The rational homology of real toric manifolds

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Toric manifolds. In a seminal paper [7] that appeared some twenty years ago, Michael Davis and Tadeusz Januszkiewicz introduced a topological version of smooth toric varieties, and showed that many properties previously discovered by means of algebro-geometric techniques are, in fact, topological in nature.

Let P be an n-dimensional simple polytope with facets $F_1, \dots, F_m$, and let χ be an integral $n \times m$ matrix such that, for each vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$, the minor of columns $i_1, \dots, i_n$ has determinant ± 1. To such data, there is associated a $2n$-dimensional toric manifold, $M_P(\chi) = T^n \times P/\sim$, where $(t,p) \sim (u,q)$ if $p = q$, and tu^{-1} belongs to the image under $\chi\colon T^m \to T^n$ of the coordinate subtorus corresponding to the smallest face of P containing q in its interior.

Here is an alternate description, using the moment-angle complex construction (see for instance [10] and references therein). Given a simplicial complex K on vertex set $[n] = \{1, \dots, n\}$, and a pair of spaces (X, A), let $\mathcal{Z}_K(X, A)$ be the subspace of the cartesian product $X^{\times n}$, defined as the union $\bigcup_{\sigma \in K} (X, A)^{\sigma}$, where $(X, A)^{\sigma}$ is the set of points for which the i-th coordinate belongs to A, whenever $i \notin \sigma$. It turns out that the quasi-toric manifold $M_P(\chi)$ is obtained from the moment angle manifold $\mathcal{Z}_K(D^2, S^1)$, where K is the dual to ∂P, by taking the quotient by the relevant free action of the torus $T^{m-n} = \ker(\chi)$.

Real toric manifolds. An analogous theory works for real quasi-toric manifolds, also known as small covers. Given a homomorphism $\chi\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ satisfying a minors condition as above, the resulting n-dimensional manifold, $N_P(\chi)$, is the quotient of the real moment angle manifold $\mathcal{Z}_K(D^1, S^0)$ by a free action of the group $\mathbb{Z}_2^{m-n} = \ker(\chi)$. The manifold $N_P(\chi)$ comes equipped with an action of $\mathbb{Z}_2^n$; the associated Borel construction is homotopy equivalent to $\mathcal{Z}_K(\mathbb{RP}^{\infty}, *)$.

If X is a smooth, projective toric variety, then $X(\mathbb{C}) = M_P(\chi)$, for some simple polytope P and characteristic matrix χ, and $X(\mathbb{R}) = N_P(\chi \bmod 2\mathbb{Z})$. Not all toric manifolds arise in this manner. For instance, $M = \mathbb{CP}^2 \sharp \mathbb{CP}^2$ is a toric manifold over the square, but it does not admit any (almost) complex structure; thus, $M \not\cong X(\mathbb{C})$.

The same goes for real toric manifolds. For instance, take P to be the dodecahedron, and use one of the characteristic matrices χ listed in [12]. Then, by a theorem of Andreev [1], the small cover $N_P(\chi)$ is a hyperbolic 3-manifold; thus, by a theorem of Delaunay [8], $N_P(\chi) \not\cong X(\mathbb{R})$.

The Betti numbers of real toric manifolds. In [7], Davis and Januszkiewicz showed that the sequence of mod 2 Betti numbers of $N_P(\chi)$ coincides with the h-vector of P. In joint work with Alvise Trevisan [18], we compute the rational cohomology groups (together with their cup-product structure) for real, quasi-toric manifolds. It turns out that the rational Betti numbers are much more subtle, depending also on the characteristic matrix χ.

More precisely, for each subset $S \subseteq [n]$, let $\chi_S = \sum_{i \in S} \chi_i$, where χ_i is the i-th row of χ, and let $K_{\chi,S}$ be the induced subcomplex of K on the set of vertices $j \in [m]$ for which the j-th entry of χ_S is non-zero. Then

$$\dim H_q(N_P(\chi), \mathbb{Q}) = \sum_{S \subseteq [n]} \dim \widetilde{H}_{q-1}(K_{\chi,S}, \mathbb{Q}). \tag{*}$$

The proof of formula (*), given in [18], relies on two fibrations relating the real toric manifold $N_P(\chi)$ to some of the aforementioned moment-angle complexes,

$$\begin{array}{ccccc} & & \mathbb{Z}_2^{m-n} & & \\ & & \downarrow & & \\ & & \mathcal{Z}_K(D^1, S^0) & & \\ & & \downarrow & & \\ \mathbb{Z}_2^n & \longrightarrow & N_P(\chi) & \longrightarrow & \mathcal{Z}_K(\mathbb{RP}^\infty, *)\,. \end{array}$$

The proof entails a detailed analysis of homology in rank 1 local systems on the space $\mathcal{Z}_K(\mathbb{RP}^\infty, *)$, exploiting at some point the stable splitting of moment-angle complexes due to Bahri, Bendersky, Cohen, and Gitler [2]. Some of the details of the proof appear in Trevisan's Ph.D. thesis [19].

As an easy application of formula (*), one can readily recover a result of Nakayama and Nishimura [14]: A real, n-dimensional toric manifold $N_P(\chi)$ is orientable if and only if there is a subset $S \subseteq [n]$ such that $K_{\chi,S} = K$.

The Hessenberg varieties. A classical construction associates to each Weyl group W a smooth, complex projective toric variety $\mathcal{T}_W$, whose fan corresponds to the reflecting hyperplanes of W and its weight lattice.

In the case when W is the symmetric group S_n, the manifold $\mathcal{T}_n = \mathcal{T}_{S_n}$ is the well-known Hessenberg variety, see [9]. Moreover, $\mathcal{T}_n$ is isomorphic to the De Concini–Procesi wonderful model $\overline{Y_{\mathcal{G}}}$, where $\mathcal{G}$ is the maximal building set for the Boolean arrangement in $\mathbb{CP}^{n-1}$. Thus, $\mathcal{T}_n$ can be obtained by iterated blow-ups: first blow up $\mathbb{CP}^{n-1}$ at the n coordinate points, then blow up along the proper transforms of the $\binom{n}{2}$ coordinate lines, etc.

The real locus, $\mathcal{T}_n(\mathbb{R})$, is a smooth, real toric variety of dimension $n-1$; its rational cohomology was recently computed by Henderson [13], who showed that

$$\dim H_i(\mathcal{T}_n(\mathbb{R}), \mathbb{Q}) = A_{2i} \binom{n}{2i},$$

where A_{2i} is the Euler secant number, defined as the coefficient of $x^{2i}/(2i)!$ in the Maclaurin expansion of $\sec(x)$. As announced in [17], we can recover this computation, using formula (*).

To start with, note that the $(n-1)$-dimensional polytope associated to $\mathcal{T}_n(\mathbb{R})$ is the permutahedron P_n. Its vertices are obtained by permuting the coordinates of the vector $(1, \ldots, n) \in \mathbb{R}^n$, while its facets are indexed by the non-empty, proper subsets $Q \subset [n]$. The characteristic matrix $\chi = (\chi^Q)$ for $\mathcal{T}_n(\mathbb{R})$ can be described as follows: χ^i is the i-th standard basis vector of $\mathbb{R}^{n-1}$ for $1 \leq i < n$, while $\chi^n = \sum_{i<n} \chi^i$ and $\chi^Q = \sum_{i\in Q} \chi^i$.

The simplicial complex K_n dual to ∂P_n is the barycentric subdivision of the boundary of the $(n-1)$-simplex. Given a subset $S \subset [n-1]$, the induced subcomplex $(K_n)_{\chi,S}$ depends only on the cardinality $r = |S|$; denote any one of these $\binom{n-1}{r}$ subcomplexes by $K_{n,r}$. It turns out that $K_{n,r}$ is the order complex associated to a rank-selected poset of a certain subposet of the Boolean lattice B_n. A result of Björner and Wachs [5] insures that such simplicial complexes are Cohen–Macaulay, and thus have the homotopy type of a wedge of spheres (of a fixed dimension); in fact, $K_{n,2r-1} \simeq K_{n,2r} \simeq \bigvee^{A_{2r}} S^{r-1}$. Hence,

$$\begin{aligned}
\dim H_i(\mathcal{T}_n(\mathbb{R}), \mathbb{Q}) &= \sum_{S\subseteq[n-1]} \dim \widetilde{H}_{i-1}((K_n)_{\chi,S}, \mathbb{Q}) \\
&= \sum_{r=1}^{n-1} \binom{n-1}{r} \dim \widetilde{H}_{i-1}(K_{n,r}, \mathbb{Q}) \\
&= \left(\binom{n-1}{2i-1} + \binom{n-1}{2i} \right) A_{2i} = \binom{n}{2i} A_{2i}.
\end{aligned}$$

Recently, Choi and Park [6] have extended this computation to a much wider class of real toric manifolds. Given a finite simple graph Γ, let $\mathcal{B}(\Gamma)$ be the building set obtained from the connected induced subgraphs of Γ, and let $P_{\mathcal{B}(\Gamma)}$ be the corresponding graph associahedron. Using formula (*), these authors compute the Betti numbers of the smooth, real toric variety $X_\Gamma(\mathbb{R})$ defined by $P_{\mathcal{B}(\Gamma)}$. When $\Gamma = K_n$ is a complete graph, $X_{K_n} = \mathcal{T}_n$, and one recovers the above calculation.

The formality question. A finite-type CW-complex X is said to be *formal* if its Sullivan minimal model is quasi-isomorphic to the rational cohomology ring of X, endowed with the 0 differential. Under a nilpotency assumption, this means that $H^*(X, \mathbb{Q})$ determines the rational homotopy type of X.

As shown by Notbohm and Ray [15], if X is formal, then $\mathcal{Z}_K(X, *)$ is formal; in particular, $\mathcal{Z}_K(S^1, *)$ and $\mathcal{Z}_K(\mathbb{CP}^\infty, *)$ are always formal. More generally, as shown by Félix and Tanré [11], if both X and A are formal, and the inclusion $A \hookrightarrow X$ induces a surjection in rational cohomology, then $\mathcal{Z}_K(X, A)$ is formal.

On the other hand, as sketched in [4], and proved with full details in [10], the spaces $\mathcal{Z}_K(D^2, S^1)$ can have non-trivial triple Massey products, and thus are not always formal. In fact, as shown in [10], there exist polytopes P and dual triangulations $K = K_{\partial P}$ for which the moment-angle manifold $\mathcal{Z}_K(D^2, S^1)$ is not formal. Using these results, as well as a construction from [3], we can exhibit real moment-angle manifolds $\mathcal{Z}_L(D^1, S^0)$ that are not formal.

In view of this discussion, the following natural question arises: are toric manifolds formal? Of course, smooth (complex) toric varieties are formal, by a classical result of Deligne, Griffith, Morgan, and Sullivan. More generally, Panov and Ray showed in [16] that all toric manifolds are formal. So we are left with the question whether real toric manifolds are always formal.

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References

[1] E.M. Andreev, *Convex polyhedra of finite volume in Lobačevskiĭ space*, Mat. Sb. **83** (1970), 256–260.

[2] A. Bahri, M. Bendersky, F.R. Cohen, S. Gitler, *The polyhedral product functor: a method of computation for moment-angle complexes, arrangements and related spaces*, Advances in Math. **225** (2010), no. 3, 1634–1668.

[3] A. Bahri, M. Bendersky, F.R. Cohen, S. Gitler, *Operations on polyhedral products and a new topological construction of infinite families of toric manifolds*, arXiv:1011.0094v4.

[4] I. Baskakov, *Triple Massey products in the cohomology of moment-angle complexes*, Russian Math. Surveys **58** (2003), no. 5, 1039–1041.

[5] A. Björner, M. Wachs, *On lexicographically shellable posets*, Trans. Amer. Math. Soc. **277** (1983), no. 1, 323–341.

[6] S. Choi, H. Park, *A new graph invariant arises in toric topology*, arXiv:1210.3776v1.

[7] M.W. Davis, T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451.

[8] C. Delaunay, *On hyperbolicity of toric real threefolds*, Int. Math. Res. Not. (2005), no. 51, 3191–3201.

[9] F. De Mari, C. Procesi, M.A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 529–534.

[10] G. Denham, A. Suciu, *Moment-angle complexes, monomial ideals and Massey products*, Pure Appl. Math. Q. **3** (2007), no. 1, 25–60.

[11] Y. Félix, D. Tanré, *Rational homotopy of the polyhedral product functor*, Proc. Amer. Math. Soc. **137** (2009), no. 3, 891–898.

[12] A. Garrison, R. Scott, *Small covers over the dodecahedron and the* 120*-cell*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 963–971

[13] A. Henderson, *Rational cohomology of the real Coxeter toric variety of type A*, in: *Configuration Spaces: Geometry, Combinatorics and Topology (Centro De Giorgi, 2010)*, 313–326, Publications of the Scuola Normale Superiore, vol. 14, Edizioni della Normale, Pisa, 2012; available at arXiv:1011.3860v1.

[14] H. Nakayama, Y. Nishimura, *The orientability of small covers and coloring simple polytopes*, Osaka J. Math. **42** (2005), no. 1, 243–256.

[15] D. Notbohm, N. Ray, *On Davis-Januszkiewicz homotopy types.* I. *formality and rationalisation*, Algebr. Geom. Topol. **5** (2005), 31–51.

[16] T. Panov, N. Ray, *Categorical aspects of toric topology*, in: *Toric Topology*, 293–322, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008.

[17] A. Suciu, *Polyhedral products, toric manifolds, and twisted cohomology*, talk at the Princeton–Rider workshop on Homotopy Theory and Toric Spaces, February 23, 2012.

[18] A. Suciu, A. Trevisan, *Real toric varieties and abelian covers of generalized Davis–Januszkiewicz spaces*, preprint, 2012.

[19] A. Trevisan, *Generalized Davis–Januszkiewicz spaces and their applications in algebra and topology*, Ph.D. thesis, Vrije University Amsterdam, 2012; available at http://dspace.ubvu.vu.nl/handle/1871/32835.