THE TOPOLOGY OF COMPACT LIE GROUP ACTIONS THROUGH THE LENS OF FINITE MODELS

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ABSTRACT. Given a compact, connected Lie group K, we use principal K-bundles to construct manifolds with prescribed finite-dimensional algebraic models. Conversely, let M be a compact, connected, smooth manifold which supports an almost free K-action. Under a partial formality assumption on the orbit space and a regularity assumption on the characteristic classes of the action, we describe an algebraic model for M with commensurate finiteness and partial formality properties. The existence of such a model has various implications on the structure of the cohomology jump loci of M and of the representation varieties of $\pi_1(M)$. As an application, we show that compact Sasakian manifolds of dimension $2n + 1$ are $(n - 1)$-formal, and that their fundamental groups are filtered-formal. Further applications to the study of weighted-homogeneous isolated surface singularities are also given.

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1. Introduction and statement of results

1.1. **Algebraic models for spaces.** The rational homotopy type of a nilpotent CW-space of finite type can be reconstructed from algebraic models associated to it. If the space in question is a smooth manifold M, the standard model is the de Rham algebra $\Omega^{\bullet}_{\mathrm{dR}}(M)$ of smooth forms on the manifold, endowed with a wedge product and differential satisfying the graded Leibniz rule. Sullivan [47] associated to any space X a commutative, differential graded $\mathbb{C}$-algebra (for short, a cdga), denoted by $\Omega^{\bullet}(X)$, which serves as the reference algebraic model for the space. In particular, $H^{\bullet}(\Omega(X)) = H^{\bullet}(X, \mathbb{C})$.

We are interested here in algebraic models of a connected CW-complex X which satisfy certain finiteness and formality conditions. As shown in [17], models with finiteness properties contain valuable topological information related to the structure around the origin of both representation varieties of fundamental groups and the loci where the corresponding twisted homology of spaces (in degrees up to a fixed $q \geqslant 1$) jumps. This requires the finiteness of the q-skeleton of X, but *no* nilpotence condition. When finiteness of models comes from stronger formality properties of X, additional nice features hold.

More precisely, we say that $A^{\bullet}$ is *q-finite* if A is connected (i.e., $A^0 = \mathbb{C}$) and A^i is finite-dimensional, for each $i \leqslant q$. Likewise, we say that $(A^{\bullet}, d)$ is a *q-model* for X if A has the same q-type as $\Omega^{\bullet}(X)$, i.e., there is a zig-zag of morphisms connecting these two cdgas, with each such morphism inducing isomorphisms in homology up to degree q and a monomorphism in degree $q + 1$. Furthermore, we say that (A, d) is *q-formal* if it can be connected to $(H^{\bullet}(A), d = 0)$ by such a zig-zag of cdga morphisms. Finally, we say X is q-formal if $\Omega^{\bullet}(X)$ has this property. All these notions have obvious analogues when $q = \infty$, in which case we drop the prefix q.

Let K be a compact, connected real Lie group. If a connected space X admits a finite model, it is known that any principal K-bundle over X has the same property. Our first result (proved in Theorem 4.3 and Corollary 7.3) uses this fact to provide a systematic way of constructing a rich variety of new, interesting spaces admitting finite models. To state this result, we identify the cohomology algebra $H^{\bullet}(K, \mathbb{Q})$ with an exterior algebra on odd-degree generators, $\bigwedge(t_1, \ldots, t_r)$, where $r = \operatorname{rank} K$.

Theorem 1.1. *Let X be a connected, finite CW-space. If X has a finite, rational model B, then any Hirsch extension $A = B \otimes_{\tau} \bigwedge(t_1, \ldots, t_r)$ can be realized as a finite, rational model of some principal K-bundle Y over X. Moreover, if the cdga B satisfies Poincaré duality in dimension n, then A has the same property in dimension $n + \dim K$.*

As is well-known, the formality property of a space X is not necessarily inherited by a principal K-bundle Y over X. Our next result gives a useful algebraic criterion on the characteristic classes of the bundle which implies the q-formality of Y.

Let $H^{\bullet}$ be a connected, commutative graded algebra, and let $\{e_{\alpha}\}$ be a sequence of homogeneous elements of degree $n_{\alpha} > 0$. We say that this sequence is *q-regular* if for

each α, the class of e_α in the quotient algebra $\overline{H}_\alpha = H/\sum_{\beta<\alpha} e_\beta H$ has trivial annihilator up to degree $q - n_\alpha + 2$.

Theorem 1.2. *Suppose $e_1, \ldots, e_r$ is an even-degree, q-regular sequence in $H^\bullet$. Then the Hirsch extension $A = H \otimes_\tau \bigwedge(t_1, \ldots, t_r)$ with $d = 0$ on H and $dt_\alpha = \tau(t_\alpha) = e_\alpha$ has the same q-type as $(H/\sum_\alpha e_\alpha H, d = 0)$. In particular, A is q-formal.*

For $q = \infty$, this is a classical property of Koszul complexes from commutative algebra, see [25, Proposition 3.6]. Work of Borel [9] and Chevalley [15] supplies examples of fully regular sequences in graded polynomial rings which arise in the context of compact connected Lie groups and finite reflection groups.

The above notion of partial regularity seems well adapted to the study of cohomology rings of finite-dimensional CW-complexes. The Hard Lefschetz Theorem provides natural examples of one-element partially regular sequences in the cohomology rings of compact Kähler manifolds, known to be formal by work of Deligne et al. [16]. Theorem 1.2 may be applied to analyze the partial formality of arbitrary homogeneous spaces of compact, connected Lie groups. As shown in Example 4.6, the resulting estimates are sharp in certain cases.

1.2. **Almost free K-actions.** We now switch to a topological context which is broader than principal bundles. Consider a compact, connected, smooth manifold M (without boundary) on which a compact, connected Lie group K acts smoothly. The K-action on M is said to be *almost free* if all its isotropy groups are finite. Let $K \to EK \times M \to M_K$ be the Borel construction on M, and let $\tau \colon H^\bullet(K, \mathbb{C}) \to H^{\bullet+1}(M_K, \mathbb{C})$ be the transgression in the Serre spectral sequence of this bundle.

Let $N = M/K$ be the orbit space. The projection map $\mathrm{pr} \colon M_K \to N$ induces an isomorphism $\mathrm{pr}^* \colon H^\bullet(N, \mathbb{C}) \to H^\bullet(M_K, \mathbb{C})$. We identify the cohomology algebra $H^\bullet(K, \mathbb{C})$ with the exterior algebra on odd-degree generators $t_1, \ldots, t_r$, as before, and we let $e_\alpha = (\mathrm{pr}^*)^{-1}(\tau(t_\alpha)) \in H^{m_\alpha+1}(N, \mathbb{C})$ be the corresponding characteristic classes.

It is known that a model for M can be built from the Sullivan model $\Omega^\bullet(N)$ and the above characteristic classes, via a suitable Hirsch extension with $\bigwedge P_K$, where $P_K = \mathrm{span}\{t_1, \ldots, t_r\}$. Our next result uses this fact, together with Theorem 1.2, to construct a q-model for M which is both q-finite and q-formal, under suitable partial formality conditions on the orbit space N and regularity conditions on the sequence $\{e_\alpha\}$, and for q in a certain range.

Theorem 1.3. *Suppose the K-action on M is almost free, the orbit space $N = M/K$ is k-formal, for some $k > \max\{m_\alpha\}$, and the characteristic classes $e_1, \ldots, e_r$ form a q-regular sequence in the graded ring $H^\bullet = H^\bullet(N, \mathbb{C})$, for some $q \leqslant k$. Then the CDGA $\left(H^\bullet/\sum_\alpha e_\alpha H^\bullet, d = 0\right)$ is a finite q-model for M. In particular, M is q-formal.*

In Example 4.10, we construct for an arbitrary K a family of almost free K-manifolds for which we are able to describe a lower bound for partial formality using the above result. In Example 4.9, this type of estimate becomes optimal.

The 1-formality property of a connected CW-complex X with finite 1-skeleton depends only on its fundamental group, $\pi = \pi_1(X)$. Furthermore, the 1-formality of π is equivalent to the Malcev Lie algebra $\mathfrak{m}(\pi)$ constructed by Quillen in [45] being filtered-isomorphic to the degree completion of a quadratic, finitely generated Lie algebra L. If L is merely assumed to have homogeneous relations, then the group π is said to be *filtered formal*.

Theorem 1.4. *Let $\pi = \pi_1(M)$ be the fundamental group of a compact, connected manifold M on which a compact, connected Lie group K acts almost freely, with 2-formal orbit space. Then:*

 (1) *The group π is filtered-formal. More precisely, the Malcev Lie algebra $\mathfrak{m}(\pi)$ is isomorphic to the degree completion of $\mathbb{L}/\mathfrak{r}$, where $\mathbb{L}$ is the free Lie algebra on $H_1(\pi, \mathbb{C})$ and $\mathfrak{r}$ is a homogeneous ideal generated in degrees 2 and 3.*
 (2) *For every complex linear algebraic group G, the germ at the trivial representation 1 of the representation variety $\mathrm{Hom}_{\mathrm{gr}}(\pi, G)$ is defined by quadrics and cubics only.*

This result has the same flavor as the weight obstructions of Morgan [35] and Kapovich–Millson [28], which are specific to fundamental groups of quasi-projective manifolds and germs of their representation varieties, respectively. On the other hand, our theorem may be applied to arbitrary principal K-bundles over formal, compact manifolds.

1.3. **Cohomology jump loci.** The *characteristic varieties* of a space X with respect to a rational representation $\iota\colon G \to \mathrm{GL}(V)$ are the sets $\mathscr{V}^i_s(X, \iota)$ consisting of those representations $\rho\colon \pi \to G$ for which the twisted cohomology group $H^i(X, V_{\iota \circ \rho})$ has $\mathbb{C}$-dimension at least s. In degree $i = 1$, these varieties depend only on the group $\pi = \pi_1(X)$, and so we may denote them as $\mathscr{V}^1_s(\pi, \iota)$. In the rank 1 case, i.e., when $G = \mathbb{C}^*$ and $\iota = \mathrm{id}_{\mathbb{C}*}$, we simply write the characteristic varieties of X as $\mathscr{V}^i_s(X)$.

When M is a closed, orientable manifold of dimension n, it is known that Poincaré duality imposes subtle restrictions on these global jump loci. Now suppose that X is a finite CW-complex admitting a finite model A, which is an n-dimensional Poincaré duality CDGA. Then, as we show in Theorem 7.7, there is an analytic involution of the germ at the origin, $\mathrm{Hom}(\pi_1(X), G)_{(1)}$, which identifies $\mathscr{V}^i_s(X, \iota)_{(1)}$ with $\mathscr{V}^{n-i}_s(X, \iota)_{(1)}$, for all i, s, where ι is the identity representation of $\mathrm{GL}(V)$. Furthermore, in the rank 1 case, this involution is induced by the involution $\rho \mapsto \rho^{-1}$ of the character group $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$.

Let again M be a smooth, closed manifold supporting an almost free K-action. The projection $p\colon M \to M/K$ induces a natural epimorphism $p_\sharp\colon \pi_1(M) \twoheadrightarrow \pi_1^{\mathrm{orb}}(M/K)$ between orbifold fundamental groups. Our next result establishes a tight connection between the germs at the origin of the rank 1 characteristic varieties and the $\mathrm{SL}_2(\mathbb{C})$ representation varieties of $\pi_1(M)$ and $\pi_1^{\mathrm{orb}}(M/K)$.

Theorem 1.5. *Suppose that the transgression $P_K^\bullet \to H^{\bullet+1}(M_K) \cong H^{\bullet+1}(M/K)$ is injective in degree 1. Then the following hold.*

(1) *If the orbit space $N = M/K$ has a 2-finite 2-model, then the epimorphism $p_\sharp$ induces a local analytic isomorphism between $\mathcal{V}_s^1(\pi_1(M))_{(1)}$ and $\mathcal{V}_s^1(\pi_1^{\mathrm{orb}}(N))_{(1)}$, for all s.*

(2) *If N is 2-formal, then $p_\sharp$ induces an analytic isomorphism between the germs at 1 of $\mathrm{Hom}(\pi_1(M), G)$ and $\mathrm{Hom}(\pi_1^{\mathrm{orb}}(N), G)$, for $G = \mathrm{SL}_2(\mathbb{C})$ or a Borel subgroup.*

For instance, let K be a compact, connected Lie group, and let N be a compact, formal manifold with $b_2(N) \geqslant \dim P_K^1$. Applying Theorem 1.1, we may use these data to construct interesting principal K-bundles $M \to N$ which fit into the purview of Theorem 1.5; see Example 5.8 for more details.

In the case when $K = S^1$, suppose M is a closed, orientable Seifert fibered 3-manifold with non-zero Euler class, and let $p\colon M \to M/S^1 = \Sigma_g$ be the projection map onto a Riemann surface of genus g. Then the above theorem insures that $\mathcal{V}_s^1(M)_{(1)} \cong \mathcal{V}_s^1(\Sigma_g)_{(1)}$, for all s, and $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))_{(1)} \cong \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{C}))_{(1)}$, as analytic germs at the trivial representation 1; see Corollary 5.9.

1.4. **Sasakian manifolds.** In many ways, Sasakian geometry is an odd-dimensional analogue of Kähler geometry. More explicitly, every compact Sasakian manifold M admits an almost-free circle action with orbit space $N = M/S^1$ a Kähler orbifold. Furthermore, the Euler class of the action coincides with the Kähler class of the base, $h \in H^2(N, \mathbb{Q})$, and this class satisfies the Hard Lefschetz property.

As shown by Tievsky in [49], every Sasakian manifold M as above has a rationally defined, finite model of the form $(H^\bullet(N) \otimes \bigwedge(t), d)$, where the differential d vanishes on $H^\bullet(N)$ and sends t to h. Using this model and Theorem 1.2, we prove in §6 the following theorem, which is the Sasakian analog of the main result from [16], namely, the formality of compact Kähler manifolds.

Theorem 1.6. *Let M be a compact Sasakian manifold of dimension $2n + 1$. Then M is $(n-1)$-formal.*

We make this statement more precise in Theorem 6.7, by providing an explicit, finite $(n-1)$-model with zero differential for M over an *arbitrary* field of characteristic 0. This result is optimal; indeed, for each $n \geqslant 1$, the $(2n+1)$-dimensional Heisenberg compact nilmanifold $\mathscr{H}_n$ is a Sasakian manifold, yet it is not n-formal. Theorem 1.6 strengthens a statement of Kasuya [30], who claimed that, for $n \geqslant 2$, a Sasakian manifold as above is 1-formal. The proof of that claim, though, has a gap, which we avoid by giving a proof based on a very different approach. The formality of compact Sasakian manifolds is also analyzed by Biswas et al. in [7], using Massey products. Furthermore, Muñoz and Tralle show in [36] that a compact, simply-connected, 7-dimensional Sasakian manifold is formal if and only if all triple Massey products vanish.

A group π is said to be a *Sasakian group* if it can be realized as the fundamental group of a compact Sasakian manifold. A major open problem in the field (cf. [10, 14]) is to characterize this class of groups. Applying the previous theorems, we obtain new, rather stringent restrictions on a finitely presented group π being Sasakian.

Theorem 1.7. *Let $\pi = \pi_1(M^{2n+1})$ be the fundamental group of a compact Sasakian manifold of dimension $2n + 1$. Then:*

(1) The group π is filtered-formal, and in fact 1-formal if $n > 1$.

(2) All irreducible components of the characteristic varieties $\mathscr{V}_s^1(\pi)$ passing through the identity are algebraic subtori of the character group $\mathrm{Hom}(\pi, \mathbb{C}^*)$.

(3) If G is a complex linear algebraic group, then the germ at the origin of $\mathrm{Hom}(\pi, G)$ is defined by quadrics and cubics only, and in fact by quadrics only if $n > 1$.

Furthermore, if $G = \mathrm{SL}_2(\mathbb{C})$ or a Borel subgroup thereof, then the germs at 1 of $\mathrm{Hom}(\pi_1(M), G)$ and $\mathscr{V}_s^1(\pi_1(M))$, for all s, depend (in an explicit way) only on the graded ring $H^\bullet(M/S^1, \mathbb{C})$, while the Tievsky model also depends in an essential manner on the Kähler class $h \in H^2(M/S^1, \mathbb{Q})$; see Corollary 6.12.

1.5. **Quasi-projective manifolds.** The infinitesimal analogue of the G-representation variety $\mathrm{Hom}(\pi, G)$ around the origin 1 is the set of g-valued flat connections on a cdga A, where g is the Lie algebra of the Lie group G. By definition, this is the set $\mathscr{F}(A, \mathfrak{g})$ of elements in $A^1 \otimes \mathfrak{g}$ which satisfy the Maurer–Cartan equation, $d\omega + \frac{1}{2}[\omega, \omega] = 0$, having as basepoint the trivial connection 0. If $\dim A^1 < \infty$, then $\mathscr{F}(A, \mathfrak{g})$ is a Zariski-closed subset of the affine space $A^1 \otimes \mathfrak{g}$, which contains the closed subvariety $\mathscr{F}^1(A, \mathfrak{g})$ consisting of all tensors of the form $\eta \otimes g$ with $d\eta = 0$.

To define the infinitesimal counterpart of characteristic varieties, let $\theta \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation. To each $\omega \in \mathscr{F}(A, \mathfrak{g})$ there is an associated covariant derivative, $d_\omega \colon A^\bullet \otimes V \to A^{\bullet+1} \otimes V$, given by $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$. By flatness, $d_\omega^2 = 0$. The *resonance varieties* of A with respect to θ are the sets $\mathscr{R}_s^i(A, \theta)$ consisting of those $\omega \in \mathscr{F}(A, \mathfrak{g})$ for which $\dim H^i(A \otimes V, d_\omega) \geqslant s$. If A is q-finite, then these subsets are Zariski-closed in $\mathscr{F}(A, \mathfrak{g})$, for all $i \leqslant q$. Furthermore, if $H^i(A) \neq 0$, then $\mathscr{R}_1^i(A, \theta)$ contains the closed subvariety $\Pi(A, \theta)$ consisting of all $\eta \otimes g \in \mathscr{F}^1(A, \mathfrak{g})$ with $\det \theta(g) = 0$.

By work of Arapura [3], the key to understanding the degree-1 characteristic varieties of rank 1 for a quasi-projective manifold M is the (finite) set $\mathscr{E}(M)$ of regular, surjective maps $f \colon M \to S$ for which the generic fiber is connected and the target is a smooth curve S with $\chi(S) < 0$, up to reparametrization at the target. All such maps extend to regular maps $\bar{f} \colon \overline{M} \to \overline{S}$, for some 'convenient' compactification $\overline{M} = M \cup D$. Letting $A(M) = A(\overline{M}, D)$ be the finite cdga model of M constructed by Morgan [35] and more generally by Dupont [19], and similarly for $A(S)$, we obtain the following structural result for the embedded first resonance varieties of a large class of quasi-projective surfaces, in the difficult rank 2 case.

Theorem 1.8. *Let M be a punctured, quasi-homogeneous surface with isolated singularity. Let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation, where $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ or a Borel subalgebra. If $b_1(M) > 0$, then there is a convenient compactification $\overline{M} = M \cup D$ such that*

$$\mathscr{F}(A(M), \mathfrak{g}) = \mathscr{F}^1(A(M), \mathfrak{g}) \cup \bigcup_{f \in \mathscr{E}(M)} f^!(\mathscr{F}(A(S), \mathfrak{g})),$$

$$\mathscr{R}_1^1(A(M), \theta) = \Pi(A(M), \theta) \cup \bigcup_{f \in \mathscr{E}(M)} f^!(\mathscr{F}(A(S), \mathfrak{g})).$$

The restriction on the first Betti number is not essential, as explained in Remark 3.5. To prove Theorem 1.8, we replace M (up to homotopy) by the singularity link, and $A(M)$ by a finite model A of this almost free S^1-manifold, for which explicit computations are done in §§9.2-9.3. We single out in Examples 8.5 and 8.6 several more classes of quasi-projective manifolds for which the conclusions of the theorem hold. Whether those conclusions hold in complete generality remains an open question, which is investigated further in [43, 44].

2. Algebraic models and formality properties

In this section, we collect some relevant facts from rational homotopy theory, following [47, 35, 20, 26, 32]. All spaces will be assumed to be path-connected, and the default coefficient ring will be a field $\Bbbk$ of characteristic 0.

2.1. Differential graded algebras and Hirsch extensions.

The basic algebraic structure we will consider in this work is that of a commutative, differential graded algebra (for short, a CDGA) over the field $\Bbbk$. This is an object of the form $A = (A^\bullet, d)$ where A is a non-negatively graded $\Bbbk$-vector space, endowed with a multiplication map $\cdot\colon A^i \otimes A^j \to A^{i+j}$ satisfying $u \cdot v = (-1)^{ij} v \cdot u$, and a differential $d\colon A^i \to A^{i+1}$ satisfying $d(u \cdot v) = du \cdot v + (-1)^i u \cdot dv$, for all $u \in A^i$ and $v \in A^j$.

We say that A is *connected* if $\dim A^0 = 1$. We also say that A is *q-finite*, for some $0 \leqslant q \leqslant \infty$, if A is connected and $\dim \bigoplus_{i \leqslant q} A^i < \infty$. When $q = \infty$, we will omit it from terminology and notation. A morphism $\varphi\colon A \to B$ between two CDGA's is a *quasi-isomorphism* if the induced homomorphism in cohomology, $\varphi^*\colon H^\bullet(A) \to H^\bullet(B)$, is an isomorphism.

Denote by $Z^\bullet(A)$ the graded vector space of d-cocycles of a CDGA A. Let $P^\bullet$ be an oddly-graded, finite-dimensional vector space, with homogeneous basis $\{t_i \in P^{m_i}\}$. Given a degree 1 linear map, $\tau\colon P^\bullet \to Z^{\bullet+1}(A)$, we define the corresponding *Hirsch extension* as the CDGA

$$(1) \qquad\qquad\qquad \left(A \otimes_\tau \bigwedge P, d \right),$$

where the differential d extends the differential on A, while $dt_i = \tau(t_i)$. The following standard fact is proved in [32, Lemmas II.2 and II.3], in the case when all the degrees m_i are equal. The same proof gives the result in the general case.

Lemma 2.1. *The isomorphism type of the* CDGA *$(A \otimes_\tau \bigwedge P, d)$ depends only on A and the homomorphism induced in cohomology, $[\tau] \colon P^\bullet \to H^{\bullet+1}(A)$. Moreover, $[\tau]$ and $[\tau] \circ g$ give isomorphic extensions, for any automorphism g of the graded vector space $P^\bullet$.*

When $\dim P = 1$, we have the following natural exact sequence, for all i (see e.g. [24, pp. 178–179]):

$$(2) \qquad H^{i-m-1}(A) \xrightarrow{[\tau(t)]\cdot} H^i(A) \longrightarrow H^i(A \otimes_\tau \bigwedge t) \longrightarrow H^{i-m}(A) \xrightarrow{[\tau(t)]\cdot} H^{i+1}(A) \ .$$

2.2. **Minimal models.** Let $\bigwedge(x)$ be the exterior (respectively, polynomial) algebra on a single generator x of odd (respectively, even) degree. A CDGA $\mathscr{M}$ is said to be *minimal* if the following three conditions are satisfied.

Freeness:	The underlying commutative graded algebra of $\mathscr{M}$ is of the form $\bigwedge(\{x_\alpha\}) = \bigotimes_\alpha \bigwedge(x_\alpha)$, with positive-degree variables $\mathcal{X} = \{x_\alpha\}$ indexed by a well-ordered set.
Nilpotence:	For any α, we have that $dx_\alpha \in \bigwedge(\{x_\beta \mid \beta < \alpha\})$.
Decomposability:	For any α, we have that $dx_\alpha \in \bigwedge^+(\mathcal{X}) \cdot \bigwedge^+(\mathcal{X})$, where $\bigwedge^+(\mathcal{X})$ is the ideal generated by $\mathcal{X}$.

A *minimal model* for a CDGA A is a minimal CDGA $\mathscr{M}$ which comes equipped with a quasi-isomorphism $\varphi \colon \mathscr{M} \to A$. Any CDGA A with $H^0(A) = \Bbbk$ has a unique minimal model, denoted by $\mathscr{M}(A)$. The notion of minimality admits the following refinement. A minimal CDGA $\mathscr{M}$ is said to be *q-minimal* (for some $q \geqslant 0$) if $\deg(x_\alpha) \leqslant q$, for all α. Such an object is a *q-minimal model* for a CDGA A if there is a *q-equivalence* $\varphi \colon \mathscr{M} \to A$, i.e., the CDGA map φ induces a homology isomorphism up to degree q and a homology monomorphism in degree $q + 1$. Again, any CDGA A with connected homology has a unique q-minimal model, denoted by $\mathscr{M}_q(A)$.

In the definition of minimality, let us denote by $V = V^\bullet$ the graded vector space generated by the set $\mathcal{X} = \{x_\alpha\}$. With this notation, $\mathscr{M}(A) = (\bigwedge V, d)$ and $\mathscr{M}_q(A) = (\bigwedge V^{\leqslant q}, d)$.

2.3. **Formality.** Two CDGAS A and B are said to be *weakly equivalent* (denoted $A \simeq B$) if there is a zig-zag of quasi-isomorphisms connecting A to B. Likewise, A and B are said to have the same *q-type* (denoted $A \simeq_q B$) if there is a zig-zag of q-equivalences connecting A to B. If $H^0(A) = \Bbbk$, then $A \simeq B$ (respectively, $A \simeq_q B$) if and only if A and B share the same minimal (respectively, q-minimal) model. Clearly, $A \simeq_q B$ implies $A \simeq_s B$, for all $s \leqslant q$.

A CDGA A is said to be *formal* if it is weakly equivalent to the cohomology algebra $H^\bullet(A)$, endowed with the zero differential. Likewise, A is said to be *q-formal* if $A \simeq_q (H^\bullet(A), d = 0)$. Plainly, q-formality implies s-formality, for all $s \leqslant q$.

As shown by Halperin and Stasheff in [26], the formality of a CDGA with connected, finite-type homology is independent of the ground field, with respect to field extensions $\Bbbk \subset \mathbb{K}$. The same descent property holds for q-formality, for all $q < \infty$, provided $H^\bullet(A)$ is $(q + 1)$-finite, see [46]. In general, it is easy to check that $A \simeq_q B$ over $\Bbbk$ implies that $A \otimes_\Bbbk \mathbb{K} \simeq_q B \otimes_\Bbbk \mathbb{K}$, but the converse does not necessarily hold.

Now let X be a topological space. An important bridge between topology and algebra is provided by the work of D. Sullivan [47], who constructed a CDGA $\Omega^\bullet(X) = \Omega^\bullet(X, \Bbbk)$ of piecewise polynomial $\Bbbk$-forms on X, inspired by the de Rham algebra of smooth forms on a manifold. This CDGA has the property that $H^\bullet(\Omega(X, \Bbbk)) \cong H^\bullet(X, \Bbbk)$, as graded rings.

A CDGA A is said to be a *model* for X if $A \simeq \Omega(X)$. Likewise, A is a *q-model* for X, for some $q \geqslant 1$, if $A \simeq_q \Omega(X)$. Finally, the space X is said to be formal (respectively, q-formal) if $\Omega^\bullet(X)$ has that property.

2.4. Formality properties of groups.

Let X be a 1-finite space, i.e., a space homotopy equivalent to a connected CW-complex with only finitely many 1-cells, and let $\pi = \pi_1(X)$. By considering a classifying map $X \to K(\pi, 1)$, it is readily seen that the 1-formality of X depends only on π.

Let $\mathfrak{m}(\pi)$ be the *Malcev Lie algebra* of π (over $\Bbbk$). This is a complete, filtered Lie algebra, constructed by Quillen in [45]. As is well-known (see e.g. [40, 46]), a finitely-generated group π is 1-formal if and only if its Malcev Lie algebra $\mathfrak{m}(\pi)$ is isomorphic, as a filtered Lie algebra, to the lower central series completion of a quadratic Lie algebra, that is, $\mathfrak{m}(\pi) \cong \widehat{L}$, where the Lie algebra L is generated in degree 1, and has relations only in degree 2.

The group π is said to be *filtered-formal* if $\mathfrak{m}(\pi) \cong \widehat{L}$, where L is generated in degree 1, and has homogeneous defining relations. This property, analyzed in detail in [46], is strictly weaker than 1-formality. For instance, if $\mathscr{H}_1$ is the 3-dimensional Heisenberg manifold from Example 6.8, then the group $\pi_1(\mathscr{H}_1)$ is filtered-formal, yet not 1-formal.

2.5. Hirsch extensions and algebraic q-type.

As before, let (A, d) be a CDGA, and let P be a graded vector space generated by finitely many odd-degree elements $t_i \in P^{m_i}$. Set $m = \max\{m_i\}$, and let $\tau\colon P \to Z^{\leqslant m+1}(A)$ be a degree one homomorphism. We then have a Hirsch extension, $(A \otimes_\tau \bigwedge P, d)$, where the differential d agrees with that of A, while taking t_i to $\tau(t_i) \in A^{m_i+1}$. The next result will be very useful in the sequel. For $q = \infty$, it follows from more general results on relative Sullivan models, see e.g. [20, Lemma 14.2]. In our simpler case, we provide an elementary proof, for $q \leqslant \infty$.

Lemma 2.2. *Suppose $A \simeq_q B$ and $q \geqslant m + 1$. There is then a degree one homomorphism $\sigma\colon P \to Z^{\leqslant m+1}(B)$ and an identification $H^{\leqslant m+1}(A) \cong H^{\leqslant m+1}(B)$ under which $[\sigma]*

corresponds to $[\tau]$*, and*

$$A \otimes_\tau \bigwedge P \simeq_q B \otimes_\sigma \bigwedge P.$$

Proof. Assume first that there is a q-equivalence $\varphi \colon A \to B$. Since $q \geqslant m+1$, the induced homomorphism, $\varphi^* \colon H^{\leqslant m+1}(A) \to H^{\leqslant m+1}(B)$, is an isomorphism. Set $\sigma = \varphi \circ \tau$. Then $\varphi^* \circ [\tau] = [\sigma]$. It remains to show that $\varphi \otimes \mathrm{id} \colon A \otimes_\tau \bigwedge P \to B \otimes_\sigma \bigwedge P$ is a q-equivalence. Since every Hirsch extension may be viewed as a finite sequence of elementary extensions with $\dim P = 1$, we can argue by induction on $\dim P$. The induction step follows from exact sequence (2) and the 5-Lemma.

Now assume that there is a q-equivalence $\psi \colon B \to A$. Arguing as above, the map $\psi^* \colon H^{\leqslant m+1}(B) \to H^{\leqslant m+1}(A)$ is an isomorphism. We may then pick a degree one homomorphism $\sigma \colon P \to Z^{\leqslant m+1}(B)$ such that $[\sigma] = (\psi^*)^{-1} \circ [\tau]$. In view of Lemma 2.1, the isomorphism type of the Hirsch extension $B \otimes_\sigma \bigwedge P$ depends only on B and $[\sigma]$, and thus not on the choice of σ. The rest of the argument is exactly as in the first case.

The general case, where A and B are connected by a zig-zag of q-equivalences, follows from the two particular cases handled above. $\square$

3. COHOMOLOGY JUMP LOCI

In this section we review and develop some material from [17, 34, 5]. Unless otherwise mentioned, we continue to work over a fixed field $\Bbbk$ of characteristic 0.

3.1. **Characteristic varieties.**
Let X be a space. We say that X is *q-finite* if it has the homotopy type of a connected CW-complex with finite q-skeleton. For $q = \infty$, this means that X is a finite, connected CW-space; in this case, we will simply say that X is a finite space.

We will assume henceforth that X is q-finite, for some $q \geqslant 1$ (possibly $q = \infty$). In this case, the fundamental group $\pi = \pi_1(X)$ is finitely generated. Fix a linear algebraic group G. The *G-representation variety* of the discrete group π is the set $\mathrm{Hom}_{\mathrm{gr}}(\pi, G)$, viewed in a natural way as an affine algebraic subvariety of $G^{\times m}$, where m is the number of generators of π. This variety comes with a distinguished basepoint 1, the trivial representation.

Now fix also a rational representation $\iota \colon G \to \mathrm{GL}(V)$, where V is a finite-dimensional $\Bbbk$-vector space. Given a representation $\rho \colon \pi \to G$, we may view V as a $\Bbbk[\pi]$-module via the action defined by $\iota \circ \rho$, and thus consider the cohomology groups of X with coefficients in this module. The *characteristic varieties* of X with respect to ι are the jump loci for these twisted cohomology groups. More precisely, in each degree $i \geqslant 0$ and depth $s \geqslant 0$, define

(3) $\mathcal{V}_s^i(X, \iota) = \{\rho \in \mathrm{Hom}(\pi, G) \mid \dim_\Bbbk H^i(X, V_{\iota \circ \rho}) \geqslant s\}.$

These sets are Zariski closed subsets of the representation variety, provided $i \leqslant q$. Note that $\mathcal{V}_s^1(\pi, \iota) := \mathcal{V}_s^1(X, \iota)$ depends only on the fundamental group π, for all s. In the rank

1 case, i.e., when $G = \Bbbk^*$, $V = \Bbbk$, and $\iota\colon \Bbbk^* \to \mathrm{GL}_1(\Bbbk)$ is the standard identification, we will simply write these jump loci as $\mathscr{V}_s^i(X)$, and view them as subvarieties of the character group $\mathrm{Hom}(\pi_1(X), \Bbbk^*) = H^1(X, \Bbbk^*)$.

3.2. Resonance varieties.
We now turn to the infinitesimal counterpart of the above setup. Let $A = (A^\bullet, d)$ be a CDGA. We will assume that A is q-finite, for some $q \geqslant 1$. Furthermore, let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Inside the affine space $A^1 \otimes \mathfrak{g}$, we shall consider the variety $\mathscr{F}(A, \mathfrak{g})$ of $\mathfrak{g}$-*valued flat connections* on A, which consists of all solutions of the Maurer–Cartan equation, $d\omega + \frac{1}{2}[\omega, \omega] = 0$. This variety is natural in both A and $\mathfrak{g}$, and has as natural basepoint the trivial flat connection, 0. If $\omega = \sum_i \eta_i \otimes g_i$, with $\eta_i \in A^1$ and $g_i \in \mathfrak{g}$, the flatness condition amounts to

$$(4) \qquad \sum_i d\eta_i \otimes g_i + \sum_{i<j} \eta_i \eta_j \otimes [g_i, g_j] = 0.$$

Now let $\theta\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation. To each flat connection $\omega \in A^1 \otimes \mathfrak{g}$ there is an associated covariant derivative, $d_\omega\colon A^\bullet \otimes V \to A^{\bullet+1} \otimes V$, given by $d_\omega = d \otimes \mathrm{id}_V + \mathrm{ad}_\omega$, where the adjoint operator ad_ω also depends on θ. Explicitly, if $\omega = \sum_i \eta_i \otimes g_i$, then

$$(5) \qquad d_\omega(\alpha \otimes v) = d\alpha \otimes v + \sum_i \eta_i \alpha \otimes \theta(g_i)(v),$$

for all $\alpha \in A^i$ and $v \in V$. It is readily verified that $d_\omega^2 = 0$, by the flatness condition.

The *resonance varieties* of A with respect to θ (in degree $i \geqslant 0$ and depth $s \geqslant 0$), are the sets

$$(6) \qquad \mathscr{R}_s^i(A, \theta) = \{\omega \in \mathscr{F}(A, \mathfrak{g}) \mid \dim_\Bbbk H^i(A \otimes V, d_\omega) \geqslant s\}.$$

These sets are Zariski-closed subsets of the variety of flat connections, provided that $i \leqslant q$. Let us define the $(q+1)$-truncation of a CDGA A to be the quotient CDGA $A^{\leqslant q+1} := A^\bullet / \bigoplus_{j>q+1} A^j$. By construction, both $\mathscr{F}(A, \mathfrak{g})$ and $\mathscr{R}_s^i(A, \theta)$ depend only on this truncation, for $i \leqslant q$ and $s \geqslant 0$.

In the rank 1 case, i.e., when $\mathfrak{g} = \Bbbk$ and $\theta\colon \Bbbk \to \mathfrak{gl}_1(\Bbbk)$ is the standard identification, we will omit $\mathfrak{g}$ and θ from the notation. In this situation, the variety $\mathscr{F}(A) = \{\omega \in A^1 \mid d\omega = 0\}$ may be identified with $H^1(A)$, by connectedness of A. Moreover, the covariant derivative is now given by $d_\omega \eta = d\eta + \omega \cdot \eta$, for $\eta \in A^\bullet$. In particular, ω belongs to $\mathscr{R}_1^1(A)$ if and only if there is an $\eta \in A^1 \backslash \Bbbk \cdot \omega$ such that $d\eta + \omega \cdot \eta = 0$.

The rank 1 resonance varieties obey the following product formula.

Proposition 3.1. *Let (A, d) and (A', d') be two connected CDGA's. Then*

$$\mathscr{R}_1^q(A \otimes A') = \bigcup_{i+j=q} \mathscr{R}_1^i(A) \times \mathscr{R}_1^j(A').$$

A proof of this statement is given in [41, Prop. 13.1] under the assumption that both d and d' vanish (see also [42, Prop. 2]). The same proof works in this wider generality.

3.3. **Germs of complex jump loci.** We now take the coefficient field to be $\Bbbk = \mathbb{C}$. Let X be a q-finite space with fundamental group π, and assume $\Omega^{\bullet}(X)$ has the same q-type as a q-finite CDGA A. Let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation of a complex linear algebraic group G, and let $\theta = d_1(\iota)\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be its tangential representation at the identity. For an affine variety $\mathcal{X}$ and a point $x \in \mathcal{X}$, we denote by $\mathcal{X}_{(x)}$ the reduced analytic germ of $\mathcal{X}$ at x.

Theorem 3.2 ([17]). *Under the above assumptions, there is an analytic isomorphism* $\mathrm{Hom}(\pi, G)_{(1)} \xrightarrow{\simeq} \mathscr{F}(A, \mathfrak{g})_{(0)}$ *which restricts to analytic isomorphisms* $\mathscr{V}_s^i(X, \iota)_{(1)} \xrightarrow{\simeq} \mathscr{R}_s^i(A, \theta)_{(0)}$, *for all $i \leqslant q$ and $s \geqslant 0$.*

Following [8, 35], we say that a CDGA (A, d) over a field $\Bbbk$ of characteristic zero has *positive weights* if, for each $i \geqslant 0$, there is a vector space decomposition, $A^i = \bigoplus_{\alpha \in \mathbb{Z}} A_\alpha^i$, such that, if we set $\mathrm{wt}(a) = \alpha$ for $a \in A_\alpha^i$, the following conditions hold:

   (1) $\mathrm{wt}(a) > 0$, for all $a \in A_\alpha^i$ with $i > 0$;
   (2) $\mathrm{wt}(da) = \mathrm{wt}(a)$, for all $a \in A_\alpha^i$;
   (3) $\mathrm{wt}(ab) = \mathrm{wt}(a) + \mathrm{wt}(b)$, for all $a \in A_\alpha^i$ and $b \in A_\beta^j$.

For instance, if $d = 0$, we may set $\mathrm{wt}(a) = i$ for all $a \in A^i$. Moreover, if (A, d) has positive weights and a d-cocycle $e \in A^{m+1}$ is homogeneous with respect to those weights, then the Hirsch extension $(A \otimes_e \bigwedge(t), d)$ also has positive weights. Indeed, we simply declare $\mathrm{wt}(t) = \mathrm{wt}(e)$, and extend the weights on the Hirsch extension accordingly. When $d = 0$, it follows that any Hirsch extension $A \otimes_\tau \bigwedge P$ has positive weights.

Now let X be a space, with Sullivan model $\Omega(X) = \Omega(X, \Bbbk)$. We say that a $\Bbbk$-CDGA (A, d) is a *q-model with positive weights* for X if the following conditions are satisfied:

   (1) A is defined over $\mathbb{Q}$;
   (2) A has positive weights over $\mathbb{Q}$;
   (3) $A \simeq_q \Omega(X)$, with the isomorphism induced on first homology by the zig-zag of q-equivalences between A and $\Omega(X)$ required to preserve $\mathbb{Q}$-structures.

As the next result shows, the existence of such a model for a space X imposes stringent conditions on the rank 1 cohomology jump loci of X.

Theorem 3.3 ([17]). *Let X be a q-finite space, and suppose X admits a q-finite, q-model A with positive weights. Then all irreducible components of $\mathscr{V}_s^i(X)$ passing through 1 are algebraic subtori of the character group $H^1(X, \mathbb{C}^*)$, for all $i \leqslant q$ and $s \geqslant 0$.*

Remark 3.4. In a very recent preprint, Budur and Wang [12] remove the positive weights hypothesis from the above theorem.

Remark 3.5. When the Betti numbers vanish, the embedded jump loci are not interesting, at least from the point of view of germs around the origin. Indeed, if $b_1(X) = 0$, then $\mathrm{Hom}(\pi_1(X), G)_{(1)} = \{1\}$, by [17, Theorem A]. Furthermore, if $b_i(A) = 0$ for some $i \leqslant q$, then $\mathscr{R}_1^i(A, \theta)_{(0)} = \varnothing$, by [34, (15)].

3.4. **Malcev completions and holonomy Lie algebras.** As before, let π be a discrete group, and let $\Bbbk$ be a field of characteristic 0. The *Malcev Lie algebra* of π (over $\Bbbk$), denoted $\mathfrak{m}(\pi)$, is a filtered, complete $\Bbbk$-Lie algebra whose filtration satisfies certain axioms, spelled out by Quillen in [45, Appendix A]. In particular, the associated graded Lie algebra of $\mathfrak{m}(\pi)$ with respect to the aforementioned filtration is isomorphic to the associated graded $\Bbbk$-Lie algebra of π with respect to the lower central series (lcs) filtration. For a detailed treatment of the various Lie algebras associated to finitely generated groups we refer to [46], and the further citations therein.

Now let $A = (A^{\bullet}, d)$ be a 1-finite CDGA over $\Bbbk$. Following [34], set $A_i = (A^i)^*$, where $(\cdot)^*$ stands for vector space duals, and let $\mathbb{L}^{\bullet}(A_1)$ be the free Lie algebra on A_1, graded by bracket length. Let $d^* \colon A_2 \to A_1 = \mathbb{L}^1(A_1)$ and $\cup^* \colon A_2 \to A_1 \wedge A_1 = \mathbb{L}^2(A_1)$ be the dual maps to the differential and the product map, respectively. The *holonomy Lie algebra* of A is the finitely presented Lie algebra

$$(7) \qquad \mathfrak{h}(A) = \mathbb{L}(A_1)/\mathrm{ideal}(\mathrm{im}(d^* + \cup^*)).$$

Clearly, this construction is functorial, and depends only on the 2-truncation $A^{\leqslant 2}$.

The following theorem strengthens a result proved by Bezrukavnikov [6] in the case when A is a quadratic algebra.

Theorem 3.6 ([5], Theorem 3.1). *Let X be a 1-finite space, and suppose A is a 1-finite 1-model for X, defined over $\Bbbk$. Then $\mathfrak{m}(\pi_1(X))$ is isomorphic (as a filtered Lie algebra) to the lcs completion of $\mathfrak{h}(A)$, over $\Bbbk$.*

The next result relates the holonomy Lie algebra to the germs of the representation varieties considered in §3.3.

Corollary 3.7. *Let $\pi = \pi_1(X)$ be the fundamental group of a 1-finite space X, and let G be a complex linear algebraic group, with Lie algebra $\mathfrak{g}$. If A is a 1-finite 1-model for X over $\mathbb{C}$, then the analytic germs $\mathrm{Hom}_{\mathrm{gr}}(\pi, G)_{(1)}$ and $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{h}(A), \mathfrak{g})_{(0)}$ are isomorphic.*

Proof. Proposition 4.5 from [34] provides a natural isomorphism of affine varieties between $\mathscr{F}(A, \mathfrak{g})$ and $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{h}(A), \mathfrak{g})$. The claim then follows from Theorem 3.2. □

Corollary 3.7 extends Theorem 17.1 from [28], from the 1-formal case to the much broader class of groups having a 1-finite 1-model.

4. Models for compact Lie group actions: Part I

In this section, we first construct from manifolds admitting finite models new examples of manifolds supporting an almost free action by a compact Lie group, which also have finite models. Secondly, we begin an algebraic study of this construction, related to the behavior of additional properties (existence of positive weights and formality). This leads to new results on the partial formality properties of such manifolds, and related spaces.

4.1. **Almost free actions and Hirsch extensions.** From now on, K will denote a compact, connected, real Lie group. Consider the universal principal K-bundle,

$$(8) \qquad\qquad K \longrightarrow EK \longrightarrow BK \; ,$$

with contractible total space EK and with base space the classifying space for K-bundles, $BK = EK/K$. By a classical result of Hopf, the cohomology ring of K (with coefficients in a characteristic zero field $\Bbbk$) is isomorphic to the cohomology ring of a finite product of odd-dimensional spheres. That is,

$$(9) \qquad\qquad H^\bullet(K) \cong \bigwedge P^\bullet$$

where $P^\bullet$ is an oddly-graded, finite-dimensional vector space, with homogeneous basis $\{t_\alpha \in P^{m_\alpha}\}$, for some odd integers $m_1, \ldots, m_r$, where $r = \operatorname{rank}(K)$.

Now let M be a compact, connected, differentiable manifold on which the compact, connected Lie group K acts smoothly. Both M and the orbit space $N = M/K$ are finite spaces. For N, this follows from triangulability of stratified spaces [23, 51]. We consider the diagonal action of K on the product $EK \times M$, and form the Borel construction,

$$(10) \qquad\qquad M_K = (EK \times M)/K.$$

Let $\operatorname{pr}\colon M_K \to N$ be the map induced by the projection $\operatorname{pr}_2\colon EK \times M \to M$.

The K-action on M is said to be *almost free* if all its isotropy groups are finite. When this assumption is met, work of Allday and Halperin [1] provides a very useful Hirsch extension model for the manifold M.

Theorem 4.1 ([1]). *Suppose M admits an almost free K-action, with orbit space $N = M/K$. There is then a map $\sigma\colon P^\bullet \to Z^{\bullet+1}(\Omega(N))$ such that $\operatorname{pr}^* \circ [\sigma]$ is the transgression in the principal bundle $K \to EK \times M \to M_K$, and*

$$\Omega(M) \simeq \Omega(N) \otimes_\sigma \bigwedge P\,.$$

This theorem may be applied for instance to the total space M of a principal K-bundle over a compact manifold $N = M/K$.

The next result adds a new interesting class of finite spaces having finite cDGA models to the known examples from [17] and [29].

Lemma 4.2. *Let M be an almost free K-manifold. Denote by m the maximum degree of $P^\bullet$, where $\bigwedge P^\bullet = H^\bullet(K)$.*

(1) *Suppose B is a q-finite q-model of the orbit space $N = M/K$, with $q \geqslant m + 1$. Then a suitable Hirsch extension $A = B \otimes_\tau \bigwedge P$ is a q-finite q-model for M.*
(2) *Suppose $N = M/K$ is q-formal. Then we may take $B^\bullet = (H^\bullet(N), d = 0)$, and $A = B \otimes_\tau \bigwedge P$ is a q-finite q-model of M with positive weights.*
(3) *Under the same formality assumption, all irreducible components of $\mathcal{V}_s^i(M)$ containing the origin are algebraic subtori of the character group $H^1(M, \mathbb{C}^*)$, for all $i \leqslant q$ and $s \geqslant 0$.*

Proof. Claim (1) and the first part of claim (2) follow from Theorem 4.1 and Lemma 2.2. The other claims follow from Theorem 3.3 and the discussion preceding it. □

In the case of principal K-bundles, we can say more. Write $H^\bullet(K, \mathbb{Q}) \cong \bigwedge P_K^\bullet = \bigwedge(t_1, \ldots, t_r)$, as before.

Theorem 4.3. *Let N be a finite space and K be a compact connected real Lie group. If N has a finite model B over $\mathbb{Q}$, then any Hirsch extension $A = B \otimes_\tau \bigwedge P_K$ can be realized as a finite model over $\mathbb{Q}$ of some principal K-bundle M over N. When B has positive weights and the image of $[\tau]$ is generated by weighted-homogeneous elements, A also has positive weights.*

Proof. Set $e_\alpha = [\tau(t_\alpha)] \in H^\bullet(B) \equiv H^\bullet(N, \mathbb{Q})$. By [39, Corollary 3.4], there is a principal K-bundle M over N having characteristic classes $\lambda_\alpha \cdot e_\alpha$, $\alpha = 1, \ldots, r$, for some $\lambda_\alpha \in \mathbb{Q}^\times$. The fact that A is a finite model of M over $\mathbb{Q}$ follows from the Hirsch Lemma. The discussion on positive weights from §3.3 completes the proof. □

This result provides a systematic way of constructing new examples of finite spaces with finite models, with interesting non-nilpotent topology. It also gives a simple criterion which ensures that the existence of positive weights on old models passes to the new models. It is known that the stronger formality property may not be inherited by Hirsch extensions of a formal CDGA. In the next subsection, we provide a criterion which solves this issue, for partial formality.

4.2. **Graded regularity and partial formality.** Fix an integer $q \geqslant 0$. Our next theorem uses graded versions of two classical notions from commutative algebra. Although the analogous result for $q = \infty$ is well-known (cf. [25]), we shall need the precise form stated here, as a crucial ingredient in the proofs of Theorems 6.6 and 6.7 below.

Let $H^\bullet$ be a connected commutative graded algebra over a characteristic zero field $\Bbbk$. We say that a homogeneous element $e \in H^k$ is a *non-zero divisor up to degree q* if the multiplication map $e \cdot : H^i \to H^{i+k}$ is injective, for all $i \leqslant q$. (For $q = 0$, this simply means that $e \neq 0$.)

Likewise, we say that a sequence $e_1, \ldots, e_r$ of homogeneous elements in H^+ is *q-regular* if the class of each e_α is a non-zero divisor up to degree $q - \deg(e_\alpha) + 2$ in

the quotient ring $H/\sum_{\beta<\alpha} e_\beta H$. (This implies in particular that the elements $e_1,\ldots,e_r$ are linearly independent over $\Bbbk$, when $q \geqslant \deg(e_\alpha) - 2$ for all α.)

Theorem 4.4. *Suppose $e_1,\ldots,e_r$ is an even-degree, q-regular sequence in $H^\bullet$. Then the Hirsch extension $A = H \otimes_\tau \bigwedge(t_1,\ldots,t_r)$ with $d = 0$ on H and $dt_\alpha = \tau(t_\alpha) = e_\alpha$ has the same q-type as $(H/\sum_\alpha e_\alpha H, d = 0)$. In particular, A is q-formal.*

Proof. The canonical projection $H \twoheadrightarrow H/\sum_\alpha e_\alpha H$ extends to a morphism of graded algebras, $\psi \colon H \otimes \bigwedge(t_1,\ldots,t_r) \to H/\sum_\alpha e_\alpha H$, by setting $\psi(t_\alpha) = 0$. In turn, this morphism defines a CDGA map, $\psi \colon H \otimes_\tau \bigwedge(t_1,\ldots,t_r) \to (H/\sum_\alpha e_\alpha H, d = 0)$. We will show that ψ is a q-equivalence.

Let $\varphi \colon (H, d = 0) \hookrightarrow A$ be the canonical CDGA inclusion, inducing $\varphi^* \colon H^\bullet/\sum_\alpha e_\alpha H \to H^\bullet(A)$. Clearly, ψ^* is surjective and $\psi^* \circ \varphi^* = \mathrm{id}$. It follows that ψ is a q-equivalence if and only if the map $\varphi^* \colon H^i \to H^i(A)$ is surjective for all $i \leqslant q + 1$.

We prove this by induction on the length of our q-regular sequence. Denote by m_α the degree of t_α. Set $A_\alpha = H \otimes_\tau \bigwedge(t_1,\ldots,t_\alpha)$ and note that $A_{\alpha+1} = A_\alpha \otimes_{e_{\alpha+1}} \bigwedge(t_{\alpha+1})$. Suppose that the map φ_α^* is surjective up to degree $q + 1$. To obtain the same property for $\varphi_{\alpha+1}$ at the induction step, it suffices to show that the map $H^i(A_\alpha) \to H^i(A_{\alpha+1})$ is surjective, for all $i \leqslant q + 1$. In view of (2), this is equivalent to showing that the map $[e_{\alpha+1}] \cdot \colon H^{i-m_{\alpha+1}}(A_\alpha) \to H^{i+1}(A_\alpha)$ is injective, for all $i \leqslant q + 1$.

Consider the following commuting diagram

(11)
$$
\begin{array}{ccc}
H^{i-m_{\alpha+1}}/\sum_{\beta\leqslant\alpha} e_\beta H & \xrightarrow{\varphi_\alpha^*} & H^{i-m_{\alpha+1}}(A_\alpha) \\
\Big\uparrow{\scriptstyle e_{\alpha+1}\cdot} & & \Big\downarrow{\scriptstyle [e_{\alpha+1}]\cdot} \\
H^{i+1}/\sum_{\beta\leqslant\alpha} e_\beta H & \xleftarrow{\psi_\alpha^*} & H^{i+1}(A_\alpha)\,.
\end{array}
$$

Since we are assuming that $i \leqslant q + 1$, the q-regularity of our sequence implies that the map $e_{\alpha+1}$ on the left side of (11) is injective. By our induction hypothesis, the map φ_α^* is an isomorphism. By commutativity of (11), the map $[e_{\alpha+1}] \cdot$ must be injective, and this completes the proof. $\qquad\square$

Classical theorems of Borel [9] and Chevalley [15] provide a machine for constructing graded algebras which satisfy the hypothesis of Theorem 4.4, in the case when $q = \infty$.

Construction 4.5. Let $H^\bullet(BK)$ be the cohomology algebra of the classifying space of a compact, connected Lie group K. Let T be a maximal torus in K, and let $W = NT/T$ be the Weyl group. The classifying space BT is the product of r copies of $\mathbb{CP}^\infty$, where r is the rank of K. Its cohomology algebra is $H^\bullet(BT) = \Bbbk[x_1,\ldots,x_r]$, with degree 2 free algebra generators, on which W acts by graded algebra automorphisms.

The natural map $\kappa \colon BT \to BK$ identifies the cohomology algebra $H^\bullet(BK)$ with the invariant subalgebra of the W-action. More precisely, $H^\bullet(BK)$ is isomorphic to a polynomial ring of the form $\Bbbk[f_1, \ldots, f_r]$, where each f_α is a W-invariant polynomial of even degree $m_\alpha + 1$, with m_α as in §4.1. Moreover, $f_1, \ldots, f_r$ forms a regular sequence in $\Bbbk[x_1, \ldots, x_r]$.

Let $U \subseteq K$ be a closed, connected subgroup of a compact, connected Lie group. As shown in [48], the Sullivan minimal model of the homogeneous space K/U is a Hirsch extension of the form $A = H \otimes_\tau \bigwedge(t_1, \ldots, t_s)$, where $H^\bullet$ is a free graded algebra on finitely many even-degree generators, with zero differential, as in Theorem 4.4. As is well-known, not all homogeneous spaces K/U are formal. Nevertheless, the criterion from the aforementioned theorem may be used to obtain valuable information on their partial formality properties.

Example 4.6. For the homogeneous space $\mathrm{Sp}(5)/\mathrm{SU}(5)$, the aforementioned algebra $H^\bullet$ has two free generators, x_6 and x_{10}, where subscripts denote degrees, and the sequence from Theorem 4.4 is $\{x_6^2, x_{10}^2, x_6 x_{10}\}$, see [21, Exercise 3.5, p. 143]. It is straightforward to deduce from our theorem that $\mathrm{Sp}(5)/\mathrm{SU}(5)$ is 19-formal. On the other hand, an easy computation with Massey triple products shows that this estimate is sharp, that is, $\mathrm{Sp}(5)/\mathrm{SU}(5)$ is not 20-formal.

4.3. **Partial formality of K-manifolds.** Now let M be an almost free K-manifold. Using the setup from §4.1, write $H^\bullet(K) = \bigwedge(t_1, \ldots, t_r)$, and denote the transgression of t_α by $e_\alpha \in H^{m_\alpha+1}(M/K)$. As before, set $m = \max\{m_\alpha\}$.

Theorem 4.7. *Suppose the K-action on M is almost free, the orbit space $N = M/K$ is k-formal, for some $k \geqslant m + 1$, and $e_1, \ldots, e_r$ is a q-regular sequence in $H^\bullet(N)$, for some $q \leqslant k$. Then*

$$\left(H^\bullet(N) / \sum_{\alpha=1}^r e_\alpha H^\bullet(N), d = 0 \right)$$

is a finite q-model for M. In particular, M is q-formal.

Proof. Since the action of K on M is almost free, Theorem 4.1 insures that the Hirsch extension $\Omega(N) \otimes_\tau \bigwedge P$ is a model for M. The k-formality assumption on N means that there is a zig-zag of k-equivalences between $\Omega(N)$ and $H(N)$. It follows from [33, Proposition 3.1] that this zig-zag may be chosen to induce the identity in homology, in degrees up to k.

Now, since $k \geqslant m + 1$ and $k \geqslant q$, Lemma 2.2 insures that $\Omega(N) \otimes_\tau \bigwedge P$ has the same q-type as $H(N) \otimes_\sigma \bigwedge(t_1, \ldots, t_r)$, where $[\sigma]$ corresponds to $[\tau]$ under the induced isomorphism $H^{\leqslant m+1}(N) \cong H^{\leqslant m+1}(\Omega(N))$. By the above, this isomorphism is the identity, and thus $\sigma(t_\alpha) = e_\alpha$, for all α.

Using now the q-regularity assumption on the sequence $\{e_\alpha\}$, Theorem 4.4 applies, and we obtain the desired conclusion. $\square$

As illustrated in the next two examples, the q-regularity assumption from Theorem 4.7 is optimal with respect to the q-formality conclusion for the manifold M, when $K = S^1$ or S^3.

Example 4.8. Let $M = \mathscr{H}_1$ be the 3-dimensional Heisenberg nilmanifold from Example 6.8. This manifold is the total space of the principal S^1-bundle over the formal manifold $N = S^1 \times S^1$, with Euler class $e \in H^2(N)$ equal to the orientation class. In this case, the sequence $\{e\}$ is 0-regular, but not 1-regular in $H^\bullet(N)$. In fact, as mentioned previously, M is not 1-formal. As explained in Example 6.8, this is the first manifold in a series, $\mathscr{H}_n$, where $(n-1)$-regularity implies $(n-1)$-formality in an optimal way.

Example 4.9. Let M to be the total space of the principal S^3-bundle over $N = S^2 \times S^2$ obtained by pulling back the Hopf bundle $S^7 \to S^4$ along a degree-one map $N \to S^4$. As above, N is formal, and the Euler class $e \in H^4(N)$ is the orientation class. In this case, $\{e\}$ is 3-regular, but not 4-regular in $H^\bullet(N)$, and Theorem 4.7 says that M is 3-formal. Direct computation with the minimal model of M shows that, in fact, M is not 4-formal.

In general, though, our lower bound for the q-formality of manifolds with almost free K-actions is not sharp. We use Construction 4.5 to illustrate this point.

Example 4.10. Let K be a compact, connected Lie group of rank r, and identify $H^\bullet(BK)$ with $\Bbbk[f_1, \ldots, f_r]$. For any $n \geqslant 1$, let M_n be the principal K-bundle over the formal manifold $N = (\mathbb{CP}^n)^{\times r}$ classified by the restriction of the natural map $\kappa\colon BT \to BK$ to N. It is immediate to check that the inclusion $j\colon N \hookrightarrow BT$ induces an isomorphism in cohomology, up to degree $2n + 1$. Furthermore, it follows from Construction 4.5 that the sequence $e_\alpha = j^*(f_\alpha)$, $\alpha = 1, \ldots, r$ is $(2n-1)$-regular in $H^\bullet(N)$. Therefore, by Theorem 4.7, the manifold M_n is $(2n-1)$-formal.

On the other hand, the Hirsch extension $H^\bullet(BT) \otimes_f \bigwedge P_K$ is fully formal, by the classical case $q = \infty$ of Theorem 4.4. The argument from Lemma 2.2 shows that the CDGA map $j^* \otimes \mathrm{id}\colon H^\bullet(BT) \otimes_f \bigwedge P_K \to H^\bullet(N) \otimes_e \bigwedge P_K$ is a $2n$-equivalence. Hence, M_n is actually $2n$-formal.

5. Models for compact Lie group actions: Part II

We continue with the setup from the previous section, and investigate several algebraic and geometric objects associated to a closed manifold M endowed with an almost free action by a compact Lie group K, to wit, the Malcev Lie algebra and the representation varieties of $\pi_1(M)$, as well as the rank 1 cohomology jump loci of M.

5.1. **Malcev completion and representation varieties.** Let H be a 2-finite CDGA with zero differential, and let $A = H \otimes_\tau \bigwedge P$ be a Hirsch extension, where P is an oddly-graded, finite-dimensional vector space.

Theorem 5.1. *The holonomy Lie algebra $\mathfrak{h}(A)$ admits a finite presentation with generators in degree 1 and relations in degrees 2 and 3.*

Proof. Clearly, the CDGAS $H \otimes_\tau \bigwedge P$ and $H \otimes_\tau \bigwedge P^1$ have the same 2-truncation; hence, we may assume that $P = P^1$. Pick bases $\{t_\alpha\} \cup \{s_\beta\}$ for P^1 and $\{e_\beta\} \cup \{f_\gamma\}$ for H^2 such that $d_A(t_\alpha) = 0$ and $d_A(s_\beta) = e_\beta$. Let $\{h_i\}$ be a basis for H^1, and denote by $\{(\cdot)^*\}$ the dual bases. Plainly, $A^1 = H^1 \oplus P$ and $A^2 = H^2 \oplus \bigwedge^2 P \oplus (H^1 \otimes P)$.

By construction, the map d_A^* is zero on A_2, except for $d_A^*(e_\beta^*) = s_\beta^*$. Moreover, we have the decomposition $\bigwedge^2 A^1 = \bigwedge^2 H^1 \oplus \bigwedge^2 P \oplus (H^1 \otimes P)$. Again by construction, $\cup_A = \cup_H$ on the first summand, and $\cup_A = \mathrm{id}$ for the others. Set $u_\beta = \cup_H^* e_\beta^* \in \mathbb{L}^2(h_i^*)$ and $v_\gamma = \cup_H^* f_\gamma^* \in \mathbb{L}^2(h_i^*)$.

By (7), the Lie algebra $\mathfrak{h}(A)$ is generated by $\{t_\alpha^*\} \cup \{s_\beta^*\} \cup \{h_i^*\}$, with the following defining relations: $s_\beta^* + u_\beta = 0 \, (\forall \beta)$; $v_\gamma = 0 \, (\forall \gamma)$; $[t_\alpha^*, t_{\alpha'}^*] = 0 \, (\forall \alpha, \alpha')$; $[s_\beta^*, s_{\beta'}^*] = 0 \, (\forall \beta, \beta')$; $[t_\alpha^*, s_\beta^*] = 0 \, (\forall \alpha, \beta)$; $[h_i^*, t_\alpha^*] = 0 \, (\forall i, \alpha)$; and $[h_i^*, s_\beta^*] = 0 \, (\forall i, \beta)$.

Using the first batch of relations to eliminate the generators $\{s_\beta^*\}$, we find that $\mathfrak{h}(A)$ is generated in degree 1 by $\{t_\alpha^*\} \cup \{h_i^*\}$, and has the following relators:

(I)	v_γ	$(\forall \gamma)$,
(II)	$[t_\alpha^*, t_{\alpha'}^*]$	$(\forall \alpha, \alpha')$,
(III)	$[h_i^*, t_\alpha^*]$	$(\forall i, \alpha)$,
(IV)	$[h_i^*, u_\beta]$	$(\forall i, \beta)$,
(V)	$[t_\alpha^*, u_\beta]$	$(\forall \alpha, \beta)$,
(VI)	$[u_\beta, u_{\beta'}]$	$(\forall \beta, \beta')$.

We claim that relations (VI) and (V) follow from the others. Indeed, $u_{\beta'} \in \mathbb{L}^2(H_1)$ and, for any i, j, the relation $[[h_i^*, h_j^*], u_\beta] = [h_i^*, [h_j^*, u_\beta]] - [h_j^*, [h_i^*, u_\beta]]$ is a consequence of (IV). Similarly, relations (V) may be eliminated using (III).

Of the remaining relations, (I)–(III) are quadratic, while (IV) are cubic. This completes the proof.   □

Corollary 5.2. *Suppose M supports an almost free K-action with 2-formal orbit space. Then:*

(1) The group $\pi = \pi_1(M)$ is filtered-formal. More precisely, the Malcev Lie algebra $\mathfrak{m}(\pi)$ is isomorphic to the lcs completion of the quotient $\mathbb{L}(H_1(\pi, \Bbbk))/\mathfrak{r}$, where $\mathfrak{r}$ is a homogeneous ideal generated in degrees 2 and 3.

(2) For every complex linear algebraic group G, the germ at the origin of the representation variety $\mathrm{Hom}_{\mathrm{gr}}(\pi, G)$ is defined by quadrics and cubics only.

Proof. By Theorem 4.1, the Hirsch extension $\Omega(N) \otimes_\tau \bigwedge P^1$ is a 1-model for M. In view of Lemma 2.2, the Hirsch extension $A^\bullet = H^\bullet(N) \otimes_\tau \bigwedge P^1$ is a 1-finite 1-model of M. Claim (1) now follows from Theorems 3.6 and 5.1. Finally, claim (2) is a consequence of Corollary 3.7 and Theorem 5.1.   □

The second statement in the above corollary is analogous to the quadraticity obstruction for fundamental groups of compact Kähler manifolds obtained by Goldman–Millson in [22, Theorem 1]. Clearly, the corollary applies to principal K-bundles over formal manifolds.

5.2. Flat connections and representation varieties.
We recall from [34] that, for any 1-finite CDGA A and any finite-dimensional Lie algebra $\mathfrak{g}$, the set

$$(12) \qquad \mathscr{F}^1(A, \mathfrak{g}) = \{\omega = \eta \otimes g \in A^1 \otimes \mathfrak{g} \mid d\eta = 0\}$$

is a Zariski closed, homogeneous subvariety of the variety of $\mathfrak{g}$-valued flat connections, $\mathscr{F}(A, \mathfrak{g})$. The subvariety $\mathscr{F}^1(A, \mathfrak{g})$ may be called the 'trivial part' of $\mathscr{F}(A, \mathfrak{g})$, since it is completely determined by the vector spaces $H^1(A)$ and $\mathfrak{g}$.

Now suppose $A = B \otimes_\tau \bigwedge P$ is a Hirsch extension of a 1-finite CDGA (B, d), and let $\varphi\colon B \hookrightarrow A$ be the canonical CDGA inclusion. By naturality, we have an inclusion,

$$(13) \qquad \mathscr{F}(A, \mathfrak{g}) \supseteq \mathscr{F}^1(A, \mathfrak{g}) \cup \varphi^!(\mathscr{F}(B, \mathfrak{g})).$$

Proposition 5.3. *If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{sl}_2$, then (13) becomes an equality.*

Proof. Key to our proof is the easily checked fact that two matrices $g, g' \in \mathfrak{sl}_2$ commute if and only if $\mathrm{rank}\{g, g'\} \leqslant 1$. Pick any $\omega \in \mathscr{F}(A, \mathfrak{g}) \backslash \mathscr{F}^1(A, \mathfrak{g})$. We have to show that $\omega \in \varphi^!(\mathscr{F}(B, \mathfrak{g}))$. Write $\omega = \eta + \eta'$, with $\eta \in B^1 \otimes \mathfrak{g}$ and $\eta' \in P^1 \otimes \mathfrak{g}$. Note that

$$A^2 = B^2 \oplus \bigwedge{}^2 P^1 \oplus (B^1 \otimes P^1).$$

Under this decomposition, the three components of the Maurer–Cartan equation (4) are:

$$d\eta' + d\eta + \tfrac{1}{2}[\eta, \eta] = 0; \quad [\eta', \eta'] = 0; \quad [\eta, \eta'] = 0.$$

Write $\eta = \sum_i h_i \otimes g_i$ and $\eta' = \sum_\alpha t_\alpha \otimes g'_\alpha$, with respect to some fixed bases for B^1 and P^1. The last two components of the flatness condition for ω then become

$$[g'_\alpha, g'_\beta] = 0, \ \forall \, \alpha, \beta; \quad [g'_\alpha, g_i] = 0, \ \forall \, \alpha, i.$$

If $\eta' \neq 0$, these conditions force $\omega \in \mathscr{F}^1(A, \mathfrak{g})$, a contradiction. Hence, $\eta' = 0$. The first component of the flatness condition for ω then becomes $d\eta + \tfrac{1}{2}[\eta, \eta] = 0$. Therefore, $\omega = \varphi^!(\eta)$, with $\eta \in \mathscr{F}(B, \mathfrak{g})$, and this completes the proof.   □

As we shall see later on, the property from the above proposition has the same flavor as similar results which hold in the context of smooth complex algebraic varieties. Proposition 5.3 also has the following topological interpretation.

Remark 5.4. Assume that K acts almost freely on M, with formal orbit space N. Working over $\mathbb{C}$, we have that $(H^\bullet(N), d = 0)$ is a finite model for N, while $H^\bullet(N) \otimes_\tau \bigwedge P$ is a finite model for M, by Lemma 4.2. Moreover, the morphism $\varphi \colon H \hookrightarrow A$ models the canonical projection, $p \colon M \to N$. Let G be either $\mathrm{SL}_2(\mathbb{C})$ or a Borel subgroup, with Lie algebra $\mathfrak{g}$. Theorem 3.2 and Proposition 5.3 then say that 'the non-trivial part of $\mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), G)_{(1)}$ pulls back from N, via the map p.' We refer to [43] for the precise description of this representation variety near 1, in the case when M is a principal K-bundle over N.

5.3. **Rank** 1 **cohomology jump loci.** We now turn our attention to the jump loci associated to algebraic models. We start with rank 1 resonance.

Proposition 5.5. *Let B be a connected* CDGA. *Fix an element $e \in B^2$ with $de = 0$, and let $A = (B \otimes_e \bigwedge(t), d)$ be the corresponding Hirsch extension.*

(1) If $[e] = 0$, then $\mathscr{R}_1^i(A) = \mathscr{R}_1^{i-1}(B) \cup \mathscr{R}_1^i(B)$, for all i.
(2) If $[e] \neq 0$, then
    (a) $\mathscr{R}_s^i(A) \subseteq \mathscr{R}_1^{i-1}(B) \cup \mathscr{R}_s^i(B)$, for all i and s.
    (b) $\mathscr{R}_s^1(A) = \mathscr{R}_s^1(B)$, for all s.

Proof. First suppose $[e] = 0$. Then A is isomorphic to $B \otimes (\bigwedge(t), d = 0)$. Applying the product formula for resonance varieties from Proposition 3.1, claim (1) follows.

Now suppose $[e] \neq 0$. Denoting by $\varphi \colon B \to A$ the canonical CDGA inclusion, an easy computation shows that $H^1(\varphi) \colon H^1(B) \to H^1(A)$ is an isomorphism, since both A and B are connected. Thus, the varieties of rank 1 flat connections on A and B may be identified.

For $\omega \in \mathscr{F}(A) \equiv \mathscr{F}(B)$, the cochain complex $(B^\bullet, d_\omega)$ is clearly a subcomplex of $(A^\bullet, d_\omega)$. Using the description of d_ω from §3.2, it is straightforward to check that the quotient complex can be identified with $(B^{\bullet-1}, -d_\omega)$. Claim (2a) then follows by examining the associated long exact sequence in homology,

$$(14) \qquad \cdots \longrightarrow H^{i-2}(B, d_\omega) \longrightarrow H^i(B, d_\omega) \longrightarrow H^i(A, d_\omega) \longrightarrow H^{i-1}(B, d_\omega) \longrightarrow \cdots .$$

For Claim (2b), the relevant portion of the long exact sequence is

$$(15) \qquad 0 \longrightarrow H^1(B, d_\omega) \longrightarrow H^1(A, d_\omega) \longrightarrow H^0(B, d_\omega) \longrightarrow \cdots .$$

If $\omega \neq 0$, then $H^0(B, d_\omega) = 0$ by connectedness of B, and therefore φ induces an isomorphism between $H^1(B, d_\omega)$ and $H^1(A, d_\omega)$. The same thing happens when $\omega = 0$, since $[e] \neq 0$ by assumption. The claim follows. $\square$

We may take in the above proposition $B = (H(\Sigma_g), d = 0)$, where Σ_g is a compact Riemann surface of genus $g > 0$, and $e \in H^2(\Sigma_g)$ equal to the orientation class.

Corollary 5.6. *Let $A = (H^\bullet(\Sigma_g) \otimes_e \bigwedge(t), d)$ be the corresponding Hirsch extension. Then $\mathscr{R}_1^1(A) = \{0\}$ if $g = 1$, and $\mathscr{R}_1^1(A) = H^1(\Sigma_g)$ if $g > 1$.*

Proof. It is immediate to check that $\mathscr{R}_1^1(B) = \{0\}$ for $g = 1$ and $\mathscr{R}_1^1(B) = H^1(\Sigma_g)$ for $g > 1$. □

5.4. **Orbifold fundamental groups.** Imposing weaker assumptions in Theorem 4.7, we can still derive interesting topological consequences. Let $f\colon \pi_1 \twoheadrightarrow \pi_2$ be an epimorphism between finitely generated groups, and let $\iota\colon G \to \mathrm{GL}(V)$ be a rational representation of a linear algebraic group. We then have a natural closed algebraic embedding, $f^!\colon \mathrm{Hom}(\pi_2, G) \hookrightarrow \mathrm{Hom}(\pi_1, G)$. An application of the Hochschild–Serre spectral sequence (see e.g. [11, §VII.6]) shows that the morphism $f^!$ preserves characteristic varieties in degree 1 and induces closed algebraic embeddings, $\mathscr{V}_s^1(\pi_2, \iota) \hookrightarrow \mathscr{V}_s^1(\pi_1, \iota)$, for all values of s.

When M is an almost free K-manifold, we recall from [10, Theorem 4.3.18] that the projection $p\colon M \to M/K$ induces a natural epimorphism $f\colon \pi_1(M) \twoheadrightarrow \pi_1^{\mathrm{orb}}(M/K)$ between orbifold fundamental groups.

Theorem 5.7. *Suppose that the K-action on M is almost free and the transgression $P^\bullet \to H^{\bullet+1}(M_K) \equiv H^{\bullet+1}(M/K)$ is injective in degree 1. Then the following hold.*

(1) If the orbit space $N = M/K$ has a 2-finite 2-model over $\Bbbk \subseteq \mathbb{C}$, then the homomorphism $f\colon \pi_1(M) \twoheadrightarrow \pi_1^{\mathrm{orb}}(N)$ induces an analytic isomorphism between $\mathscr{V}_s^1(\pi_1^{\mathrm{orb}}(N))_{(1)}$ and $\mathscr{V}_s^1(\pi_1(M))_{(1)}$, for all s.

(2) If N is 2-formal, then f induces an analytic isomorphism between the germs at 1 of $\mathrm{Hom}(\pi_1^{\mathrm{orb}}(N), G)$ and $\mathrm{Hom}(\pi_1(M), G)$, for $G = \mathrm{SL}_2(\mathbb{C})$ or a Borel subgroup.

Proof. Let B be a 2-finite 2-model of N. By Theorem 4.1, $\Omega(M) \simeq_1 \Omega(N) \otimes_\sigma \bigwedge P^1$. By Lemma 2.2, the Hirsch extension $A = B \otimes_\tau \bigwedge P^1$ is a 1-finite 1-model for $\pi_1(M)$. Using [10, pp. 117–118], we deduce that $\pi_1^{\mathrm{orb}}(N)$ and $\pi_1(N)$ share the same 1-minimal model. Therefore, B is a 1-finite 1-model for both $\pi_1(N)$ and $\pi_1^{\mathrm{orb}}(N)$. We infer from Theorem 3.2 that

$$\mathscr{V}_s^1(\pi_1(M))_{(1)} \cong \mathscr{R}_s^1(A)_{(0)} \quad \text{and} \quad \mathscr{V}_s^1(\pi_1^{\mathrm{orb}}(N))_{(1)} \cong \mathscr{R}_s^1(B)_{(0)}.$$

Now choose a basis $\{t_1, \dots, t_r\}$ of P^1, and set $e_\alpha = \tau(t_\alpha)$. Our hypothesis on the transgression implies that the classes $[e_1], \dots, [e_r]$ are linearly independent in $H^2(B)$. Let $\varphi\colon B \hookrightarrow A$ be the canonical CDGA inclusion. Setting $A_\alpha = B \otimes_\tau \bigwedge(t_1, \dots, t_\alpha)$, we infer from Proposition 5.5(2b) that the map $\varphi^*\colon H^1(B) \to H^1(A)$ is an isomorphism, and that $\mathscr{R}_s^1(A_\alpha) = \mathscr{R}_s^1(A_{\alpha+1})$. Hence, $\mathscr{R}_s^1(B) = \mathscr{R}_s^1(A)$, for all s. Consequently, the germs $\mathscr{V}_s^1(\pi_1(M))_{(1)}$ and $\mathscr{V}_s^1(\pi_1^{\mathrm{orb}}(N))_{(1)}$ have the same analytic local ring $R_M \cong R_N$ (up to isomorphism).

Moreover, the morphism $f^!\colon \mathcal{V}_s^1(\pi_1^{\mathrm{orb}}(N))_{(1)} \hookrightarrow \mathcal{V}_s^1(\pi_1(M))_{(1)}$ induces a surjection, $f^!\colon R_M \twoheadrightarrow R_N$ between the corresponding analytic local rings. By the well-known Hopfian property for Noetherian rings (see for instance [50, p. 65]), the map $f^!$ must be an isomorphism, which completes the proof of claim (1).

To prove claim (2), we use a similar argument, with the CDGA $H^\bullet := (H^\bullet(N), d = 0)$ replacing B. By Theorem 3.2, there exist isomorphisms $\mathrm{Hom}(\pi_1(M), G)_{(1)} \cong \mathscr{F}(A, \mathfrak{g})_{(0)}$ and $\mathrm{Hom}(\pi_1^{\mathrm{orb}}(N), G)_{(1)} \cong \mathscr{F}(H, \mathfrak{g})_{(0)}$. By Proposition 5.3, $\mathscr{F}(A, \mathfrak{g})$ equals $\mathscr{F}^1(A, \mathfrak{g}) \cup \varphi^!(\mathscr{F}(H, \mathfrak{g}))$. Since φ^* is an isomorphism in degree one, $\mathscr{F}^1(A, \mathfrak{g}) = \varphi^!(\mathscr{F}^1(H, \mathfrak{g}))$. Therefore, $\mathscr{F}(A, \mathfrak{g}) = \varphi^!(\mathscr{F}(H, \mathfrak{g})) \cong \mathscr{F}(H, \mathfrak{g})$. We may then take germs at 0 and conclude as before, by a Hopfian argument. $\qquad\square$

Example 5.8. As usual, let K be a compact, connected Lie group, and identify $H^\bullet(K, \mathbb{Q})$ with $\bigwedge P_K^\bullet$. Let N be a compact, formal manifold, and assume $b_2(N) \geqslant s$, where $s = \dim P_K^1$ (for instance, take N to be the product of at least s compact Kähler manifolds). There is then a degree-preserving linear map, $\tau\colon P_K^\bullet \to H^{\bullet+1}(N, \mathbb{Q})$, which is injective in degree 1. By Theorem 4.3, such a map can be realized as the transgression in a principal K-bundle $M_\tau \to N$, and the manifold M_τ satisfies the assumptions from Theorem 5.7.

Theorem 5.7 may also be applied to a Seifert fibered 3-manifold with non-zero Euler class, $p\colon M \to M/S^1 = \Sigma_g$. We obtain the following relations between rank 1 jump loci and rank 2 representation varieties of M and the corresponding, intensively studied, objects for the compact Riemann surface Σ_g.

Corollary 5.9. *In the above setup, $\mathcal{V}_s^1(M)_{(1)}$ is isomorphic to $\mathcal{V}_s^1(\Sigma_g)_{(1)}$, for all s, while $\mathrm{Hom}(\pi_1(M), G)_{(1)} \cong \mathrm{Hom}(\pi_1(\Sigma_g), G)_{(1)}$, for $G = \mathrm{SL}_2(\mathbb{C})$ or a Borel subgroup.*

6. Sasakian manifolds

In this section, we apply the results from the preceding chapter to obtain significant consequences for the topology of compact Sasakian manifolds, related to formality properties, representation varieties and cohomology jump loci.

6.1. **Sasakian geometry.** A general reference for Sasakian geometry is the monograph of Boyer and Galicki [10].

Let M^{2n+1} be a compact Sasakian manifold of dimension $2n + 1$. By work of Ornea and Verbitsky [38], we may assume that the Sasakian structure is quasi-regular. A basic structural result in Sasakian geometry (Theorem 7.1.3 from [10]) guarantees that, in this case, M supports an almost free circle action. Furthermore, the quotient space, $N = M/S^1$, is a compact Kähler orbifold, with Kähler class $h \in H^2(N, \mathbb{Q})$ satisfying the Hard Lefschetz property, that is, multiplication by h^k defines an isomorphism

$$(16) \qquad\qquad H^{n-k}(N) \xrightarrow{\;\cong\;} H^{n+k}(N)$$

for each $1 \leqslant k \leqslant n$; see [10, Proposition 7.2.2 and Theorem 7.2.9].

The thesis of Tievsky [49, §4.3] provides a very useful model for a Sasakian manifold.

Theorem 6.1 ([49]). *Every compact Sasakian manifold M admits as a finite model over* $\mathbb{R}$ *the Hirsch extension*

$$A^\bullet(M) = (H^\bullet(N) \otimes_h \bigwedge(t), d),$$

where d is zero on $H^\bullet(N)$ *and* $dt = h$*, the Kähler class of N.*

Sasakian geometry is an odd-dimensional analogue of Kähler geometry. From this point of view, the above theorem is a rough analogue of the main result on the algebraic topology of compact Kähler manifolds from [16], guaranteeing that such manifolds are formal. Theorem 6.1 only says that M behaves like an almost free compact S^1-manifold with formal orbit space. A recent result from [4] establishes the formality of the orbifold de Rham algebra of a compact Kähler orbifold. Unfortunately, this is not enough for applying Theorem 4.7, since the authors of [4] do *not* prove that the orbifold de Rham algebra is weakly equivalent to the Sullivan de Rham algebra.

By construction, the Tievsky model $A^\bullet(M)$ is a real CDGA defined over $\mathbb{Q}$. Nevertheless, it does *not* follow from [49] that $A^\bullet(M)$ is a model for M over $\mathbb{Q}$. To understand why that is the case, let us make a parenthetical remark.

Remark 6.2. It follows from Sullivan's work [47] that there exist smooth manifolds that have non-quasi-isomorphic rational models which become quasi-isomorphic when tensored with $\mathbb{R}$ (for concrete examples of this sort, see [37]). Such failure of descent from real homotopy type to rational homotopy type may even occur with models endowed with 0-differentials.

However, we can say something very useful regarding rational models for Sasakian manifolds. We start with a lemma, and will come back to this point in Theorem 6.7.

Lemma 6.3. *The Tievsky model* $A^\bullet_{\mathbb{R}}(M) = (H^\bullet(N, \mathbb{R}) \otimes_h \bigwedge(t), d)$ *is a finite model with positive weights for M, in the sense from* §3.3.

Proof. By Theorem 6.1, we have that $\Omega(M, \mathbb{R}) \simeq A_{\mathbb{R}}$, where $A_{\Bbbk} = H_{\Bbbk} \otimes_h \bigwedge(t)$ and $H_{\Bbbk}$ denotes the CDGA $(H^\bullet(N, \Bbbk), d = 0)$. Note that $\Omega(M, \mathbb{R})$, $A_{\mathbb{R}}$, and $H_{\mathbb{R}}$ all have rational forms, namely, $\Omega(M, \mathbb{Q})$, $A_{\mathbb{Q}}$, and $H_{\mathbb{Q}}$. The fact that $A_{\mathbb{R}}$ has positive weights follows from the discussion in §3.3. We only need to show that Tievsky's identification, $H^1(A_{\mathbb{R}}) \equiv H^1(M, \mathbb{R})$, preserves $\mathbb{Q}$-structures.

Let $\varphi \colon H_{\mathbb{R}} \to A_{\mathbb{R}}$ be the canonical CDGA inclusion. It follows from the construction of the Tievsky model that the homomorphism $\varphi^* \colon H^\bullet(N, \mathbb{R}) \to H^\bullet(A_{\mathbb{R}}) \equiv H^\bullet(M, \mathbb{R})$ coincides with the induced homomorphism $p^* \colon H^\bullet(N, \mathbb{R}) \to H^\bullet(M, \mathbb{R})$, where $p \colon M \to N = M/S^1$ is the canonical projection. In particular, the respective $\mathbb{Q}$-structures are preserved. Moreover, since $h \neq 0$, the map $\varphi^* \colon H^1(N, \mathbb{R}) \to H^1(A_{\mathbb{R}})$ is an isomorphism, and we are done. □

Applying now Theorem 3.3, we obtain the following corollary.

Corollary 6.4. *Let M be a compact Sasakian manifold. For each $i, s \geqslant 0$, all irreducible components of the characteristic variety $\mathcal{V}_s^i(M)$ passing through 1 are algebraic subtori of the character group $H^1(M, \mathbb{C}^*)$.*

A well-known, direct relationship between Kähler and Sasakian geometry is as follows. Let N be a compact Kähler manifold such that the Kähler class is integral, i.e., $h \in H^2(N, \mathbb{Z})$, and let M be the total space of the principal S^1-bundle classified by h. Then M is a regular Sasakian manifold. A concrete class of examples is provided by the Heisenberg manifolds $\mathcal{H}_n$ from Example 6.8 below.

6.2. **Partial formality of Sasakian manifolds.** Let M^{2n+1} be a compact Sasakian manifold, with fundamental group $\pi = \pi_1(M)$. A basic question one can ask is: Is the group π (or, equivalently, the manifold M) 1-formal? When $n = 1$, clearly the answer is negative, a simple example being provided by the Heisenberg manifold $\mathcal{H}_1$. In [30, Theorem 1.1], H. Kasuya claims that the case $n = 1$ is exceptional, in the following sense.

Claim 6.5. *Every compact Sasakian manifold of dimension $2n + 1$ is 1-formal over $\mathbb{R}$, provided $n > 1$.*

It turns out that the proof from [30] has a gap, which we now proceed to explain. Given a CDGA A, the (degree 2) decomposable part is the subspace $DH^2(A) \subseteq H^2(A)$ defined as the image of the product map in homology, $H^1(A) \wedge H^1(A) \to H^2(A)$. What Kasuya actually shows is that

(17) $$DH^2(\mathcal{M}_1(M)) = H^2(\mathcal{M}_1(M)),$$

for a compact Sasakian manifold M^{2n+1} with $n > 1$, where $\mathcal{M}_1(M)$ is the Sullivan 1-minimal model of M, over $\mathbb{R}$.

Equality (17) is an easy consequence of 1-formality. Kasuya deduces the 1-formality of M from (17), by invoking in [30, Proposition 4.1] as a crucial tool Lemma 3.17 from [2]. Unfortunately, though, this lemma is false, as shown by Măcinic in Example 4.5 and Remark 4.6 from [33]. Nevertheless, the next theorem proves Claim 6.5 in a stronger form, while also recovering equality (17).

Theorem 6.6. *Every compact Sasakian manifold M of dimension $2n+1$ is $(n-1)$-formal, over an arbitrary field $\Bbbk$ of characteristic 0.*

Proof. Let $N = M/S^1$. By Theorem 6.1, the manifold M admits the Tievsky model $A_{\mathbb{C}} = (H^\bullet(N) \otimes_h \bigwedge(t), d)$, with $d = 0$ on $H^\bullet(N)$ and $dt = h$. Recall now that the Kähler orbifold N satisfies the Hard Lefschetz Theorem, as explained in (16). It follows that the sequence $\{h\}$ is $(n-1)$-regular in $H^\bullet(N)$, in the sense from §4.2. Hence, by Theorem 4.4, the manifold M is $(n-1)$-formal over $\mathbb{C}$. By descent of partial formality (cf. [46]), M is $(n-1)$-formal over $\mathbb{Q}$, and hence over $\Bbbk$. □

The next result makes Theorem 6.6 more precise, by constructing an explicit finite, $(n-1)$-model with zero differential for M over an *arbitrary* field of characteristic 0.

Theorem 6.7. *Let M be a compact Sasakian manifold M of dimension $2n + 1$. The Sullivan model of M over a field $\Bbbk$ of characteristic 0 has the same $(n-1)$-type over $\Bbbk$ as the CDGA $(H^\bullet(N)/h \cdot H^\bullet(N), d = 0)$, where $N = M/S^1$ and $h \in H^2(N, \Bbbk)$ is the Kähler class.*

Proof. As before, let $\Omega(M, \Bbbk)$ be the Sullivan model of M over $\Bbbk$, let $A_\mathbb{R}$ be Tievsky model of M over $\mathbb{R}$, and let $H_\Bbbk = (H^\bullet(N, \Bbbk), d = 0)$. Recall from the proof of Theorem 4.4 that there is a CDGA map $\psi \colon A_\mathbb{R} \to (H_\mathbb{R}/hH_\mathbb{R}, d = 0)$ which induces a graded ring isomorphism between the truncations $H^{\leqslant n}(A_\mathbb{R})$ and $(H_\mathbb{R}/hH_\mathbb{R})^{\leqslant n}$, with inverse induced by φ^*. Identify the graded rings $H^{\leqslant n}(M, \mathbb{R})$ and $H^{\leqslant n}(A_\mathbb{R})$, using the zig-zag of quasi-isomorphisms provided by the weak equivalence $\Omega(M, \mathbb{R}) \simeq A_\mathbb{R}$. The proof of Lemma 6.3 shows that the composed graded ring isomorphism, $H^{\leqslant n}(M, \mathbb{R}) \cong (H_\mathbb{R}/hH_\mathbb{R})^{\leqslant n}$, respects $\mathbb{Q}$-structures. Therefore, we obtain an isomorphism of graded rings,

$$\text{(18)} \qquad\qquad H^{\leqslant n}(M, \mathbb{Q}) \cong (H_\mathbb{Q}/hH_\mathbb{Q})^{\leqslant n}.$$

The following version of the truncation construction from §3.2 will be helpful in the sequel. Let A be a connected CDGA over a field of characteristic zero, and let $q \geqslant 0$. Write $A^{q+1} = Z^{q+1}(A) \oplus U^{q+1}$. Plainly, $U^{q+1} \oplus \bigoplus_{j > q+1} A^j$ is a differential ideal. Denote by $A[q + 1]$ the quotient CDGA, and let $\kappa \colon A \to A[q + 1]$ be the canonical CDGA projection. It is immediate to check that the map $\kappa^* \colon H^\bullet(A) \to H^\bullet(A[q + 1])$ is an isomorphism up to degree $q + 1$, and that $A^\bullet[q + 1] = 0$ in degrees $> q + 1$. In particular, $A \simeq_q A[q + 1]$ and the truncated cohomology ring $H^{\leqslant q+1}(A)$ is isomorphic to $H^\bullet(A[q + 1])$.

Let $\mathscr{M}_\mathbb{Q}$ be the minimal model of $\Omega(M, \mathbb{Q})$. As we explained previously, $\Omega(M, \mathbb{Q}) \simeq_{n-1} \mathscr{M}_\mathbb{Q}[n]$. This clearly implies that $(H^\bullet(M, \mathbb{Q}), d = 0) \simeq_{n-1} (H^\bullet(\mathscr{M}_\mathbb{Q}[n]), d = 0)$. On the other hand, $\Omega(M, \mathbb{Q}) \simeq_{n-1} (H^\bullet(M, \mathbb{Q}), d = 0)$, by partial formality over $\mathbb{Q}$. Hence,

$$\Omega(M, \mathbb{Q}) \simeq_{n-1} (H^\bullet(\mathscr{M}_\mathbb{Q}[n]), d = 0)$$

$$\text{(19)} \qquad\qquad \cong (H^{\leqslant n}(M, \mathbb{Q}), d = 0)$$

$$\cong ((H_\mathbb{Q}/hH_\mathbb{Q})^{\leqslant n}, d = 0),$$

where the last two isomorphisms are given by modified truncation and (18). Plainly, $(H_\mathbb{Q}/hH_\mathbb{Q}, d = 0) \simeq_{n-1} ((H_\mathbb{Q}/hH_\mathbb{Q})^{\leqslant n}, d = 0)$, by standard truncation. Putting things together, we conclude that $\Omega(M, \mathbb{Q}) \simeq_{n-1} (H_\mathbb{Q}/hH_\mathbb{Q}, d = 0)$. By extension of scalars, the same conclusion holds over $\Bbbk$. This completes the proof of the theorem. $\qquad\square$

As illustrated by the next example, the conclusion of Theorem 6.6 is optimal.

Example 6.8. Let $E = S^1 \times S^1$ be an elliptic complex curve, and let $N = E^{\times n}$ be the product of n such curves, with Kähler form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. The corresponding Sasakian manifold is the $(2n + 1)$-dimensional Heisenberg nilmanifold $\mathscr{H}_n$. Theorem 6.6 guarantees that $\mathscr{H}_n$ is $(n - 1)$-formal. As noted in [33, Remark 5.4], though, the manifold $\mathscr{H}_n$ is *not n-formal*. We refer to [33] for a detailed study of the partial formality properties of this family of manifolds.

Remark 6.9. The compact Sasakian manifold $\mathscr{H}_n$ from the above example may be alternatively described as the quotient of a certain nilpotent Lie group $H(1, n)$ by a suitable discrete, cocompact subgroup. The Tievsky model was used in [13] to show that a compact, $(2n + 1)$-dimensional nilmanifold admits a Sasakian structure if and only if it is the quotient of $H(1, n)$ by a discrete, cocompact subgroup Γ.

6.3. **Sasakian groups.** A group π is said to be a *Sasakian group* if it can be realized as the fundamental group of a compact, Sasakian manifold. A major open problem in the field (see e.g. [10, Chapter 7] or [14]) is: "Which finitely presented groups are Sasakian?"

A first, well-known obstruction is that the first Betti number $b_1(\pi)$ must be even, see for instance the references listed by Chen in [14]. Much more subtle obstructions are provided by the following result. Fix a field $\Bbbk$ of characteristic 0.

Corollary 6.10. *Let $\pi = \pi_1(M^{2n+1})$ be a Sasakian group. Then:*

(1) *The Malcev Lie algebra $\mathfrak{m}(\pi, \Bbbk)$ is the lcs completion of the quotient of the free Lie algebra $\mathbb{L}(H_1(\pi, \Bbbk))$ by an ideal generated in degrees 2 and 3. Moreover, this Lie algebra presentation can be explicitly described in terms of the graded ring $H^\bullet(M/S^1, \Bbbk)$ and the Kähler class $h \in H^2(M/S^1, \Bbbk)$.*
(2) *The group π is filtered-formal.*
(3) *For every complex linear algebraic group G, the germ at the origin of the representation variety $\mathrm{Hom}(\pi, G)$ is defined by quadrics and cubics only.*

Proof. If $n > 1$, Theorems 6.6 and 3.6 imply that $\mathfrak{m}(\pi, \Bbbk)$ is isomorphic to the lcs completion of the holonomy Lie algebra of $(H/hH, d = 0)$, where $H^\bullet := H^\bullet(M/S^1, \Bbbk)$. Pick $\Bbbk$-bases for H^1 and H^2, say, $\{h_i\}$ and $\{h, f_\gamma\}$. Set $v_\gamma = \cup_H^* f_\gamma^* \in \mathbb{L}^2(h_i^*)$. It follows from the definitions that the above holonomy Lie algebra has quadratic presentation

$$(20) \qquad \mathfrak{h}((H/hH, d = 0)) = \mathbb{L}(h_i^*)/\mathrm{ideal}\,(v_\gamma).$$

If $n = 1$, it is well-known that the compact Kähler orbifold M/S^1 is a genus g smooth projective curve Σ_g (see e.g. [10, Proposition 4.4.4]). By [16], the manifold Σ_g is formal. We infer from Lemma 4.2 that $\Omega(M, \Bbbk) \simeq A$, where $A := H \otimes_e \bigwedge(t)$, for some $e \in H^2(\Sigma_g, \Bbbk)$. Note that M and M/S^1 have the same first Betti number, since $h \neq 0$. This implies that $e \neq 0$, since otherwise $b_1(M) = b_1(M/S^1) + 1$. Normalizing if necessary, we may thus assume that e is the orientation class of Σ_g.

By Theorem 3.6, the Malcev Lie algebra $\mathfrak{m}(\pi, \Bbbk)$ is isomorphic to the lcs completion of $\mathfrak{h}(A)$, which in turn can be computed as in Theorem 5.1. Plainly, $\mathfrak{h}(A) = 0$ if $g = 0$. If $g > 0$, let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a dual symplectic basis for H_1. Set $u = \sum_i [a_i, b_i]$. We obtain the following cubic presentation:

$$(21) \qquad \mathfrak{h}(A) \cong \mathbb{L}(a_1, b_1, \ldots, a_g, b_g) / \text{ideal}\,([a_i, u], [b_i, u]).$$

As shown in [46], the same result as in the case $n = 1$ holds for all orientable Seifert fibered 3-manifolds. The claim on representation varieties is a consequence of Corollary 3.7. □

As an application of Corollary 6.4, we obtain another (independent) obstruction to Sasakianity.

Corollary 6.11. *Let π be a Sasakian group. For each $s \geqslant 0$, all irreducible components of the characteristic variety $\mathcal{V}_s^1(\pi)$ passing through 1 are algebraic subtori of the character group $\mathrm{Hom}(\pi, \mathbb{C}^*)$.*

By Theorem 6.1, the $\mathbb{R}$-homotopy type of a compact Sasakian manifold M depends only on the cohomology ring $H^\bullet(M/S^1, \mathbb{R})$ and the Kähler class $h \in H^2(M/S^1, \mathbb{Q})$. Surprisingly enough, it turns out that the germs at 1 of certain representation varieties and jump loci of $\pi_1(M)$ depend only on the graded cohomology ring of M/S^1.

Corollary 6.12. *Let M be a compact Sasakian manifold, and let G be either $\mathrm{SL}_2(\mathbb{C})$, or a Borel subgroup. Then the germ at 1 of $\mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), G)$ depends only on the graded ring $H^\bullet(M/S^1, \mathbb{C})$ and the Lie algebra of G, in an explicit way. Similarly, the germs $\mathcal{V}_s^1(\pi_1(M))_{(1)}$ depend (explicitly) only on the graded ring $H^\bullet(M/S^1, \mathbb{C})$, for all s.*

Proof. Set $H^\bullet = H^\bullet(M/S^1, \mathbb{C})$, and let $A = H \otimes_h \bigwedge(t)$ be the complex Tievsky model of M. Denote by $\varphi \colon (H, d = 0) \hookrightarrow A$ the canonical cdga inclusion. Let $\mathfrak{g} \subseteq \mathfrak{sl}_2(\mathbb{C})$ be the Lie algebra of G. Theorem 3.2 and Proposition 5.3 give the following isomorphism of analytic germs:

$$(22) \qquad \mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), G)_{(1)} \cong (\mathcal{F}^1(A, \mathfrak{g}) \cup \varphi^!(\mathcal{F}(H, \mathfrak{g})))_{(0)}.$$

Since $h \neq 0$, the map $\varphi^* \colon H^1 \to H^1(A)$ is an isomorphism. Hence, $\mathcal{F}^1(A, \mathfrak{g}) = \varphi^!(\mathcal{F}^1(H, \mathfrak{g})) \subseteq \varphi^!(\mathcal{F}(H, \mathfrak{g}))$. Therefore, $\mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), G)_{(1)} \cong \mathcal{F}(H, \mathfrak{g})_{(0)}$. This proves the first claim.

Again by Theorem 3.2, $\mathcal{V}_s^1(\pi_1(M))_{(1)} \cong \mathcal{R}_s^1(A)_{(0)}$, for all s. We infer from Proposition 5.5(2b) that

$$(23) \qquad \mathcal{V}_s^1(\pi_1(M))_{(1)} \cong \mathcal{R}_s^1((H, d = 0))_{(0)}, \text{ for all } s,$$

and this proves the second claim. □

7. Poincaré duality and cohomology jump loci

In this section, we prove that Poincaré duality at the level of cochains implies twisted Poincaré duality. We illustrate this phenomenon with examples coming from almost free K-actions.

Let A be a finite-dimensional, commutative graded algebra over a characteristic zero field $\Bbbk$. We say that A is a *Poincaré duality algebra* of dimension n (for short, an n-PDA) if $A^i = 0$ for $i > n$ and $A^n = \Bbbk$, while the bilinear form

$$(24) \qquad\qquad A^i \otimes A^{n-i} \longrightarrow A^n = \Bbbk$$

given by the product is non-degenerate, for all $0 \leqslant i \leqslant n$ (in particular, A is connected). If M is a closed, connected, orientable, n-dimensional manifold, then, by Poincaré duality, the cohomology algebra $A = H^\bullet(M, \Bbbk)$ is an n-PDA.

Now let $A = (A^\bullet, d)$ be a CDGA. We say that A is a *Poincaré duality differential graded algebra* of dimension n (for short, an n-PD-CDGA) if the underlying algebra A is an n-PDA, and, moreover, $H^n(A) = \Bbbk$, or, equivalently, $dA^{n-1} = 0$.

Clearly, if A is an n-PDA, then $(A, d = 0)$ is an n-PD-CDGA. Hasegawa showed in [27] that the minimal model for the classifying space of a finitely-generated nilpotent group π is a PD-CDGA. Let $A^\bullet$ be a CDGA with $H^1(A) = 0$ for which $H^\bullet(A)$ is an n-PDA. Lambrechts and Stanley showed in [31] that A is weakly equivalent to an n-PD-CDGA. The next result is probably known to the experts. For the reader's convenience, we include a proof.

Lemma 7.1. *Let $A^\bullet = B^\bullet \otimes_\tau \bigwedge(t_i)$ be a Hirsch extension with variables t_i of degree m_i. If $(B^\bullet, d_B)$ is an n-PD-CDGA, then $(A^\bullet, d_A)$ is an m-PD-CDGA, where $m = n + \sum m_i$.*

Proof. Clearly $A^\bullet$ is an m-PDA. It remains to check that $d_A(b \otimes t_{i_1} \wedge \cdots \wedge t_{i_r}) = 0$, for all $b \in B^q$ such that $q + \sum_j m_{i_j} = m - 1$. Note that the condition on degrees forces $q \geqslant n - 1$. Indeed,

$$(25) \quad d_A(b \otimes t_{i_1} \wedge \cdots \wedge t_{i_r}) = d_B(b) \otimes t_{i_1} \wedge \cdots \wedge t_{i_r} + \sum_j \pm b \cdot \tau(t_{i_j}) \otimes t_{i_1} \wedge \cdots \widehat{t_{i_j}} \cdots \wedge t_{i_r}.$$

In the above, all elements of the form $b \cdot \tau(t_{i_j})$ belong to $B^{>n}$. These elements must be equal to zero, since B is an n-PDA. By the same argument, $d_B(b) = 0$ if $q > n - 1$. Finally, if $q = n - 1$ then again $d_B(b) = 0$, by our n-PD-CDGA assumption on B. $\qquad\square$

Corollary 7.2. *Let M be an almost free K-manifold. If B is a finite model of the orbit space $N = M/K$ and an n-PD-CDGA, then the finite model of M from Lemma 4.2, $A = B \otimes_\tau \bigwedge P$, is an $(n + \dim K)$-PD-CDGA. If N is a formal, closed, orientable, n-manifold, we may take $B^\bullet = (H^\bullet(N), d = 0)$.*

In the case of principal K-bundles, we obtain a more precise result.

Corollary 7.3. *Let N be a finite space having an n-PD-CDGA finite model B, over $\mathbb{Q}$. Let K be an arbitrary compact connected Lie group. Any Hirsch extension, $A = B \otimes_\tau \bigwedge P_K$, may be realized as an $(n + \dim K)$-PD-CDGA finite model of a principal K-bundle M_τ over N. When N is a formal, closed, orientable, n-manifold, we may take $B^\bullet = (H^\bullet(N), d = 0)$.*

Proof. The existence of the principal K-bundle M_τ with prescribed finite model $A = B \otimes_\tau \bigwedge P_K$ follows from Theorem 4.3. In turn, Lemma 7.1 yields the claimed PD-CDGA property. □

Let (A, d) be a finite CDGA . For a finite-dimensional vector space V, define an isomorphism $\sigma \colon A^1 \otimes \mathfrak{gl}(V) \xrightarrow{\simeq} A^1 \otimes \mathfrak{gl}(V^*)$ by $\sigma(\eta \otimes g) = -\eta \otimes g^*$ for $\eta \in A^1$ and $g \in \mathfrak{gl}(V)$. Identifying V with $\mathbb{k}^m$, this isomorphism coincides with the involution $- \operatorname{id}_{A^1} \otimes T$, where $T \colon \mathfrak{gl}_m(\mathbb{k}) \to \mathfrak{gl}_m(\mathbb{k})$ is matrix transposition. It is straightforward to verify that σ induces an isomorphism between the corresponding varieties of flat connections,

$$(26) \qquad \sigma \colon \mathscr{F}(A, \mathfrak{gl}(V)) \xrightarrow{\simeq} \mathscr{F}(A, \mathfrak{gl}(V^*)) .$$

In the next result, covariant derivatives are taken with respect to the identity representations of $\mathfrak{gl}(V)$ and $\mathfrak{gl}(V^*)$.

Lemma 7.4. *Let $(A^\bullet, d)$ be an n-PD-CDGA, and let $\omega \in \mathscr{F}(A, \mathfrak{gl}(V))$ be a $\mathfrak{gl}(V)$-valued flat connection. Then, for all $0 \leqslant i \leqslant n$, the diagram*

$$
\begin{array}{ccc}
(A^i)^* \otimes V^* & \xleftarrow{\quad d_\omega^* \quad} & (A^{i+1})^* \otimes V^* \\[2pt]
{\scriptstyle \mathrm{PD}} \Big\uparrow {\scriptstyle \simeq} & & {\scriptstyle \mathrm{PD}} \Big\uparrow {\scriptstyle \simeq} \\[2pt]
A^{n-i} \otimes V^* & \xleftarrow{\quad d_{\sigma(\omega)} \quad} & A^{n-i-1} \otimes V^* ,
\end{array}
$$

with vertical arrows induced by Poincaré duality isomorphisms, commutes up to a $(-1)^{n-i}$ sign.

Proof. Write $\omega = \sum_\alpha \eta_\alpha \otimes g_\alpha \in A^1 \otimes \mathfrak{gl}(V)$. Then $\sigma(\omega) = \sum_\alpha -\eta_\alpha \otimes g_\alpha^*$. Pick $a \otimes v^* \in A^{n-i-1} \otimes V^*$ and $b \otimes u \in A^i \otimes V$. Denoting by $\langle \text{-} , \text{-} \rangle$ the evaluation maps and using formula (5) for the covariant derivative, we find that

$$(27) \qquad \langle \mathrm{PD} \circ d_{\sigma(\omega)}(a \otimes v^*), b \otimes u \rangle = da \cdot b \langle v^*, u \rangle - \sum_\alpha \eta_\alpha \cdot a \cdot b \langle v^*, g_\alpha u \rangle$$

and

$$(28) \qquad \langle d_\omega^* \circ \mathrm{PD}(a \otimes v^*), b \otimes u \rangle = \langle \mathrm{PD}(a \otimes v^*), d_\omega(b \otimes u) \rangle$$
$$= a \cdot db \langle v^*, u \rangle + \sum_\alpha a \cdot \eta_\alpha \cdot b \langle v^*, g_\alpha u \rangle.$$

In view of our n-PD-CDGA assumption on A, and the fact that $0 = d(a \cdot b) = da \cdot b + (-1)^{n-i-1} a \cdot db$, the first terms from (27) and (28) differ by a factor of $(-1)^{n-i}$. Moreover,

by graded-commutativity of A, we also have that $-\eta_\alpha ab = (-1)^{n-i} a\eta_\alpha b$, for all α, and this completes the proof. $\qquad\square$

The previous lemma leads to the following twisted Poincaré duality result.

Corollary 7.5. *Let $(A^\bullet, d)$ be an n-PD-CDGA, and let $\omega \in \mathscr{F}(A, \mathfrak{gl}(V))$. Then*
$$H^i(A \otimes V, d_\omega)^* \cong H^{n-i}(A \otimes V^*, d_{\sigma(\omega)}), \, \forall\, i.$$

This in turn gives rise to Poincaré duality for embedded resonance varieties.

Lemma 7.6. *Let $A^\bullet$ be an n-PD-CDGA. Let $\mathfrak{g}$ be either $\mathfrak{gl}_m(\Bbbk)$ or $\mathfrak{sl}_m(\Bbbk)$. Denote by θ either* id: $\mathfrak{gl}_m(\Bbbk) \to \mathfrak{gl}_m(\Bbbk)$ *or the inclusion* $\mathfrak{sl}_m(\Bbbk) \hookrightarrow \mathfrak{gl}_m(\Bbbk)$. *Then the involution* $\sigma\colon A^1 \otimes \mathfrak{gl}_m(\Bbbk) \xrightarrow{\cong} A^1 \otimes \mathfrak{gl}_m(\Bbbk)$ *induces an algebraic isomorphism of embedded varieties,*
$$\sigma\colon (\mathscr{F}(A, \mathfrak{g}), \mathscr{R}_s^i(A, \theta)) \xrightarrow{\cong} (\mathscr{F}(A, \mathfrak{g}), \mathscr{R}_s^{n-i}(A, \theta)), \, \forall\, i, s.$$

Proof. The first case is a direct consequence of Corollary 7.5.

In the second case, the equality $\mathscr{F}(A, \mathfrak{sl}_m(\Bbbk)) = \mathscr{F}(A, \mathfrak{gl}_m(\Bbbk)) \cap A^1 \otimes \mathfrak{sl}_m(\Bbbk)$ is an instance of the well-known general formula describing the behavior of flat connections with respect to Lie subalgebras; see (4). In view of Remark 2.5 from [34], the resonance variety $\mathscr{R}_s^i(A, \theta)$ is the intersection of $\mathscr{R}_s^i(A, \mathrm{id}_{\mathfrak{gl}_m(\Bbbk)})$ with $A^1 \otimes \mathfrak{sl}_m(\Bbbk)$. On the other hand, the involution σ leaves invariant the subspace $A^1 \otimes \mathfrak{sl}_m(\Bbbk) \subseteq A^1 \otimes \mathfrak{gl}_m(\Bbbk)$. The desired conclusion follows at once. $\qquad\square$

We deduce the following topological consequence. Let G be either $\mathrm{GL}_m(\mathbb{C})$ or $\mathrm{SL}_m(\mathbb{C})$. Denote by ι either the identity $\mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ or the inclusion $\mathrm{SL}_m(\mathbb{C}) \hookrightarrow \mathrm{GL}_m(\mathbb{C})$.

Theorem 7.7. *Let X be a finite space admitting a finite model A which is an n-PD-CDGA over $\mathbb{C}$. There is then an analytic involution of $\mathrm{Hom}(\pi_1(X), G)_{(1)}$ which identifies $\mathscr{V}_s^i(X, \iota)_{(1)}$ with $\mathscr{V}_s^{n-i}(X, \iota)_{(1)}$, for all i, s. Furthermore, in the rank 1 case, this involution is induced by the involution $\rho \mapsto \rho^{-1}$ of the character group $H^1(X, \mathbb{C}^*)$.*

Proof. The general case follows from Theorem 3.2 and Lemma 7.6. In the rank one case, Theorem B(2) from [17] guarantees that the identification between $\mathscr{F}(A)_{(0)} \equiv H^1(X, \mathbb{C})_{(0)}$ and $H^1(X, \mathbb{C}^*)_{(1)}$ is given by the exponential map. On the other hand, in this case $\sigma = -\,\mathrm{id}$. $\qquad\square$

Remark 7.8. Given a finitely generated group π, the correspondence $\rho \mapsto (\rho^*)^{-1}$ defines an algebraic involution of the representation variety, $\alpha\colon \mathrm{Hom}_{\mathrm{gr}}(\pi, \mathrm{GL}_m(\mathbb{C})) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{gr}}(\pi, \mathrm{GL}_m(\mathbb{C}))$. If M is an n-dimensional closed, orientable manifold with $\pi = \pi_1(M)$, well-known results about Poincaré duality with local coefficients (see for instance [52, §2]) imply that the global involution α identifies $\mathscr{V}_s^i(M, \iota)$ with $\mathscr{V}_s^{n-i}(M, \iota)$, for all i, s, where ι is the identity map of $\mathrm{GL}_m(\mathbb{C})$. Theorem 7.7, then, can be viewed as a local analogue of this classical result.

8. Quasi-projective manifolds

Another class of examples where our techniques developed so far give strong topological consequences is provided by certain complex quasi-projective manifolds, closely related to classical 3-manifold theory. In this section we establish the general setup.

8.1. **Admissible maps and rank** 1 **jump loci.** Let M be a *quasi-projective manifold*, i.e., an irreducible, smooth, complex quasi-projective variety, and let S be a smooth complex curve, i.e., a 1-dimensional quasi-projective manifold. A regular, surjective map $f \colon M \to S$ is said to be an *admissible map* if the generic fiber of f is connected. The curve S is said to be of general type if $\chi(S) < 0$. The set $\mathscr{E}(M)$ of admissible maps onto curves of general type (modulo reparametrization at the target) is finite.

It is readily seen that $\mathscr{V}_1^1(S) = H^1(S, \mathbb{C}^*)$, for every curve S of general type. A celebrated theorem of Arapura describes the geometry of the characteristic variety $\mathscr{V}_1^1(M)$, largely in terms of pull-backs along admissible maps of the character tori of the target curves of general type.

Theorem 8.1 ([3])**.** *The correspondence $f \rightsquigarrow f^!(H^1(S, \mathbb{C}^*))$ gives a bijection between the set $\mathscr{E}(M)$ and the set of positive-dimensional irreducible components of $\mathscr{V}_1^1(M)$ passing through the identity of the character group $H^1(M, \mathbb{C}^*)$.*

In particular, the non-trivial part of $(\mathrm{Hom}(\pi_1(M), \mathbb{C}^*), \mathscr{V}_1^1(M))_{(1)}$ pulls back from curves of general type, via admissible maps. The infinitesimal counterpart of this result is a consequence of [17, Theorem C].

Theorem 8.2 ([17])**.** *For a quasi-projective manifold M with finite model A with positive weights, the set $\mathscr{E}(M)$ is in bijection with the set of positive-dimensional irreducible components of $\mathscr{R}_1^1(A) \subseteq H^1(A) \equiv H^1(M)$ via the correspondence $f \rightsquigarrow f^!(H^1(S, \mathbb{C}))$.*

8.2. **Embedded rank** 2 **jump loci and admissible maps.** We now formulate a rank-2 analog of Theorem 8.1.

Question 8.3. Let M be a quasi-projective manifold. Given a rational representation $\iota \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V)$, does the non-trivial part of the germ

$$(29) \qquad (\mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), \mathrm{SL}_2(\mathbb{C})), \mathscr{V}_1^1(M, \iota))_{(1)}$$

pull back from curves of general type, via admissible maps?

We start by giving a precise infinitesimal analog of this question. To do this, we first need to review a couple of relevant facts about compactifications and Gysin models.

Following [19], we say that a divisor D in a projective manifold $\overline{M}$ is a *hypersurface arrangement* if all irreducible components of D are smooth, and D coincides locally with the union of an arrangement of linear hyperplanes. When all these hyperplanes are actually coordinate hyperplanes, D is a *normal crossing* divisor.

Now let M be a quasi-projective manifold. There is then a *convenient compactification* $M \subset \overline{M}$. This means that the complement $D = \overline{M} \setminus M$ is a hypersurface arrangement, and every element of $\mathscr{E}(M)$ is represented by an admissible map $f \colon M \to S$ which is the restriction of a regular map $\overline{f} \colon \overline{M} \to \overline{S}$, where $\overline{S} = S \cup F$ is the canonical compactification of the curve S (obtained by adding a finite set of points F), such that $\overline{f}^{-1}(F) \subseteq D$. (If needed, one may also assume that D is a normal crossing divisor.)

In the case when D is a normal-crossing divisor, Morgan constructed in [35] a *Gysin model*, $A(\overline{M}, D)$, for the quasi-projective manifold M. In [19], Dupont extends this construction to the case when D is an arbitrary hypersurface arrangement. As with Morgan's original Gysin model, Dupont's model $A(\overline{M}, D)$ is a finite model, and is functorial in the appropriate sense.

We may now rephrase Question 8.3. Let M be a quasi-projective manifold. As explained in Remark 3.5, we may assume $b_1(M) > 0$, to avoid trivialities. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ or one of its Borel subalgebras, and let $\theta \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation.

We will need one more definition, extracted from [34]. For a 1-finite CDGA $A^\bullet$ and a finite-dimensional Lie representation $\theta \colon \mathfrak{g} \to \mathfrak{gl}(V)$, the set

$$(30) \qquad \Pi(A, \theta) = \{\omega = \eta \otimes g \in \mathscr{F}^1(A, \mathfrak{g}) \mid \det \theta(g) = 0\}$$

is a Zariski-closed, homogeneous subvariety of $\mathscr{F}^1(A, \mathfrak{g})$. Furthermore, if $H^i(A) \neq 0$, then $\Pi(A, \theta)$ is contained in $\mathscr{R}_1^i(A, \theta)$, cf. [34, Theorem 1.2]. Plainly, the variety $\Pi(A, \theta)$ depends only on the vector space $H^1(A)$ and the representation θ.

Question 8.4. Is there a convenient compactification $\overline{M} = M \cup D$ such that the following equalities hold?

$$(31) \qquad \mathscr{F}(A(\overline{M}, D), \mathfrak{g}) = \mathscr{F}^1(A(\overline{M}, D), \mathfrak{g}) \cup \bigcup_{f \in \mathscr{E}(M)} f^!(\mathscr{F}(A(\overline{S}, F), \mathfrak{g})).$$

$$(32) \qquad \mathscr{R}_1^1(A(\overline{M}, D), \theta) = \Pi(A(\overline{M}, D), \theta) \cup \bigcup_{f \in \mathscr{E}(M)} f^!(\mathscr{F}(A(\overline{S}, F), \mathfrak{g})).$$

In the above, we view $\mathscr{F}^1(A(\overline{M}, D), \mathfrak{g})$ and $\Pi(A(\overline{M}, D), \theta)$ as the 'trivial parts' of the respective varieties, since they depend only on $b_1(M)$ and θ. In the case when $\mathfrak{g} = \mathbb{C}$ and $\theta = \text{id}_{\mathbb{C}}$, we clearly have $\mathscr{F}^1(A(\overline{M}, D), \mathfrak{g}) = H^1(M)$ and $\Pi(A(\overline{M}, D), \theta) = \{0\}$.

In view of a recent result from [44], a positive answer to the global Question 8.4 would imply a positive answer to the local Question 8.3. Equalities (31) and (32) are known to hold for several interesting classes of quasi-projective manifolds M. Let $W_\bullet$ denote the Deligne weight filtration on $H^\bullet(M)$.

Example 8.5. If M is a projective manifold (in which case $W_1(H^1(M)) = H^1(M)$), we may take $D = \varnothing$ and $F = \varnothing$ in the above formulas. By [35], we have that $A(M, \varnothing) =$

$(H^\bullet(M), d = 0)$, and similarly for $S = \overline{S}$. Using now Corollary 7.2 from [34], we conclude that Question 8.4 has a positive answer in this case.

Example 8.6. As shown in Theorem 4.2 and Proposition 4.1 from [5], in the case when $W_1(H^1(M)) = 0$ there is a convenient compactification $\overline{M} = M \cup D$, where D is a normal-crossings divisor, such that the equalities from Question 8.4 hold. Furthermore, according to Theorem 1.3 from [5], Question 8.4 has a positive answer also for partial configuration spaces of smooth projective curves.

Question 8.4 is analyzed in much detail in follow-up work, [43, 44]. In particular, in [44, Theorem 1.2], we reinterpret this question in terms of the local structure of the representation variety $\mathrm{Hom}(\pi_1(M), G)$ and of the embedded cohomology jump loci $\mathcal{V}_1^1(\pi_1(M), \iota)$.

9. ISOLATED SURFACE SINGULARITIES

In this section, we will describe another large class of quasi-projective manifolds for which Question 8.4 has a positive answer.

9.1. **A Hirsch extension model.** Let X be a complex affine surface endowed with a good $\mathbb{C}^*$-action and having a normal, isolated singularity at 0. Let M be its singularity link (a closed, oriented, smooth 3-dimensional manifold). The punctured surface $X^* = X \backslash \{0\}$ is a quasi-projective manifold which deform-retracts onto M. Moreover, the almost free good $\mathbb{C}^*$-action on X^* restricts to an S^1-action on M with finite isotropy subgroups. In particular, M is an orientable Seifert fibered 3-manifold. The orbit space, $M/S^1 = X^*/\mathbb{C}^*$, is a smooth projective curve Σ_g, of genus $g = \frac{1}{2}b_1(M)$, and the regular canonical projection, $f \colon X^* \to X^*/\mathbb{C}^*$, induces an isomorphism on first homology. See for instance [18] and references therein.

Proposition 9.1. *Let $e \in H^2(\Sigma_g)$ be the orientation class of Σ_g. The quasi-projective manifold X^* has a finite model with positive weights of the form*

$$A^\bullet = (H^\bullet(\Sigma_g) \otimes_e \bigwedge(t), d),$$

where d vanishes on $H^\bullet(\Sigma_g)$ and $dt = e$. Moreover, $(A^\bullet, d)$ is a 3-PD-CDGA.

Proof. The smooth projective curve Σ_g is a formal space, and thus admits the finite model $B^\bullet = (H^\bullet(\Sigma_g), d = 0)$. Let $e' \in H^2(\Sigma_g)$ be the Euler class from Lemma 4.2. The $e' \neq 0$, since otherwise $b_1(M) = 1 + 2g$. Hence, after normalization if necessary, we may assume $e' = e$. The assertion on Poincaré duality follows from Corollary 7.2. $\square$

From now on, we will assume that $g > 0$ (the reason for this restriction on the genus is explained in Remark 3.5).

Proposition 9.2. *For X^* as above, $\mathcal{E}(X^*) = \varnothing$ if $g = 1$ and $\mathcal{E}(X^*) = \{f\}$ if $g > 1$.*

Proof. The orbit map $f\colon X^* \to X^*/\mathbb{C}^* = \Sigma_g$ is an orbifold bundle map, with connected generic fiber $\mathbb{C}^*$; thus, f is an admissible map. Our claims follow from Theorem 8.2, Proposition 9.1 and Corollary 5.6. $\qquad\square$

Remark 9.3. By Proposition 9.1, A is a 3-PD-CDGA. In particular, $\mathscr{R}_s^i(A) = \varnothing$ for $i > 3$ and $s > 0$. For $i = 0$, it follows directly from definitions (6) and (5) that $\mathscr{R}_1^0(A) = \{0\}$ and $\mathscr{R}_s^0(A) = \varnothing$ for $s > 1$. By Proposition 5.5(2b), $\mathscr{R}_s^1(A)$ equals $\mathscr{R}_s^1((H^\bullet(\Sigma_g), d = 0))$, for all s. Again from the definitions, and using Poincaré duality in $H^\bullet(\Sigma_g)$, it is straightforward to check that $\mathscr{R}_s^1(A)$ equals $H^1(\Sigma_g)$ for $s \leqslant 2g - 2$, while it is $\{0\}$ for $2g - 1 \leqslant s \leqslant 2g$ and it is empty for $s > 2g$. By Lemma 7.6, $\mathscr{R}_s^i(A)$ is thus computed for all values of i and s. By Theorem 3.2, this gives a description of $\mathscr{V}_s^i(M)_{(1)}$ for all i, s, which complements the computations from [18, §8].

9.2. **Rank** 2 **flat connections.** Set $\pi = \pi_1(M) \cong \pi_1(X^*)$ and $H^\bullet = H^\bullet(\Sigma_g)$. Furthermore, let $A^\bullet = (H^\bullet \otimes_e \bigwedge(t), d)$ be the complex finite model for $M \cong X^*$ described in Proposition 9.1. Let $\iota\colon \mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$ be the defining representation, and let $\theta\colon \mathfrak{sl}_2 \hookrightarrow \mathfrak{gl}_2$ be its tangential representation. By Theorem 3.2, there is an analytic isomorphism of germ pairs,

$$(33) \qquad (\mathrm{Hom}_{\mathrm{gr}}(\pi, \mathrm{SL}_2), \mathscr{V}_1^i(X^*, \iota))_{(1)} \xrightarrow{\ \simeq\ } (\mathscr{F}(A, \mathfrak{sl}_2), \mathscr{R}_1^i(A, \theta))_{(0)} \ ,$$

for all $i \geqslant 0$. Our next goal is to describe completely these germs of embedded rank 2 jump loci.

We start with the varieties of flat connections. We denote by $\varphi\colon (H^\bullet, d = 0) \hookrightarrow A^\bullet$ the CDGA inclusion. Note that $H^1(\varphi)\colon H^1 \to H^1(A)$ is an isomorphism, since $e \neq 0$; see §5.3.

Corollary 9.4. *Suppose* $\mathfrak{g}$ *is a Lie subalgebra of* $\mathfrak{sl}_2$. *If* $g = 1$*, then* $\mathscr{F}(A, \mathfrak{g}) = \mathscr{F}^1(A, \mathfrak{g})$*, whereas if* $g > 1$*, then* $\mathscr{F}(A, \mathfrak{g}) = \varphi^!(\mathscr{F}(H, \mathfrak{g}))$.

Proof. Our assertions follow from Proposition 5.3. In the second case, we use the fact that $\mathscr{F}^1(A, \mathfrak{g}) = \varphi^!(\mathscr{F}^1(H, \mathfrak{g}))$, since $H^1(\varphi)$ is an isomorphism. In the first case, the CDGA $(H, d = 0)$ is the cochain algebra of a two-dimensional abelian Lie algebra. Hence, $\mathscr{F}(H, \mathfrak{g}) = \mathscr{F}^1(H, \mathfrak{g})$, by [34, Lemma 4.14]. $\qquad\square$

This corollary and Lemma 7.3 from [34] provide an explicit description of the variety $\mathscr{F}(A, \mathfrak{sl}_2)$.

9.3. **Rank** 2 **resonance.** To complete the picture, we turn to the resonance varieties, $\mathscr{R}_1^i(A, \theta)$. We know from Proposition 9.1 that $A^\bullet$ is a 3-PD-CDGA; in particular, it is concentrated in degrees $0 \leqslant i \leqslant 3$. Consequently, $\mathscr{R}_1^i(A, \theta) = \varnothing$ for $i > 3$. In the remaining degrees, we use Poincaré duality, cf. Lemma 7.6.

Lemma 3.4 from [34] takes care of the first case ($i = 0$):

$$(34) \qquad\qquad \mathscr{R}_1^0(A, \theta) = \Pi(A, \theta).$$

In the last case ($i = 1$), we obtain the following explicit description.

Proposition 9.5. *With notation as in §9.2, we have*

$$\mathscr{R}_1^1(A, \theta) = \begin{cases} \Pi(A, \theta) & \text{if } g = 1, \\ \varphi^!(\mathscr{F}(H, \mathfrak{sl}_2)) & \text{if } g > 1. \end{cases}$$

Proof. For $g > 1$, we apply Proposition 4.1 from [5] to the one-element family of CDGA maps $\{\varphi : H \hookrightarrow A\}$. We have to check that $\mathscr{R}_1^1(A) = \operatorname{im} H^1(\varphi)$ and that equality holds in (13) for $\mathfrak{g} = \mathfrak{sl}_2$. The first property follows from Corollary 5.6, and the second property is verified in Proposition 5.3. We infer that $\mathscr{R}_1^1(A, \theta) = \Pi(A, \theta) \cup \varphi^!(\mathscr{F}(H, \mathfrak{sl}_2))$. Our claim then follows from Corollary 9.4.

The genus 1 formula is a consequence of Corollary 3.8 from [34], since in this case $\mathscr{F}(A, \mathfrak{sl}_2) = \mathscr{F}^1(A, \mathfrak{sl}_2)$, by Corollary 9.4, and $\mathscr{R}_1^1(A) = \{0\}$, by Corollary 5.6. □

We are now in a position to prove Theorem 1.8 from the Introduction, which may be rephrased as follows.

Theorem 9.6. *For a punctured, quasi-homogeneous surface with isolated singularity, Question 8.4 has a positive answer.*

Proof. For the purpose of this proof, we will denote by M the given punctured, quasi-homogeneous surface. We will show that (31) and (32) hold, for a convenient compactification $\overline{M}$ obtained by adding a normal crossings divisor D. By Proposition 4.1 from [5] and the discussion following it, it is enough to check only equality (31).

Set $\overline{A}^\bullet = A^\bullet(\overline{M}, D)$ and $H^\bullet = (H^\bullet(\Sigma_g), d = 0)$, and note that $A^\bullet(\Sigma_g, \varnothing) = H^\bullet$. We need to consider the two cases appearing in Proposition 9.2. In the case when $g = 1$, formula (31) reduces to

$$(35) \qquad\qquad \mathscr{F}(\overline{A}, \mathfrak{g}) = \mathscr{F}^1(\overline{A}, \mathfrak{g}).$$

Since the map $f^* : H^1(\Sigma_g) \to H^1(M)$ is an isomorphism, $\mathscr{F}^1(\overline{A}^\bullet, \mathfrak{g}) = f^!(\mathscr{F}^1(H^\bullet, \mathfrak{g}))$. Hence, in genus $g > 1$, formula (31) becomes

$$(36) \qquad\qquad \mathscr{F}(\overline{A}, \mathfrak{g}) = f^!(\mathscr{F}(H^\bullet, \mathfrak{g})).$$

Next, we claim it is enough to show that the backward inclusions from both (35) and (36) become equalities around 0. This is due to the fact that all varieties in sight have the property that all their irreducible components pass through 0. This property in turn is an easy consequence of the fact that all the above varieties are endowed with a positive weight $\mathbb{C}^*$-action. We recall from [17, Example 5.3] that the Gysin model $\overline{A}$ has positive weights. Moreover, as explained in [17, §9.17], the associated $\mathbb{C}^*$-action on $\overline{A} \otimes \mathfrak{g}$ leaves $\mathscr{F}(\overline{A}, \mathfrak{g})$ invariant. The varieties $\mathscr{F}^1(\overline{A}, \mathfrak{g})$ and $f^!(\mathscr{F}(H^\bullet, \mathfrak{g}))$ are defined by quadratic equations; thus, they are also invariant with respect to the standard, weight-1 $\mathbb{C}^*$-action. Therefore, we have reduced our proof to the corresponding germs at the origin.

Let $A^\bullet = (H^\bullet \otimes_e \bigwedge(t), d)$ be the finite model for M from Proposition 9.1. We know from Corollary 9.4 that $\mathscr{F}(A^\bullet, \mathfrak{g}) = \mathscr{F}^1(A^\bullet, \mathfrak{g})$ for $g = 1$ and $\mathscr{F}(A^\bullet, \mathfrak{g}) = \varphi^!(\mathscr{F}(H^\bullet, \mathfrak{g}))$ for $g > 1$. In particular, these equalities hold in a neighborhood of 0.

By Theorem 3.2, $\mathscr{F}(\overline{A}^\bullet, \mathfrak{g})_{(0)} \cong \mathscr{F}(A^\bullet, \mathfrak{g})_{(0)}$. Clearly, $\mathscr{F}^1(\overline{A}^\bullet, \mathfrak{g})_{(0)} \cong \mathscr{F}^1(A^\bullet, \mathfrak{g})_{(0)}$ and $f^!(\mathscr{F}(H^\bullet, \mathfrak{g}))_{(0)} \cong \varphi^!(\mathscr{F}(H^\bullet, \mathfrak{g}))_{(0)}$. Hence, $\mathscr{F}(\overline{A}^\bullet, \mathfrak{g})_{(0)} \cong \mathscr{F}^1(\overline{A}^\bullet, \mathfrak{g})_{(0)}$ for $g = 1$ and $\mathscr{F}(\overline{A}^\bullet, \mathfrak{g})_{(0)} \cong f^!(\mathscr{F}(H^\bullet, \mathfrak{g}))_{(0)}$ for $g > 1$. In both cases, the Hopfian argument from the proof of Theorem 5.7 shows that equality holds in a neighborhood of 0, in both (35) and (36). This completes the proof. □

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