightarrow 34$
$\gamma \leftrightarrow 56$

non-trivial triple Massey product

$$H^3 \wedge H^3 \wedge H^3 \longrightarrow H^8$$
$$(\alpha, \beta, \gamma) \longmapsto \langle \alpha, \beta, \gamma \rangle$$

This Massey product is "decomposable" ($= uv$, $u, v \in H^4$)

Simplest example with indecomposable Massey products (modelled on an ideal defined by Backelin)

$$I_K = (x_1x_2,\ x_1x_3x_4x_5,\ x_3x_4x_5x_6,\ x_3x_5x_6x_7,\ x_7x_8)$$

has

$\alpha, \gamma \in H^3(Z_K)$, $\beta \in H^7(Z_K)$

$\langle \alpha, \beta, \gamma \rangle \in H^{12}(Z_K)$ $\quad$ indecomposable

This gives a non-trivial differential in Eilenberg–Moore ss of $\Omega Z \to PZ \to Z$, so $\pi^*(\Omega Z_K) \otimes \mathbb{Q}$ does not equal $\operatorname{Prim}(\operatorname{Ext}_{H^*(Z_K,\mathbb{Q})}(\mathbb{Q},\mathbb{Q})) = \pi^*(H^*(Z_K,\mathbb{Q}))$

Subspace arrangements

$$\mathcal{Z}_K \simeq \mathcal{Z}_K(\mathbb{C}, \mathbb{C}^*) \qquad \text{since } (D^2, S^1) \simeq (\mathbb{C}, \mathbb{C}^*)$$

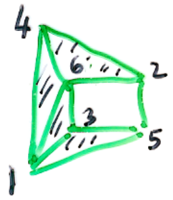
$$= \mathbb{C}^n \setminus \bigcup_{\emptyset \neq \sigma \in K} H_\sigma \qquad \text{the complement of a coordinate subspace arrangement}$$

where $H_\sigma = \{ z \in \mathbb{C}^n : z_i = 0 \text{ if } i \notin \sigma \}$

Thm (Arnol'd, Brieskorn) $\mathcal{A}$ hyperplane arrangement in $\mathbb{C}^\ell$
$\Rightarrow$ $\mathbb{C}^\ell - \mathcal{A}$ formal

Thm [Feichtner–Yuzvinsky 05] $\mathcal{A}$ subspace arrangement in $\mathbb{C}^\ell$ with $L(\mathcal{A})$ geometric $\Rightarrow$ $\mathbb{C}^\ell - \mathcal{A}$ formal

Using the above, it follows that complements of coordinate subspace arrangements are not formal, in general:

K 

$$\rightsquigarrow \mathcal{A}_K = \{ H_{12}, H_{23}, H_{34}, H_{45}, H_{56} \}$$

$$H_{ij} = \{ z \in \mathbb{C}^6 : z_i = z_j = 0 \}$$

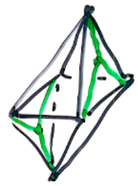
$\rightarrow$ $\mathbb{C}^6 \setminus \bigcup H_{ij}$ not formal [$L(\mathcal{A})$ not geometric]

Non-formal $\mathcal{Z}_K$-manifolds

Recall: K triangulation of S^ℓ on n vertices $\Rightarrow$
$\mathcal{Z}_K$ is a compact manifold of $\dim = n+\ell+1$
(with $\pi_1 = \pi_2 = 0$)

If $n+\ell \leq 9$, then $\mathcal{Z}_K$ is formal - use [Miller 79]

In particular: K triangulation of S^2 on $n \leq 7$ vertices
$\Rightarrow \mathcal{Z}_K$ formal

$K =$  $\rightsquigarrow$ $\mathcal{Z}_K$ not formal

triangulation of S^2 on 8 vertices

(has [decomposable] non-trivial, triple Massey product)

Remark We can construct a 16-vertex triangulation K of S^6 such that the 23-dim manifold $\mathcal{Z}_K$ has an indecomposable triple Massey product $\alpha, \gamma \in H^3$, $\beta \in H^7 \rightsquigarrow \langle \alpha, \beta, \gamma \rangle \in H^{12}$

Using [McDiarmid, Steger, Welsh - JCTB 05], we get:

Thm (Denham-S. 07)

$$\lim_{n\to\infty} \frac{\#\{\mathcal{Z}_K \mid K \text{ triangulation of } S^2 \text{ on } n \text{ vertices with } \mathcal{Z}_K \text{ formal}\}}{\#\{\mathcal{Z}_K \mid K \text{ triangulation of } S^2 \text{ on } n \text{ vertices}\}} = 0$$