Topology and combinatorics of polyhedral products

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I Construction

- Input: • K, a simplicial complex on $\{1,\dots,n\}$
  • (X,A), a pair of spaces
- Output: $Z_K(X,A) := \bigcup_{\sigma\in K} (X,A)^\sigma \subset X^{\times n}$

  $(X,A)^\sigma = \{x\in X^{\times n} : x_i\in A \text{ if } i\notin\sigma\}$

— the "polyhedral product" of (X,A) interpolates between

$$Z_\emptyset(X,A) = Z_K(A,A) = A^{\times n}$$

and

$$Z_{\Delta^{n-1}}(X,A) = Z_K(X,X) = X^{\times n}$$

- Particular case:

$$Z_K(X) := Z_K(X,*) \qquad * = \text{basepoint}$$

eg: • $Z_{\{n \text{ points}\}}(X) = \bigvee^n X$ wedge

• $Z_{\partial(\Delta^{n-1})}(X) = T^n(X)$ fat wedge

II Properties

(a) $L\subset K$ subcomplex $\rightsquigarrow Z_L(X,A)\subset Z_K(X,A)$

(b) (X,A) (finite) CW-pair $\rightsquigarrow Z_K(X,A)$ (finite) CW-complex

(c) $Z_{K*L}(X,A) = Z_K(X,A)\times Z_L(X,A)$

(d) $f:(X,A) \longrightarrow (Y,B)$ continuous $\rightsquigarrow$

$$\begin{array}{ccc} Z^f : \mathcal{Z}_K(X,A) & \longrightarrow & \mathcal{Z}_K(Y,B) \\ \cap & & \cap \\ X^{\times n} & \xrightarrow{f^{\times n}} & Y^{\times n} \end{array}$$

hence: $(X,A) \simeq (Y,B) \Rightarrow \mathcal{Z}_K(X,A) \simeq \mathcal{Z}_K(Y,B)$

(e) If X comm. topological monoid & A submonoid
then [Strickland]: (eg: $(X,A) = (D^2, S^1)$)

$f: K \to L$ simplicial $\rightsquigarrow \mathcal{Z}_f : \mathcal{Z}_K(X,A) \longrightarrow \mathcal{Z}_L(X,A)$

III Fibrations

__Theorem__ (Denham–S. PAMQ 07)

Let
$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \cup & & \cup & & \cup \\ F' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$
fibrations (or G-bundles) with $F = F'$ or $B = B'$

Then $\mathcal{Z}_K(F,F') \longrightarrow \mathcal{Z}_K(E,E') \longrightarrow \mathcal{Z}_K(B,B')$

is also a fibration (or G-bundle)

__Example__

$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & BG \\ \parallel & & \cup & & \cup \\ G & \longrightarrow & G & \longrightarrow & * \end{array}$$

$\overset{\rightsquigarrow}{\text{thm}}$ $G^{\times n} \longrightarrow \mathcal{Z}_K(EG, G) \longrightarrow \mathcal{Z}_K(BG)$

Furthermore:

__Lemma__ The homotopy fiber of $\mathcal{Z}_K(BG) \hookrightarrow BG^{\times n}$ is $\mathcal{Z}_K(EG, G)$.

IV Manifolds

Prop (Denham–S. in preparation)
Let $(M, \partial M)$ be a d-dimensional compact manifold and K a PL-triangulation of S^m on n vertices
[K is PL-homeo to $\partial\Delta^{m+1}$]
Then: $Z_K(M, \partial M)$ is a compact manifold of dimension $= (d-1)n + m + 1$

Also, a similar result for PL-triangulations of D^{m+1}

Examples: $\cdot\ M_K = Z_K([0,1], \{0,1\})$ has $\dim = m+1$
$\cdot\ Z_K = Z_K(D^2, S^1)$ has $\dim = n+m+1$

— · —

Theorem (Bosio–Meersseman, Acta 06)
Let K be a polytopal triangulation of S^m
[K = boundary complex of a convex polytope of $\dim = m+1$]
Then:
- Z_K if $n+m+1$ even
- $Z_K \times S^1$ if $n+m+1$ odd

admits a complex structure.

- This generalizes classical constructions of Hopf ($S^1 \times S^{2n+1}$), Calabi–Eckmann ($S^k \times S^\ell$, k, ℓ odd), and Lopez de Medrano–Verjovsky.

- These compact, complex manifolds are almost never Kähler (as B–M note). In fact, we show that, quite often, they are not even formal.

V Toric complexes

$T_K := Z_K(S^1)$ a subcomplex of $T^n = (S^1)^{\times n}$ (with product cell structure from $S^1 = e^0 \cup e^1$)

- Let $C_*(T_K)$ be the cellular chain complex. Since $T_K \subset T^n$, all $\partial_k : C_k \to C_{k-1}$ vanish. Since k cells of $T_K \longleftrightarrow (k-1)$ simplices of K,

$$H_k(T_K) = C_{k-1}(K) \cong \mathbb{Z}^{d_k(K)}$$

where $d_k(K) = \#\{\sigma \in K : \underbrace{|\sigma|}_{\dim\sigma+1} = k\}$

- <u>Cohomology ring</u> [Kim–Roush 1980]:

$$H^*(T_K) = \Lambda(x_1,\dots,x_n) / \text{ideal}\langle x_{i_1}\cdots x_{i_r} : \{i_1,\dots,i_r\} \notin K\rangle$$

i.e.,

$$H^*(T_K, \Bbbk) = \Bbbk\langle K\rangle = E/J_K$$

— the exterior Stanley–Reisner ring

In particular, if

$K = \Delta(\Gamma)$ $\qquad$ $\Gamma = K^{(1)}$ 1-skeleton of K, $\Delta(\Gamma)$ flag complex of Γ

then J_K is a quadratic monomial ideal, and so $H^*(T_{\Delta(\Gamma)}; \Bbbk)$ is a Koszul algebra [Fröberg 1975]

- <u>Fundamental group</u>

$$\pi_1(T_K) = G_\Gamma := \langle v \in V_\Gamma : vw = wv \text{ if } \{v,w\} \in E_\Gamma \rangle$$

right-angled Artin group of Γ

K	T_K	G_Γ
•	S^1	$\mathbb{Z}$
• •	$S^1 \vee S^1$	F_2
•—•	$S^1 \times S^1$	$\mathbb{Z}^2$

K	T_K	G_Γ
• • •	$S^1 \vee S^1 \vee S^1$	F_3
/ •	$T^2 \vee S^1$	$\mathbb{Z}^2 * \mathbb{Z}$
∧	$S^1 \times (S^1 \vee S^1)$	$\mathbb{Z} \times F_2$

- Classifying space for G_Γ is also a toric complex:

$$K(G_\Gamma, 1) = T_{\Delta(\Gamma)}$$

[Meier-VanWyck, Charney-Davis based on a lemma of Gromov]

In general, though, T_K is not aspherical.

Theorem (Papadima-S. 08) Let Δ be the flag complex of K. Set

$$p := \sup\{k : d_i(\Delta) = d_i(K) \text{ for } i \le k\}$$

Then:

$$T_K \text{ aspherical} \Leftrightarrow K \text{ flag complex} \Leftrightarrow p = \infty$$

Moreover, if $p < \infty$, then:

- $\pi_2(T_K) = \cdots = \pi_{p-1}(T_K) = 0$
- $\pi_p(T_K)$ has (explicit) finite presentation as $\mathbb{Z}G_\Gamma$-module
- $\pi_p(T_K)_{G_\Gamma} = \mathbb{Z}^{d_{p+1}(\Delta) - d_{p+1}(K)} \neq 0$

[The implication $T_K \simeq K(G_\Gamma, 1) \Rightarrow K = \Delta(\Gamma)$ recovers a recent result of Leary and Saadetoḡlu]

- Formality

Theorem [Notbohm-Ray AGT 05] $T_K = Z_K(S^1)$ formal, $\forall K$

[A different proof is given in [PS MathAnn 06] in the case when $K = \Delta(\Gamma)$:

- G_Γ is 1-formal [Kapovich-Millson IHES 98]
- $\mathbb{Q}\langle\Delta(\Gamma)\rangle$ Koszul $\Rightarrow T_{\Delta(\Gamma)}$ formal
- [$T_{\Delta(\Gamma)}$ aspherical]

use [Papadima-Yuzvinsky JPAA 99]]

Lie algebras associated to RAAG's

G finitely generated group $\rightsquigarrow$ graded Lie algebras:

- $gr(G) = \bigoplus_{k\geq 1} G_k/G_{k+1}$ $\qquad G_1 = G, \dots, G_{k+1} = [G, G_k], \dots$
  $\leftarrow$ associated graded Lie algebra
- $gr(G/G'')$ $\leftarrow$ Chen Lie algebra
- $h_G = \mathrm{Lie}(H_1 G) / \mathrm{ideal}(H_2 G \xrightarrow{\Delta} H_1 G \wedge H_1 G)$
  $\uparrow$ holonomy Lie algebra

In general:
- $h_G \twoheadrightarrow gr(G)$ with iso in deg ≤ 3
- $h_G \otimes \mathbb{Q} \xrightarrow{\cong} gr(G) \otimes \mathbb{Q}$ if G is 1-formal [Sullivan]
- $gr(G) \twoheadrightarrow gr(G/G'')$ with iso in deg ≤ 3
- $h_G / h_G'' \twoheadrightarrow gr(G/G'')$
- $h_G \otimes \mathbb{Q} / h_G'' \otimes \mathbb{Q} \xrightarrow{\cong} gr(G/G'') \otimes \mathbb{Q}$ if G is 1-formal [Papadima-S. IMRN 04]

Now let $G = G_\Gamma$, $\Gamma = (V, E)$. Then:

Theorem (PS – Math Ann 06)

- $gr(G) \cong h_G = \mathrm{Lie}(V) / \langle [v,w] = 0 \text{ if } \{v,w\} \in E \rangle$
  and is torsion-free
- The ranks $\phi_k = \mathrm{rank}\, gr_k(G)$ are given by

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t)$$

where $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma)\, t^k$ is clique polynomial

[recovers and sharpens results of Duchamp & Krob – 1992]

7)

Theorem (PS 06) For $G = G_\Gamma$:

- $gr(G/G'') \cong \mathfrak{h}_G/\mathfrak{h}_G''$ and is torsion-free
- The Chen ranks $\theta_k = \operatorname{rank} gr_k(G/G'')$ given by

$$\sum_{k=2}^{\infty} \theta_k(G)t^k = Q_\Gamma\left(t/_{1-t}\right)$$

where $Q_\Gamma(t) = \sum_{j\geq 2} c_j(\Gamma)t^j$ is the "cut polynomial"

$$c_j(\Gamma) = \sum_{\substack{W\subset V\\ |W|=j}} \tilde{b}_0(\Gamma_W)$$

Γ_W: induced subgraph

Proof uses

Thm (Fröberg – Löfwall, HHA 02)

A graded algebra$/\Bbbk$ with:
- A^1 finite-dim
- $a\cdot a = 0$, $\forall a \in A^1$
- $E := \Lambda A^1 \twoheadrightarrow A$ (A gen. in deg 1)

Then:

$$\left(\mathfrak{h}_A'/\mathfrak{h}_A''\right)_k \cong \operatorname{Tor}^E_{k-1}(A,\Bbbk)_k, \quad \forall k\geq 2$$

8) Cohomology jumping loci

- A graded algebra $/\Bbbk$ $\quad$ ($A^0=\Bbbk$, $\dim_{\Bbbk} A^i<\infty$, char $\Bbbk\neq 2$)

For each $a\in A^1$, have a cochain complex

$$(A,a):\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

The resonance varieties of A:

$$R^i_d(A):=\{a\in A^1 :\ \dim_{\Bbbk} H^i(A,a)\geq d\}$$

- X connected, finite-type CW-complex

The characteristic varieties of X:

$$V^i_d(X,\Bbbk):=\{\rho\in \mathrm{Hom}(\pi_1 X,\Bbbk^{\times}) :\ \dim_{\Bbbk} H_i(X,\Bbbk_\rho)\geq d\}$$

Also write:
- $R^i_d(X,\Bbbk):=R^i_d(H^*(X,\Bbbk))$
- $R_d(G,\Bbbk)=R^1_d(X,\Bbbk)$, $V_d(G,\Bbbk)=V^1_d(X,\Bbbk)$ for $G=\pi_1 X$

Now let K simplicial complex, $\quad \Gamma=K^{(1)}=(V,E)$

$A=H^*(T_K;\Bbbk)=\Bbbk\langle K\rangle$, $\quad A^1=\Bbbk^V$ (spanned by $x_v: v\in V$)

$\mathrm{Hom}(G_\Gamma,\Bbbk^{\times})=(\Bbbk^{\times})^V$

Theorem (PS 08)

$$R^i_d(T_K,\Bbbk)=\bigcup_{\substack{W\subset V\\ \sum_{\sigma\in K_{V-W}} \dim_{\Bbbk} \tilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{K_W}(\sigma),\Bbbk)\geq d}} \Bbbk^W$$

use [AAH – TAMS 00]

e.g.:
$$R^1_1(G_\Gamma,\Bbbk)=\bigcup_{\substack{W\subset V\\ \Gamma_W \text{ disconnected}}} \Bbbk^W$$

[PS – Math Ann 06]

$$V^i_d(T_K,\Bbbk)=\bigcup_{W\subset V} (\Bbbk^{\times})^V$$

... same ...

9) <u>Quasi-projectivity</u>

<u>Theorem</u> [Dimca–Papadima–S. 05] TFAE:

- $G_\Gamma = \pi_1$ (smooth, quasi-projective variety)
- $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$ [a product of f.g. free groups]
- $\Gamma = K_{n_1,\dots,n_r}$ [a complete multipartite graph]

Uses previous computation of $R_d(G_\Gamma)$ & $V_d(G_\Gamma)$, plus:

<u>Thm</u> [Arapura – JAG97] X smooth, quasi-projective
$\Rightarrow V_d^i(X,\mathbb{C})$ is a union of translated subtori in $(\mathbb{C}^*)^n$ $(n=b_1X)$

<u>Thm</u> [DPS 05] G 1-formal $\Rightarrow TC_1(V_d(G,\mathbb{C})) = R_d(G,\mathbb{C})$

which together imply

<u>Thm</u> [DPS 05] G 1-formal & $G=\pi_1$(smooth, quasi-projective)

$\Rightarrow R_d(G,\mathbb{C}) = \bigcup_\alpha R^\alpha$

- R^α linear subspace in $\mathbb{C}^n$
- $R^\alpha \cap R^\beta = 0$ if $\alpha \neq \beta$
- and more ...

<u>Example</u> $\Gamma =$

max disconnected subgraphs: Γ_{134} & Γ_{124}

$\therefore R_1(G_\Gamma,\mathbb{C}) = \mathbb{C}^3 \cup \mathbb{C}^3 \subset \mathbb{C}^4$

spanned by e_1, e_3, e_4 spanned by e_1, e_2, e_4

intersect in $\mathbb{C}^2 = \mathrm{span}(e_1, e_4) \neq 0$

But G_Γ is 1-formal $\Rightarrow G_\Gamma$ not quasiprojective

Moment-angle complexes

$$\mathcal{Z}_K := \mathcal{Z}_K(D^2, S^1)$$

a subcomplex of polydisk $(D^2)^{\times n} \cong D^{2n}$, with product cell structure from $D^2 = e^0 \cup e^1 \cup e^2$

e.g.:

- $K = \partial\Delta^{n-1} \rightsquigarrow \mathcal{Z}_K = S^{2n-1}$
- $K = \{n \text{ points}\} \rightsquigarrow \mathcal{Z}_K = \bigvee_{k=2}^{n} (k-1)\binom{n}{k} S^{k+1}$
- K a "shifted complex" $\rightsquigarrow \mathcal{Z}_K \simeq$ wedge of spheres
  [Grbić & Theriault, Top07]
- K an n-gon $\rightsquigarrow \mathcal{Z}_K = \#_{j=1}^{n-3}\, j\binom{n-2}{j+1} S^{j+2} \times S^{n-j}$
  [McGavran, TAMS 79]

Closely related is the Davis-Januszkiewicz space [DJ, Duke 91]

$$DJ_K = \mathcal{Z}_K(BS^1) \subset (\mathbb{CP}^\infty)^{\times n}$$

- Fibrations:

$$\begin{tikzcd}
(S^1)^{\times n} \arrow[r] \arrow[d, equal] & \mathcal{Z}_K(ES^1, S^1) \arrow[r] \arrow[d, "\wr"] & \mathcal{Z}_K(BS^1) \arrow[r, hook] \arrow[d, equal] & B(S^1)^{\times n} \arrow[d, equal] \\
T^n \arrow[r] & \mathcal{Z}_K \arrow[r] & DJ_K \arrow[r, hook] & (\mathbb{CP}^\infty)^{\times n}
\end{tikzcd}$$

Clearly:
- $\pi_1(DJ_K) = 0$
- $\pi_1(\mathcal{Z}_K) = \pi_2(\mathcal{Z}_K) = 0$
- $\pi_i(\mathcal{Z}_K) \cong \pi_i(DJ_K)$ for $i > 2$

- Cohomology ring:

- $H^*(DJ_K, \mathbb{Z}) = S/I_K$ $\quad S = \mathbb{Z}[x_1, \dots, x_n]$, $\deg x_i = 2$, I_K = Stanley-Reisner ideal of K ← [DJ]

- $H^*(\mathcal{Z}_K, \mathbb{Z}) = \mathrm{Tor}^S(S/I_K; \mathbb{Z})$ ← [Buchstaber-Panov 00]

$\xrightarrow{\text{Hochster}}$ $H^s(\mathcal{Z}_K) = \bigoplus_{w \subset [n], |w| = p} \tilde{H}^{s-p-1}(K_w)$

· Homotopy groups

Let $\phi_r := \operatorname{rank} \pi_r(Z_K) = \dim_{\Bbbk} \pi_r(Z_K) \otimes \Bbbk$ (char $\Bbbk = 0$)

Thm [Denham–S 07]:

$$\prod_{r=1}^{\infty} \frac{(1+t^{2r-1})^{\phi_{2r}}}{(1-t^{2r})^{\phi_{2r+1}}} = (1+t)^{-n} \sum_{p,q} \dim_{\Bbbk} \operatorname{Tor}_p^{S/I_K}(\Bbbk,\Bbbk)_q\, t^q$$

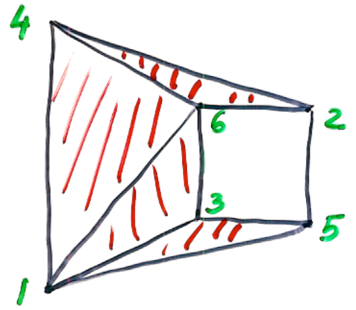
given by Berglund's formula

· Formality

DJ_K always formal [Notbohm–Ray]

Z_K is not formal in general [Baskakov 03] [DS 07]

Simplest example:

K 

$H^3(Z_K) = \mathbb{Z}^3$, generated by

$\alpha \leftrightarrow 12$
$\beta \leftrightarrow 34$
$\gamma \leftrightarrow 56$

non-trivial triple Massey product

$$H^3 \wedge H^3 \wedge H^3 \longrightarrow H^8$$
$$(\alpha,\beta,\gamma) \longmapsto \langle \alpha,\beta,\gamma \rangle$$

This Massey product is "decomposable" ($= uv$, $u,v \in H^4$)

Simplest example with indecomposable Massey products (modelled on an ideal defined by Backelin)

$$I_K = (x_1x_2,\ x_1x_3x_4x_5,\ x_3x_4x_5x_6,\ x_3x_5x_6x_7,\ x_7x_8\rangle$$

has

$\alpha, \gamma \in H^3(Z_K)$, $\beta \in H^7(Z_K)$

$\langle \alpha,\beta,\gamma \rangle \in H^{12}(Z_K)$ indecomposable

This gives a non-trivial differential in Eilenberg–Moore ss of $\Omega Z \to PZ \to Z$, so $\pi^*(\Omega Z_K) \otimes \mathbb{Q}$ does not equal $\operatorname{Prim}(\operatorname{Ext}_{H^*(Z_K,\mathbb{Q})}(\mathbb{Q},\mathbb{Q})) = \pi^*(H^*(Z_K,\mathbb{Q}))$

Subspace arrangements

$$Z_K \simeq Z_K(\mathbb{C}, \mathbb{C}^*)$$ since $(D^2, S^1) \simeq (\mathbb{C}, \mathbb{C}^*)$

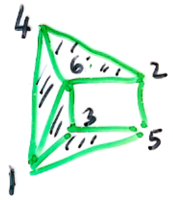
$$= \mathbb{C}^n \setminus \bigcup_{\emptyset \neq \sigma \in K} H_\sigma$$ the complement of a coordinate subspace arrangement

where $H_\sigma = \{ z \in \mathbb{C}^n : z_i = 0 \text{ if } i \notin \sigma \}$

Thm (Arnol'd, Brieskorn) $\mathcal{A}$ hyperplane arrangement in $\mathbb{C}^\ell$
$\Rightarrow \mathbb{C}^\ell - \mathcal{A}$ formal

Thm [Feichtner - Yuzvinsky 05] $\mathcal{A}$ subspace arrangement in $\mathbb{C}^\ell$ with $L(\mathcal{A})$ geometric $\Rightarrow \mathbb{C}^\ell - \mathcal{A}$ formal

Using the above, it follows that complements of coordinate subspace arrangements are not formal, in general:

K 

$\rightsquigarrow \mathcal{A}_K = \{ H_{12}, H_{23}, H_{34}, H_{45}, H_{56} \}$

$H_{ij} = \{ z \in \mathbb{C}^6 : z_i = z_j = 0 \}$

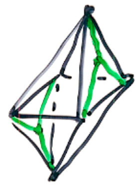
$\rightarrow \mathbb{C}^6 \setminus \bigcup H_{ij}$ not formal [$L(\mathcal{A})$ not geometric]

Non-formal Z_K-manifolds

Recall: K triangulation of S^ℓ on n vertices $\Rightarrow$ Z_K is a compact manifold of dim $= n+\ell+1$ (with $\pi_1 = \pi_2 = 0$)

If $n+\ell \leq 9$, then Z_K is formal – use [Miller 79]

In particular: K triangulation of S^2 on $n \leq 7$ vertices $\Rightarrow Z_K$ formal

$K =$  $\rightsquigarrow$ Z_K not formal

triangulation of S^2 on 8 vertices

(has [decomposable] non-trivial, triple Massey product)

Remark We can construct a 16-vertex triangulation K of S^6 such that the 23-dim manifold Z_K has an indecomposable triple Massey product $\alpha, \gamma \in H^3$, $\beta \in H^7 \rightsquigarrow \langle \alpha, \beta, \gamma \rangle \in H^{12}$

Using [McDiarmid, Steger, Welsh – JCTB 05], we get:

Thm (Denham–S. 07)

$$\lim_{n\to\infty} \frac{\#\{Z_K \mid K \text{ triangulation of } S^2 \text{ on } n \text{ vertices with } Z_K \text{ formal}\}}{\#\{Z_K \mid K \text{ triangulation of } S^2 \text{ on } n \text{ vertices}\}} = 0$$