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POINCARÉ DUALITY AND RESONANCE VARIETIES

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Abstract. We explore the constraints imposed by Poincaré duality on the resonance varieties of a graded algebra. For a 3-dimensional Poincaré duality algebra A, we obtain a fairly precise geometric description of the resonance varieties $\mathscr{R}^i_k(A)$.

Contents

1. Introduction

1.1. **Resonance varieties.** The cohomology ring of a space captures deep, albeit incomplete information about the homotopy type of the space. Suppose we are given a connected, finite CW-complex X and a coefficient field $\Bbbk$ of characteristic different from 2. Finding a presentation for the $\Bbbk$-algebra $A = H^{\bullet}(X, \Bbbk)$, in and of itself, is not the end of the story. One still would like to extract further information from this graded algebra, such as the Betti numbers, $b_i(A) = \dim_{\Bbbk} A^i$, the bigraded Betti numbers

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$b_{ij} = \dim_{\Bbbk} \operatorname{Tor}_i^A(\Bbbk, \Bbbk)_j$, or the cup-length. Such numerical invariants, though, are oftentimes too coarse to tell apart graded algebras which may differ in quite subtle ways.

Enter the resonance varieties, $\mathscr{R}^i_k(A)$, which are the main focus of attention in this paper. These varieties are homogenous algebraic subsets of the affine space $A^1 = H^1(X, \Bbbk)$ which keep track of vanishing cup products in the cohomology ring of X. More precisely, for each $a \in A^1$, consider the cochain complex (A, δ_a) with differentials $\delta^i_a\colon A^i \to A^{i+1}$ given by $\delta^i_a(u) = au$. Then the degree i, depth k resonance variety $\mathscr{R}^i_k(A)$ consists of those points $a \in A^1$ for which $H^i(A, \delta_a)$ has dimension at least k. In particular, $\mathscr{R}^1_1(A)$ is the union of all isotropic planes in A^1.

In general, the resonance varieties can be quite complicated. On the other hand, if A is the cohomology ring of a formal space, then the resonance varieties of A are unions of rationally defined, linear subspaces of A^1, see [10, 9]. Our main goal here is to see what kind of restrictions another topological property, namely, Poincaré duality, puts on the resonance varieties.

1.2. **Poincaré duality algebras.** A graded, locally finite, graded commutative algebra A is said to be a Poincaré duality algebra of dimension m if there exists a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ such that all the bilinear forms $A^i \otimes A^{m-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular. For such a PD_m algebra, the Betti numbers satisfy the well-known equality $b_i(A) = b_{m-i}(A)$. A similar phenomenon holds for the resonance varieties; more precisely, we show in Theorem 5.3 that

$$\mathscr{R}^i_k(A) = \mathscr{R}^{m-i}_k(A), \tag{1.1}$$

for all i and k. Most interesting to us is the case when $m = 3$. For a PD_3 algebra A, we have that $\mathscr{R}^1_k(A) = \mathscr{R}^2_k(A)$, and $\mathscr{R}^i_k(A) \subseteq \{0\}$ for $i = 0$ or 3. So we are left with computing the degree 1 resonance varieties.

To that effect, we start by noting that the multiplicative structure of A is encoded by the alternating 3-form $\mu_A\colon \bigwedge^3 A^1 \to \Bbbk$ given by $\mu_A(a \wedge b \wedge c) = \varepsilon(abc)$. Fixing a basis $\{e_1, \dots, e_n\}$ for A^1, and setting $\mu_{ijk} = \mu_A(e_i \wedge e_j \wedge e_k)$, this information can be stored dually in the trivector $\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k$ belonging to $\bigwedge^3 (A^1)^*$. Conversely, any 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ on a finite-dimensional $\Bbbk$-vector space V determines in an obvious fashion a PD_3 algebra A over $\Bbbk$ for which $\mu_A = \mu$. As shown in Theorem 4.7, this construction yields a one-to-one correspondence, $A \leftrightsquigarrow \mu_A$, between isomorphism classes of PD_3 algebras and equivalence classes of alternating 3-forms.

The rank of a 3-form μ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$. The computation of the degree 1 resonance varieties of a PD_3 algebra reduces to the case when the associated 3-form has maximal rank. More precisely, let A be any PD_3 algebra, and write $A^1 = B^1 \oplus C^1$, where the restriction of μ_A to $\bigwedge^3 B^1$ has rank equal to the rank of μ_A. Letting B the PD_3 algebra with associated

3-form equal to this restriction, we show in Theorem 6.2 that

$$\mathscr{R}^1_k(A) \cong \mathscr{R}^1_{k-r+1}(B) \times C^1 \cup \mathscr{R}^1_{k-r}(B) \times \{0\} \tag{1.2}$$

for all $k \geq 0$, where $r = \operatorname{corank} \mu_A$. In particular, $\mathscr{R}^1_k(A) = A^1$ for all $k < \operatorname{corank} \mu_A$.

In Theorem 6.6 we give a lower bound on the dimension of the degree-1 resonance varieties up to a certain depth. Letting ν denote the nullity of μ_A, we show that

$$\dim \mathscr{R}^1_{\nu-1}(A) \geq \nu \geq 2, \tag{1.3}$$

provided $\overline{\Bbbk} = \Bbbk$ and $b_1(A) \geq 4$; in particular, $\dim \mathscr{R}^1_1(A) \geq \nu$. Finally, in Theorem 6.7 we use a result from [15] to show that, with a few exceptions, $\mathscr{R}^1_1(A) \neq \{0\}$, provided $\Bbbk = \mathbb{R}$.

1.3. **Pfaffians and resonance.** Consider now the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$, and let θ be the $n \times n$ skew-symmetric matrix of S-linear forms with entries $\theta_{ik} = \sum_{j=1}^n \mu_{jik} x_j$. It turns out that the resonance varieties of A are the degeneracy loci of this matrix, that is,

$$\mathscr{R}^1_k(A) = V(I_{n-k}(\theta)), \tag{1.4}$$

the vanishing locus of the ideal of codimension k minors of θ. Using known facts about Pfaffian ideals of skew-symmetric matrices, we show in Theorem 7.3 that

$$\mathscr{R}^1_{2k}(A) = \begin{cases} \mathscr{R}^1_{2k+1}(A) & \text{if } n \text{ is even,} \\ \mathscr{R}^1_{2k-1}(A) & \text{if } n \text{ is odd.} \end{cases} \tag{1.5}$$

We also show in Theorem 7.5 that the bottom resonance varieties vanish, provided $n \geq 3$ and μ_A has maximal rank:

$$\mathscr{R}^1_{n-2}(A) = \mathscr{R}^1_{n-1}(A) = \mathscr{R}^1_n(A) = \{0\}. \tag{1.6}$$

In this case, we have the following chains of inclusions for the varieties $\mathscr{R}_k = \mathscr{R}^1_k(A)$:

$$\begin{aligned} &A^1 = \mathscr{R}_0 = \mathscr{R}_1 \supseteq \mathscr{R}_2 = \mathscr{R}_3 \supseteq \cdots \supseteq \mathscr{R}_{n-3} \supseteq \mathscr{R}_{n-2} = \{0\} \qquad \text{if } n \text{ is even,} \\ &A^1 = \mathscr{R}_0 \supseteq \mathscr{R}_1 = \mathscr{R}_2 \supseteq \mathscr{R}_3 \supseteq \cdots \supseteq \mathscr{R}_{n-3} \supseteq \mathscr{R}_{n-2} = \{0\} \qquad \text{if } n \text{ is odd.} \end{aligned} \tag{1.7}$$

1.4. **The top resonance varieties.** By way of contrast, the top resonance varieties of a PD_3 algebra A have a much more interesting geometry. Without essential loss of generality, we may assume that $n = \dim A^1$ is at least 4 (the cases when $n \leq 3$ are easily dealt with). We then show in Theorem 8.6 that

$$\mathscr{R}^1_1(A) = \begin{cases} V(\mathrm{Pf}(\mu_A)) & \text{if } n \text{ is odd and } \mu_A \text{ is generic in the sense of [1],} \\ A^1 & \text{otherwise.} \end{cases} \tag{1.8}$$

Finally, suppose μ_A is generic in the sense of [6]. If n is odd, then $\mathscr{R}^1_1(A)$ is a hypersurface which is smooth if $n \leq 7$, and singular in codimension 5 if $n \geq 9$. On the other hand, if n is even, then $\mathscr{R}^1_2(A)$ is a subvariety of codimension 3, which is smooth if $n \leq 10$, and is singular in codimension 7 if $n \geq 12$.

In Appendix A we list the irreducible 3-forms μ_A of rank at most 8, according to the classification from [22, 13], together with the corresponding resonance varieties $\mathscr{R}^1_k(A)$.

This work is pursued in [34], where we provide further applications to the study of cohomology jump loci of 3-manifolds.

2. The resonance varieties of a graded algebra

2.1. **Resonance varieties.** Let A be a graded, graded commutative algebra over a field $\Bbbk$ of characteristic different from 2. Throughout, we will assume that A is non-negatively graded, that A is of finite-type (i.e., each graded piece A^i is finite-dimensional), and that A is connected (i.e., $A^0 = \Bbbk$, generated by the unit 1). We will write $b_i = b_i(A)$ for the Betti numbers of A, and we will generally assume that $b_1 > 0$, so as to avoid trivialities.

By graded-commutativity of the product and the assumption that $\operatorname{char} \Bbbk \neq 2$, each element $a \in A^1$ squares to zero. We thus obtain a cochain complex,

$$(A, \delta_a)\colon\ A^0 \xrightarrow{\delta^0_a} A^1 \xrightarrow{\delta^1_a} A^2 \xrightarrow{\delta^2_a} \cdots , \tag{2.1}$$

with differentials $\delta^i_a(u) = a \cdot u$, for all $u \in A^i$. The *resonance varieties* of A (in degree $i \geq 0$ and depth $k \geq 0$) are defined as

$$\mathscr{R}^i_k(A) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, a) \geq k\}. \tag{2.2}$$

In other words, the resonance varieties record the locus of points a in the affine space $A^1 = \Bbbk^{b_1}$ where the 'twisted' Betti numbers $b_i(A, a) := \dim_{\Bbbk} H^i(A, \delta_a)$ jump by at least k. We will allow at times $k \leq 0$, in which case we will set $\mathscr{R}^i_k(A) = A^1$. Clearly, the sets $\mathscr{R}^i_k(A)$ are homogeneous subsets of A^1. Here is a more concrete description of these sets, which follows at once from the definitions.

Lemma 2.1. *An element $a \in A^1$ belongs to $\mathscr{R}^i_k(A)$ if and only if there exist $u_1, \dots, u_k \in A^i$ such that $au_1 = \cdots = au_k = 0$ in A^{i+1}, and the set $\{au, u_1, \dots, u_k\}$ is linearly independent in A^i, for all $u \in A^{i-1}$.*

Consequently, $\mathscr{R}^i_{b_i}(A) = \{0\}$ and $\mathscr{R}^i_k(A) = \emptyset$ for $k > b_i$; in particular, if $b_1 = 0$, then $\mathscr{R}^1_k(A) = \emptyset$ for all $k \geq 1$. Moreover, for each $i \geq 0$, we have a descending filtration,

$$A^1 = \mathscr{R}^i_0(A) \supseteq \mathscr{R}^i_1(A) \supseteq \cdots \supseteq \mathscr{R}^i_{b_i}(A) = \{0\} \supset \mathscr{R}^i_{b_{i+1}}(A) = \emptyset. \tag{2.3}$$

Therefore,

$$b_i(A) = \max\{k \mid 0 \in \mathscr{R}^i_k(A)\}. \tag{2.4}$$

2.2. **Isotropic subspaces.** We say that a linear subspace $U \subset A^1$ is *isotropic* if the restriction of the multiplication map $A^1 \wedge A^1 \to A^2$ to $U \wedge U$ is the zero map; that is, $ab = 0$, for all $a, b \in U$.

Lemma 2.2. *Let A be a graded algebra as above.*

(1) *If $U \subseteq A^1$ is an isotropic subspace of dimension k, then $U \subseteq \mathscr{R}^1_{k-1}(A)$.*

(2) *$\mathscr{R}^1_1(A)$ is the union of all isotropic planes in A^1.*

Proof. The first claim follows straight from the definitions. To prove claim (2), let $\mathscr{Q}(A)$ be the union of all isotropic planes in A^1. By claim (1), we have that $\mathscr{Q}(A) \subseteq \mathscr{R}^1_1(A)$; it remains to establish the reverse inclusion.

So let $a \in \mathscr{R}^1_1(A)$; there is then a vector $b \in A^1$, not proportional to a, such that $ab = 0$ in A^2. Let U be the plane spanned by a and b. Then U is isotropic (if $\alpha = \lambda_1 a + \nu_1 b$ and $\beta = \lambda_2 a + \nu_2 b$ are two vectors in U, then clearly $\alpha\beta = 0$), and we are done. □

Remark 2.3. The resonance varieties $\mathscr{R}^1_1(A)$ were first considered by Falk [18] in the case when A is the Orlik–Solomon algebra attached to a hyperplane arrangement and $\Bbbk = \mathbb{C}$. It was noted in that paper that Lemma 2.2 holds in that setting, while subsequent work of Falk [19] highlighted and made use of the fact that these rulings by isotopic planes hold over fields $\Bbbk$ of arbitrary characteristic, even when $\mathscr{R}^1_1(A)$ is not a union of linear subspaces, as is the case when $\operatorname{char}(\Bbbk) = 0$.

Remark 2.4. The resonance varieties of a graded algebra A do not depend in an essential way on the field $\Bbbk$, but rather, just on its characteristic. More precisely, if $\Bbbk \subset \mathbb{K}$ is a field extension, then the $\Bbbk$-points on $\mathscr{R}^i_k(A \otimes_{\Bbbk} \mathbb{K})$ coincide with $\mathscr{R}^i_k(A)$. Nonetheless, as we shall see in Example 6.8, this subtle difference between the two varieties can be quite meaningful.

2.3. **Resonance varieties of products.** One of the more pleasant properties of resonance varieties is the way they behave with respect to tensor products of graded algebras. This topic is treated in various levels of generality in [26, 27, 35]. We summarize here the relevant result.

Proposition 2.5. *Let $A = B \otimes_{\Bbbk} C$ be the tensor product of two connected, finite-type graded $\Bbbk$-algebras. Then*

$$\mathscr{R}^1_k(B \otimes_{\Bbbk} C) = \mathscr{R}^1_k(B) \times \{0\} \cup \{0\} \times \mathscr{R}^1_k(C),$$

$$\mathscr{R}^i_1(B \otimes_{\Bbbk} C) = \bigcup_{p \geq 0} \mathscr{R}^p_1(B) \times \mathscr{R}^{i-p}_1(C), \qquad \text{if } i \geq 2.$$

Proof. As in [26, 35], the claim easily follows from the following fact: if $a = (b, c)$ is an element in $A^1 = B^1 \oplus C^1$, then the cochain complex (A, a) splits as a tensor product of cochain complexes, $(B, b) \otimes (C, c)$, and thus $b_i(A, a) = \sum_{p+q=i} b_p(B, b) b_q(C, c)$. □

2.4. **Naturality properties.** The resonance varieties enjoy several nice naturality properties with respect to morphisms of graded algebras. To describe some of these properties, we start with a Lemma/Definition, following the approach from [7], where a more general situation is studied.

Lemma 2.6. *Let $\varphi\colon A \to B$ be a morphism of graded $\Bbbk$-algebras. For each $a \in A^1$, there is an induced homomorphism*

$$\varphi_a\colon H^{\bullet}(A, \delta_a) \to H^{\bullet}(B, \delta_{\varphi(a)}). \tag{2.5}$$

Proof. Let $[b] \in H^i(A, a)$, represented by an element $b \in A^i$ such that $ab = 0$ in A^{i+1}. Since $\varphi(a)\varphi(b) = 0$, we may define a map φ_a from $H^\bullet(A, \delta_a)$ to $H^\bullet(B, \delta_{\varphi(a)})$ by sending $[b]$ to $[\varphi(b)]$. To verify this map is well-defined, suppose $b = ac$, for some $c \in A^{i-1}$; then $\varphi(b) = \varphi(a)\varphi(c)$, and so $[\varphi(b)] = [\varphi(c)]$. □

Proposition 2.7. *Let $\varphi\colon A \to B$ be a morphism of graded algebras such that $\varphi^i\colon A^i \to B^i$ is injective and φ^{i-1} is surjective, for some $i \geq 1$. Then*

(1) *The homomorphisms $\varphi^i_a\colon H^i(A, \delta_a) \to H^i(B, \delta_{\varphi(a)})$ are injective, for all $a \in A^1$.*
(2) *Suppose further that the map $\varphi^1\colon A^1 \to B^1$ is injective. Then this map restricts to inclusions $\varphi^1\colon \mathscr{R}^i_k(A) \hookrightarrow \mathscr{R}^i_k(B)$, for all $k \geq 0$.*

Proof. To prove part (1), suppose that $\varphi^i_a([b]) = 0$, for some $b \in A^i$. Then $\varphi^i(b) = \varphi^1(a)v$, for some $v \in B^{i-1}$. By our surjectivity assumption on φ^{i-1}, there is an element $u \in A^{i-1}$ such that $\varphi^{i-1}(u) = v$, and so $\varphi^i(b) = \varphi^i(av)$. Our injectivity assumption on φ^i now implies that $b = av$, and so $[b] = 0$.

Part (2) follows at once from part (1) and the definition of resonance varieties. □

As a particular case, we recover a result from [25, 33].

Corollary 2.8. *Let $\varphi\colon A \to B$ be a morphism of graded, connected algebras. If the map $\varphi^1\colon A^1 \to B^1$ is injective, then $\varphi^1(\mathscr{R}^1_k(A)) \subseteq \mathscr{R}^1_k(B)$, for all $k \geq 0$.*

It follows that the resonance varieties of a graded, connected algebra A depend only on the isomorphism type of A. More precisely, if $\varphi\colon A \xrightarrow{\simeq} B$ is an isomorphism between two such algebras, then the linear isomorphism $\varphi^1\colon A^1 \xrightarrow{\simeq} B^1$ restricts to isomorphisms $\varphi^1\colon \mathscr{R}^i_k(A) \xrightarrow{\simeq} \mathscr{R}^i_k(B)$ for all $k \geq 0$.

In general, though, even if $\varphi\colon A \to B$ is an injective morphism between two graded algebras, the set $\varphi^1(\mathscr{R}^i_k(A))$ may not be included in $\mathscr{R}^i_k(B)$, for some $i > 1$ and $k > 0$.

Example 2.9. Let $f\colon S^1 \times S^1 \to S^1 \vee S^2$ be the map obtained (up to homotopy) by pinching a meridian circle of the torus to a point, and let $\varphi\colon A \to B$ be the induced morphism between the respective cohomology $\Bbbk$-algebras. It is readily seen that φ is injective, yet $\mathscr{R}^2_1(A) = \Bbbk$, whereas $\mathscr{R}^2_1(B) = \{0\}$.

3. Resonance and the BGG correspondence

In this section we explain how the BGG correspondence can be used to find equations for the resonance varieties of a graded algebra, and discuss the behavior of these varieties under coproducts, and under injective morphisms of algebras.

3.1. **Equations for the resonance varieties.** Once again, let A be a connected, finite-type cga over a field $\Bbbk$. Without essential loss of generality, we will assume that $n := b_1(A)$ is at least 1. Let us pick a basis $\{e_1, \dots, e_n\}$ for the $\Bbbk$-vector space A^1, and let $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for the dual vector space $A_1 = (A^1)^*$. These choices allow us to identify the symmetric algebra $\mathrm{Sym}(A_1)$ with the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$.

The Bernstein–Bernstein–Gelfand correspondence (see for instance [17, §7B]) yields a cochain complex of finitely generated, free S-modules,

$$\mathbf{L}(A) = (A \otimes_{\Bbbk} S, \delta)\colon \ \cdots \longrightarrow A^{i-1} \otimes_{\Bbbk} S \xrightarrow{\delta_A^{i-1}} A^i \otimes_{\Bbbk} S \xrightarrow{\delta_A^{i}} A^{i+1} \otimes_{\Bbbk} S \longrightarrow \cdots, \tag{3.1}$$

with differentials given by $\delta_A^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j$ for $u \in A^i$. By construction, the matrices associated to these differentials have entries that are linear forms in the variables of S.

It is readily verified that the evaluation of the cochain complex $\mathbf{L}(A)$ at an element $a \in A^1$ coincides with the cochain complex (A, δ_a) from (2.1), that is to say, $\delta_A^i\big|_{x_j = a_j} = \delta_a^i$. By definition, an element $a \in A^1$ belongs to $\mathscr{R}_k^i(A)$ if and only if

$$\operatorname{rank} \delta_a^{i-1} + \operatorname{rank} \delta_a^i \le b_i(A) - k, \tag{3.2}$$

where recall $b_i(A) = \dim_{\Bbbk} A^i$. Let $I_r(\psi)$ denote the ideal of $r \times r$ minors of a $p \times q$ matrix ψ with entries in S, with the convention that $I_0(\psi) = S$ and $I_r(\psi) = 0$ if $r > \min(p, q)$. Using the well-known fact that $I_r(\phi \oplus \psi) = \sum_{s+t=r} I_s(\phi) \cdot I_t(\psi)$, we infer that

$$\begin{aligned} \mathscr{R}_k^i(A) &= V\Big(I_{b_i(A)-k+1}(\delta_A^{i-1} \oplus \delta_A^i)\Big) \\ &= \bigcap_{s+t=b_i(A)-k+1} \Big(V(I_s(\delta_A^{i-1})) \cup V(I_t(\delta_A^i))\Big). \end{aligned} \tag{3.3}$$

The degree 1 resonance varieties admit an even simpler description. Clearly, the map $\delta_A^0\colon S \to S^n$ has matrix $(x_1 \cdots x_n)$, and so $V(I_1(\delta_A^0)) = \{0\}$; hence,

$$\mathscr{R}_k^1(A) = V(I_{n-k}(\delta_A^1)) \tag{3.4}$$

for $0 \le k < n$ and $\mathscr{R}_n^1(A) = \{0\}$.

Remark 3.1. It is sometimes useful to consider the *resonance schemes* $\boldsymbol{\mathcal{R}}_k^i(A)$ of a graded algebra A as above. These schemes are defined by the ideals $I_{b_i(A)-k+1}(\delta_A^{i-1} \oplus \delta_A^i)$ from (3.3), and have as underlying sets the resonance varieties $\mathscr{R}_k^i(A)$.

3.2. **Induced morphisms in cohomology.** Given an arbitrary morphism $\varphi\colon A \to B$ of connected, finite-type graded $\Bbbk$-algebras, it is not clear how to define an induced chain map, $\mathbf{L}(\varphi)\colon \mathbf{L}(A) \to \mathbf{L}(B)$. Nevertheless, when φ is injective, this can be done (after making some non-canonical choices), following the approach from [7].

Since each map $\varphi^i\colon A^i \hookrightarrow B^i$ is injective, the $\Bbbk$-dual map, $\varphi_i\colon B_i \twoheadrightarrow A_i$, is surjective. Let $\psi_i\colon A_i \hookrightarrow B_i$ be a $\Bbbk$-linear splitting of φ_i, so that $\varphi_i \circ \psi_i = \mathrm{id}_{A_i}$.

Lemma 3.2. *The map of S-modules $\mathbf{L}(\varphi)\colon \mathbf{L}(A) \to \mathbf{L}(B)$ defined by*

$$
\begin{tikzcd}
\mathbf{L}(A): \arrow[d, "\mathbf{L}(\varphi)"] & A^0 \otimes_{\Bbbk} \mathrm{Sym}(A_1) \arrow[r, "\delta^0_A"] \arrow[d, "\varphi^0\otimes \mathrm{Sym}(\psi_1)"] & A^1 \otimes_{\Bbbk} \mathrm{Sym}(A_1) \arrow[r, "\delta^1_A"] \arrow[d, "\varphi^1\otimes \mathrm{Sym}(\psi_1)"] & A^2 \otimes_{\Bbbk} \mathrm{Sym}(A_1) \arrow[r] \arrow[d, "\varphi^2\otimes \mathrm{Sym}(\psi_1)"] & \cdots \\
\mathbf{L}(B): & B^0 \otimes_{\Bbbk} \mathrm{Sym}(B_1) \arrow[r, "\delta^0_B"] & B^1 \otimes_{\Bbbk} \mathrm{Sym}(B_1) \arrow[r, "\delta^1_B"] & B^2 \otimes_{\Bbbk} \mathrm{Sym}(B_1) \arrow[r] & \cdots
\end{tikzcd}
\tag{3.5}
$$

is a chain map.

Proof. Pick bases $\{e_1, \dots, e_n\}$ for A^1 and $\{f_1, \dots, f_p\}$ for B^1 so that $\varphi^1(e_j) = f_j$ for $j \le p$ and $\varphi^1(e_j) = 0$, otherwise. Letting $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_p\}$ be the dual bases for A_1 and B_1, respectively, we find that

$$
\begin{aligned}
(\varphi^{i+1} \otimes \mathrm{Sym}(\psi_1)) \circ \delta^i_A(u \otimes 1) &= \varphi^{i+1} \otimes \mathrm{Sym}(\psi_1)\Big(\sum_{j=1}^n e_j u \otimes x_j\Big) \\
&= \sum_{j=1}^n \varphi^1(e_j)\varphi^i(u) \otimes \psi_1(x_j) \\
&= \sum_{j=1}^p f_j\varphi^i(u) \otimes y_j \\
&= \delta^i_B(\varphi^i(u) \otimes 1) \\
&= \delta^i_B \circ (\varphi^i \otimes \mathrm{Sym}(\psi_1))(u \otimes 1),
\end{aligned}
$$

thus verifying our claim. ◻

The chain map defined above induces a morphism in cohomology, $\mathbf{L}(\varphi)^*\colon H^{\bullet}(\mathbf{L}(A)) \to H^{\bullet}(\mathbf{L}(B))$. The next proposition follows at once.

Proposition 3.3. *For each $i \ge 0$, the evaluation of the morphism $\mathbf{L}(\varphi)^*\colon H^i(\mathbf{L}(A)) \to H^i(\mathbf{L}(B))$ at a point $a \in A^1$ yields the map $\varphi^i_a\colon H^i(A, \delta_a) \to H^i(B, \delta_{\varphi(a)})$ from (2.5).*

3.3. **Resonance varieties of coproducts.** Let B and C be two connected cga's. Their wedge sum, $B \vee C$, is a new connected cga, whose underlying graded vector space in positive degrees is $B^+ \oplus C^+$, with multiplication $(b, c) \cdot (b', c') = (bb', cc')$. The next proposition sharpens results from [26, 35]. Since this is a new proof, and since we will use the same approach to prove Theorem 6.2 below, we give complete details.

Proposition 3.4. *Let $A = B \vee C$ be the wedge sum of two connected, finite-type graded $\Bbbk$-algebras with $b_1(B) > 0$ and $b_1(C) > 0$. Identifying $A^1 = B^1 \oplus C^1$, we have*

$$
\mathcal{R}^i_k(A) = \begin{cases} \bigcup_{s+t=k-1} \mathcal{R}^1_s(B) \times \mathcal{R}^1_t(C) & \text{if } i = 1, \\ \bigcup_{s+t=k} \mathcal{R}^i_s(B) \times \mathcal{R}^i_t(C) & \text{if } i \ge 2. \end{cases}
$$

Proof. Note that $\mathbf{L}(A)^+ = \mathbf{L}(B)^+ \oplus \mathbf{L}(C)^+$. Thus, for $i > 0$ the matrix of δ^i_A is the block sum of the matrices of δ^i_B and δ^i_C, and so $I_r(\delta^i_A) = \sum_{s+t=r} I_s(\delta^i_B) \cdot I_t(\delta^i_C)$, where $I_s(\delta^i_B)$ and $I_t(\delta^i_C)$ are viewed as ideals of $S = \mathrm{Sym}(A_1)$ by extension of scalars. When $i = 1$, we get

$$
\begin{aligned}
\mathscr{R}^1_k(A) &= V(I_{b_1(A)-k}(\delta^1_A)) \\
&= V(I_{b_1(A)-k}(\delta^1_B \oplus \delta^1_C) \\
&= V\Big(\sum_{s+t=b_1(A)-k} I_s(\delta^1_B) \cdot I_t(\delta^1_C)\Big) \\
&= \bigcap_{s+t=b_1(A)-k} \Big(V(I_s(\delta^1_B)) \cup V(I_t(\delta^1_C))\Big) \\
&= \bigcap_{u+v=k} \Big(V(I_{b_1(B)-u}(\delta^1_B)) \cup V(I_{b_1(C)-v}(\delta^1_C))\Big) \\
&= \bigcap_{u+v=k} \Big((\mathscr{R}^1_u(B) \times C^1) \cup (B^1 \times \mathscr{R}^1_v(C))\Big) \\
&= \bigcup_{s+t=k-1} \mathscr{R}^1_s(B) \times \mathscr{R}^1_t(C)
\end{aligned}
$$

where the last step is set-theoretical, based solely on the resonance filtrations (2.3) for the algebras B and C. The proof for the case $i > 1$ is similar. ◻

4. Poincaré duality algebras and alternating forms

In this section we consider a restricted class of graded algebras which abstract the notion of Poincaré duality for closed, oriented topological manifolds, and we discuss the alternating form naturally associated to such an algebra.

4.1. **Poincaré duality.** Let A be a non-negatively graded, graded-commutative algebra over a field $\Bbbk$. We will assume throughout that A is connected and locally finite. We say that A is a *Poincaré duality* $\Bbbk$*-algebra* of formal dimension m if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that all the bilinear forms

$$A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk, \quad a \otimes b \mapsto \varepsilon(ab) \tag{4.1}$$

are non-singular. It follows ε is an isomorphism, and that $A^i = 0$ for $i > m$. Furthermore, for each $0 \le i \le m$, there is an isomorphism

$$\mathrm{PD}^i\colon A^i \to (A^{m-i})^*, \quad \mathrm{PD}^i(a)(b) = \varepsilon(ab). \tag{4.2}$$

Consequently, each element $a \in A^i$ has a "Poincaré dual," $a^\vee \in A^{m-i}$, which is uniquely determined by the formula $\varepsilon(aa^\vee) = 1$. We define the orientation class $\omega_A \in A^m$ as the Poincaré dual of $1 \in A^0$, that is, $\omega_A = 1^\vee$. Conversely, a choice of orientation class $\omega_A \in A^m$ defines an orientation $\varepsilon\colon A^m \to \Bbbk$ by setting $\varepsilon(\omega_A) = 1$.

In more algebraic terms, a PD_m algebra is a graded, graded-commutative Gorenstein Artin algebra of socle degree m.

The main motivation for these definitions comes from topology: if M is a compact, connected, orientable, m-dimensional manifold, then, by Poincaré duality, the cohomology algebra $A = H^{\bullet}(M, \Bbbk)$ is a PD_m algebra over $\Bbbk$, with the orientation class $[M] \in H_m(M, \Bbbk)$ determining the orientation on A by setting $\omega_A([M]) = 1$.

4.2. **Tensor products and connected sums.** The class of Poincaré duality algebras is closed under taking tensor products and connected sums.

Indeed, if A and B are Poincaré duality algebras of dimension m and n, respectively, then their tensor product, $A \otimes_{\Bbbk} B$, is a Poincaré duality algebra of dimension $m + n$. Conversely, if the tensor product of two graded algebras is a PD algebra, then each factor must be a PD algebra, see for instance [23, p. 188] or [32, Prop. 3.3].

Now let A and B be two PD_m algebras, with orientation classes ω_A and ω_B, respectively. Much as in [24], let us define their *connected sum*, $C = A\#B$, as the pushout

$$\begin{tikzcd} \bigwedge(\omega) \arrow[r, "\omega\mapsto\omega_A"] \arrow[d, "\omega\mapsto\omega_B"'] & A \arrow[d] \\ B \arrow[r] & A\#B \end{tikzcd} \tag{4.3}$$

In other words, $C^0 = \Bbbk \cdot 1$, $C^i = A^i \oplus B^i$ for $0 < i < m$, and $C^m = \Bbbk \cdot \omega_C$, with ω_A and ω_B identified to ω_C, and with multiplication defined in the obvious way.

The motivation and terminology for the above notions comes from manifold topology. Indeed, if M and N are two closed, oriented manifolds, then $M \times N$ is again a closed, oriented manifold, and $H^{\bullet}(M \times N, \Bbbk) \cong H^{\bullet}(M, \Bbbk) \otimes_{\Bbbk} H^{\bullet}(N, \Bbbk)$. Moreover, the cohomology algebra of the connected sum of two closed, oriented manifolds of the same dimension is the connected sum of the respective cohomology algebras, that is, $H^{\bullet}(M\#N, \Bbbk) \cong H^{\bullet}(M, \Bbbk)\#H^{\bullet}(N, \Bbbk)$.

4.3. **The alternating form of a PD_m algebra.** Associated to a PD_m algebra over a field $\Bbbk$ there is an alternating m-form,

$$\mu_A \colon \textstyle\bigwedge^m A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m). \tag{4.4}$$

Let us specialize now to the case when $m = 3$. In this instance, the multiplicative structure of A can be recovered from the 3-form $\mu = \mu_A$ and the orientation ε, as follows. As before, set $n = b_1(A)$, and fix a basis $\{e_1, \dots, e_n\}$ for A^1. Let $\{e_1^{\vee}, \dots, e_n^{\vee}\}$ be the Poincaré dual basis for A^2, and take as generator of $A^3 = \Bbbk$ the class $\omega = 1^{\vee}$. The multiplication in A, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^{n} \mu_{ijk}\, e_k^{\vee}, \quad e_i e_j^{\vee} = \delta_{ij}\omega, \tag{4.5}$$

where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$ and δ_{ij} is the Kronecker delta. An alternate way to encode this information is to let $A_i = (A^i)^*$ be the dual $\Bbbk$-vector space and to let $e^i \in A_1$ be the

(Kronecker) dual of e_i. We may then view $\mu = \mu_A$ dually as a trivector,

$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \textstyle\bigwedge^3 A_1, \tag{4.6}$$

and will sometimes abbreviate this as $\mu = \sum \mu_{ijk}\, e^i e^j e^k$.

Example 4.1. It is readily seen that the trivector associated to a connected sum of two PD_3 algebras is the sum of the corresponding trivectors; that is,

$$\mu_{A\#B} = \mu_A + \mu_B. \tag{4.7}$$

Any alternating 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ on a finite-dimensional $\Bbbk$-vector space V determines a PD_3 algebra A over $\Bbbk$ for which $\mu_A = \mu$: simply take $A^0 = A^3 = \Bbbk$ and $A^1 = A^2 = V$, choose dual bases as above, and define the multiplication map as in (4.5).

Remark 4.2. In [29], Roos outlined procedures for writing down a presentation for the algebra A in terms of the trivector μ, and for determining whether A is a Koszul algebra.

Remark 4.3. In [36], Sullivan showed that every alternating 3-form over a field $\Bbbk$ of characteristic 0 can be realized as the 3-form associated to the cohomology algebra $A = H^\bullet(M, \Bbbk)$ of a closed, oriented 3-manifold M.

4.4. **Classification of alternating forms.** Let V be a $\Bbbk$-vector space of dimension n, and let $\bigwedge^m(V^*)$ be the vector space of alternating m-forms on V. The general linear group $\mathrm{GL}(V)$ acts on this affine space by

$$(g \cdot \mu)(a_1 \wedge \cdots \wedge a_m) := \mu\,(g^{-1}a_1 \wedge \cdots \wedge g^{-1}a_m). \tag{4.8}$$

The orbits of this action are the equivalence classes of alternating m-forms on V. (We write $\mu \sim \mu'$ if $\mu' = g \cdot \mu$.) Over $\overline{\Bbbk}$, the Zariski closures of these orbits define affine algebraic varieties. A standard dimension argument with algebraic groups (see e.g. [5]) shows that there can be finitely many orbits over $\overline{\Bbbk}$ only if $n^2 \geq \binom{n}{m}$, that is, $m \leq 2$ or $m = 3$ and $n \leq 8$. Furthermore, when $\Bbbk = \mathbb{R}$ and $\overline{\Bbbk} = \mathbb{C}$, each complex orbit has only finitely many real forms, by [2, Prop. 2.3].

Let us specialize now to the case of most interest to us, to wit, $m = 3$. For $\Bbbk = \mathbb{C}$, the classification of alternating trilinear forms was carried out by Schouten [30] in dimensions $n \leq 7$ and by Gurevich [22] for $n = 8$. For $\Bbbk = \mathbb{R}$, the classification was done by Gurevich, Revoy, and Westwick for $n \leq 7$ and by Djoković [13] for $n = 8$. The classification in dimensions $n \leq 7$ was extended to arbitrary fields by Cohen and Helminck [5];

Over $\mathbb{C}$ there are 23 orbits in dimension $n = 8$. Lying in the closure of another orbit defines a partial order on the set of orbits; the corresponding Hasse diagram is given in [14]. Those 23 complex orbits split into either 1, 2, or 3 real orbits, for a total of 35 orbits, as indicated in [13]. Representative trivectors for each one of these $\mathbb{C}$-orbits (and the corresponding $\mathbb{R}$-orbits for $n \leq 7$) are given in the tables from Appendix A.

4.5. **Maps of non-zero degree.** Let A and B be two PD_m algebras. We say that a morphism of graded algebras $\varphi\colon A \to B$ has *non-zero degree* if the linear map $\varphi^m\colon A^m \to B^m$ is non-zero. In this case, we may pick orientation classes such that

$$\varphi^m(\omega_A) = \omega_B. \tag{4.9}$$

Consequently, φ is compatible with the Poincaré duality isomorphisms from (4.2), that is, $(\varphi^{m-i})^* \circ \mathrm{PD}^i_A = \mathrm{PD}^i_B \circ \varphi^i$, for $0 \le i \le m$. It follows that

$$\mu_B \circ \textstyle\bigwedge^m \varphi^1 = \mu_A. \tag{4.10}$$

Once again, the terminology comes from topology: if $f\colon M \to N$ is a map of degree $d \neq 0$ between two closed, oriented manifolds of dimension m, then the induced morphism in cohomology, $f^*\colon H^\bullet(N, \Bbbk) \to H^\bullet(M, \Bbbk)$ will restrict to multiplication by d in degree m. Thus, if the characteristic of $\Bbbk$ does not divide d (for instance, if $\operatorname{char} \Bbbk = 0$), then the morphism f^* has non-zero degree.

We shall need the following alternate way to express the naturality of Poincaré duality with respect to non-zero degree morphisms (compare with [24, Lemma I.3.1]).

Lemma 4.4. *Let $\varphi\colon A \to B$ be a non-zero degree morphism between two PD_m algebras. Then $\varphi(a^\vee) = \varphi(a)^\vee$, for all homogeneous elements $a \in A$.*

Proof. We have $\varphi(a) \cdot \varphi(a^\vee) = \varphi(aa^\vee) = \varphi(\omega_A) = \omega_B$, and the claim follows at once. □

Proposition 4.5. *A morphism $\varphi\colon A \to B$ between two PD_m algebras is injective if and only if φ has non-zero degree.*

Proof. If φ is injective, then in particular φ^m is injective, and thus is non-zero. For the converse, suppose φ has non-zero degree. By the proof of the above lemma, $\varphi(a) \neq 0$, for all homogeneous elements $a \in A$, and the claim follows. □

For instance, if $A = B\#C$, then the canonical morphisms $B \to A$ and $B \to C$ are injective, and thus have non-zero degree.

An isomorphism of PD_m algebras is a map $\varphi\colon A \to B$ between two PD_m algebras which preserves both the graded algebra structures and the orientation classes.

Proposition 4.6. *Two PD_m algebras A and B are isomorphic as PD_m algebras if and only if they are isomorphic as graded algebras. Furthermore, either of these conditions implies that $\mu_A \sim \mu_B$.*

Proof. By Proposition 4.5, if $\varphi\colon A \to B$ is an isomorphism between the two underlying graded algebras, then condition (4.9) is satisfied, and so φ is an isomorphism of PD_m algebras. The converse is obvious.

Suppose now that $\varphi\colon A \to B$ is an isomorphism of PD_m algebras. Then, by (4.10), we have that $\mu_B \circ \bigwedge^m \varphi^1 = \mu_A$, that is, $\mu_B = \varphi^1 \cdot \mu_A$, and so $\mu_A \sim \mu_B$. □

Theorem 4.7. *For two PD_3 algebras A and B, the following are equivalent.*

(1) *$A \cong B$, as* PD_m *algebras.*
(2) *$A \cong B$, as graded algebras.*
(3) *$\mu_A \sim \mu_B$.*

Proof. In view of Proposition 4.6, we only need to show that (3) $\Rightarrow$ (2). Suppose $\mu_A \sim \mu_B$. There is then a linear isomorphism $g\colon A^1 \to B^1$ such that $\mu_B = g \cdot \mu_A$, that is, $\omega_B = (\bigwedge^3 g)(\omega_A)$. Define a map $\varphi\colon A \to B$ by requiring $\varphi^0 = \mathrm{id}$, $\varphi^1 = g$, $\varphi^2 = g^\vee$, and $\varphi^3 = \bigwedge^3 g$, where $g^\vee\colon A^2 \to B^2$ is given by $g^\vee(a^\vee) = (g(a))^\vee$. Clearly, φ is also a linear isomorphism. Now let $a, b \in A^1$ be two non-zero elements. Setting $c = (ab)^\vee$, we have

$$\omega_B = \varphi^3(g)(\omega_A) = \varphi^3(g)(abc) = g(a)g(b)g(c),$$

and so

$$\varphi(ab) = g^\vee(ab) = g^\vee(c^\vee) = g(c)^\vee = g(a)g(b) = \varphi(a)\varphi(b).$$

It follows that φ is an isomorphism of graded algebras, and we are done. $\square$

In conclusion, the constructions from §4.3 together with the above theorem establish a one-to-one correspondence between isomorphism classes of 3-dimensional Poincaré duality algebras and equivalence classes of alternating 3-forms, given by $A \leftrightsquigarrow \mu_A$.

5. Poincaré duality and resonance

In this section we explore some of the constraints imposed by Poincaré duality on the resonance varieties of a PD algebra. Henceforth, the ground field $\Bbbk$ will be assumed to be of characteristic different from 2.

5.1. Resonance varieties of PD_m algebras.

We start with a lemma expressing the compatibility between Poincaré duality and the BGG correspondence. A similar statement is proved in [28, Lemma 7.3], in a more general context. For completeness, we provide a short proof.

Lemma 5.1. *Let A be a* PD_m *algebra. Then, for all $0 \le i \le m$ and all $a \in A^1$, we have a commuting square,*

$$\begin{tikzcd}
(A^{m-i})^* \arrow[r, "(\delta^{m-i-1}_{-a})^*"] & (A^{m-i-1})^* \\
A^i \arrow[u, "\cong"', "\Phi_i"] \arrow[r, "\delta^i_a"] & A^{i+1} \arrow[u, "\cong"', "\Phi_{i+1}"],
\end{tikzcd}$$

where $\Phi_i = (-1)^i \,\mathrm{PD}^i$.

Proof. Let $b \in A^i$ and $c \in A^{m-i-1}$. Then $\mathrm{PD} \circ \delta_a(b)(c) = \mathrm{PD}(ab)(c) = \varepsilon(abc)$, while $\delta^*_{-a} \circ \mathrm{PD}(b)(c) = \mathrm{PD}(b)(\delta_{-a}(c)) = -\mathrm{PD}(b)(ac) = -\varepsilon(bac)$. Since $ab = (-1)^i ba$, we are done. $\square$

The next corollary follows at once.

Corollary 5.2. *Let A be a PD_m algebra. Then, for all $0 \le i \le m$ and all $a \in A^1$,*

$$\left(H^i(A, \delta_a)\right)^* \cong H^{m-i}(A, \delta_{-a}).$$

Furthermore, if $\varphi\colon A \to B$ is a morphism between two PD_m algebras, then the map $\varphi^i_a\colon H^i(A, \delta_a) \to H^i(B, \delta_{\varphi(a)})$ from (2.5) is dual to $\varphi^{m-i}_{-a}\colon H^{m-i}(A, \delta_{-a}) \to H^{m-i}(B, \delta_{-\varphi(a)})$.

We are now ready to state and prove the resonance analogue of the palindromicity of the Betti numbers of a Poincaré duality algebra.

Theorem 5.3. *Let A be a PD_m-algebra. Then, for all i and k,*

$$\mathscr{R}^i_k(A) = \mathscr{R}^{m-i}_k(A).$$

Proof. By Corollary 5.2, the $\Bbbk$-vector space $H^i(A, \delta_a)$ is dual to $H^{m-i}(A, \delta_{-a})$. The claimed equality follows straight from the definition of resonance. ◻

This theorem shows that it is enough to compute the resonance varieties of a PD_m algebra in degrees up to the middle dimension: the other ones are then essentially given by Poincaré duality.

As a consequence of Theorem 5.3, we deduce that $\mathscr{R}^m_1(A) = \{0\}$, a fact which was proved in a somewhat different fashion in [8, Prop. 5.14]. Moreover, in view of formula (2.4), we recover the fact that $b_i(A) = b_{m-i}(A)$. Thus, the above theorem may be regarded as a generalization of the palindromicity of the Poincaré polynomial of a closed, orientable manifold.

5.2. **Connected sums and resonance.** The resonance varieties of a connected sum of two Poincaré duality algebras can be computed in terms of the resonance varieties of the factors. Arguing as in the proof of Proposition 3.4, we obtain the following result.

Proposition 5.4. *Let $A = B\#C$ be the connected sum of two PD_m algebras with positive first Betti numbers. Then, for all $k \ge 0$,*

$$\mathscr{R}^i_k(A) = \begin{cases} \bigcup_{s+t=k-1} \mathscr{R}^i_s(B) \times \mathscr{R}^i_t(C) & \text{if } i = 1 \text{ or } m-1, \\ \bigcup_{s+t=k} \mathscr{R}^i_s(B) \times \mathscr{R}^i_t(C) & \text{if } 1 < i < m-1, \\ \{0\} & \text{if } i = 0 \text{ or } m, \text{ and } k = 1, \end{cases} \tag{5.1}$$

and $\mathscr{R}^i_k(A) = \emptyset$, otherwise.

Corollary 5.5. *Let $A = B\#C$ be the connected sum of two PD_m algebras. If $b_1(B) > 0$ and $b_1(C) > 0$, then $\mathscr{R}^1_1(A) = A^1$.*

Example 5.6. Let $A = H^{\bullet}(\Sigma_g, \Bbbk)$ be the cohomology algebra of a closed, orientable surface of genus $g \ge 2$. Since $\Sigma_g \cong \Sigma_{g-1} \# S^1 \times S^1$, the above corollary yields $\mathscr{R}^1_1(A) = A^1$.

5.3. **A resonance obstruction to domination.** A fundamental question in manifold topology (studied by Gromov [21] and others) is to decide whether there exists a map $f\colon M \to N$ of non-zero degree between two closed, oriented manifolds M and N of the same dimension. If such a map exists, one says that M *dominates* N.

By analogy, given two PD_m algebras A and B, we say that B dominates A if there is a non-zero degree morphism $A \to B$. By Proposition 4.5, this is equivalent to saying there is an injective morphism $A \to B$; in particular, we must have $b_i(A) \le b_i(B)$ for all $i \ge 0$. Applying Corollary 2.8, we obtain a geometric obstruction to domination.

Corollary 5.7. *Suppose $\mathscr{R}^1_k(A)$ has larger dimension (or more irreducible components) than $\mathscr{R}^1_k(B)$, for some $k \ge 1$. Then B does not dominate A.*

Example 5.8. The exterior algebra $E = \bigwedge(\Bbbk^m)$ is a Poincaré duality algebra of dimension m. Since the Koszul complex $\mathbf{L}(E) = E \otimes_{\Bbbk} S$ is exact, the resonance varieties of E vanish; more precisely, $\mathscr{R}^i_k(E) = \{0\}$ if $1 \le k \le \binom{m}{i}$ and is empty, otherwise. It follows that E does not dominate any PD_m algebra A for which $\mathscr{R}^1_1(A)$ has positive dimension.

6. The resonance varieties of a PD_3 algebra

We analyze now in more detail the structural properties of the resonance varieties of a 3-dimensional Poincaré duality algebra.

6.1. **Reduction to degree 1 resonance.** The next proposition reduces the computation of the resonance varieties of a PD_3 algebra to those in degree 1.

Proposition 6.1. *Let A be a PD_3 algebra with $b_1(A) = n$. Then*

(1) $\mathscr{R}^i_0(A) = A^1$.
(2) $\mathscr{R}^3_1(A) = \mathscr{R}^0_1(A) = \{0\}$ *and* $\mathscr{R}^2_n(A) = \mathscr{R}^1_n(A) = \{0\}$.
(3) $\mathscr{R}^2_k(A) = \mathscr{R}^1_k(A)$ *for* $0 < k < n$.
(4) *In all other cases,* $\mathscr{R}^i_k(A) = \emptyset$.

Proof. Statements (1), (2), and (4) follow straight from the definitions and previous remarks, while (3) follows from Theorem 5.3. ◻

Thus, in order to understand the resonance varieties of a PD_3 algebra A, it suffices to describe the resonance varieties $\mathscr{R}^1_k(A)$, in depths $0 < k < b_1(A)$. As a trivial example, suppose $\mu_A = 0$; then $\mathscr{R}^1_k(A) = A^1$ for $k < b_1(A)$.

6.2. **Decomposable and irreducible forms.** The next result further reduces the computation of the resonance varieties of an arbitrary PD_3 algebra to those of a PD_3 algebra whose associated 3-form is irreducible.

Let $\mu\colon \bigwedge^3 V \to \Bbbk$ be an alternating 3-form on a finite-dimensional $\Bbbk$-vector space V. The *rank* of μ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$; we write $\operatorname{corank}\mu = \dim V - \operatorname{rank}\mu$. The 3-form μ is said to be *irreducible* if it has maximal rank, that is, $\operatorname{corank}\mu = 0$.

Theorem 6.2. *Every* PD_3 *algebra* A *decomposes as* $A \cong B\#C$*, where* B *are* C *are* PD_3 *algebras such that* μ_B *is irreducible and has the same rank as* μ_A*, and* $\mu_C = 0$*. Furthermore, the isomorphism* $A^1 \cong B^1 \oplus C^1$ *restricts to isomorphisms*

$$\mathscr{R}^1_k(A) \cong \mathscr{R}^1_{k-r+1}(B) \times C^1 \cup \mathscr{R}^1_{k-r}(B) \times \{0\} \tag{6.1}$$

for all $k \geq 0$*, where* $r = \operatorname{corank} \mu_A$.

Proof. Let $W \subset A^1$ be a subspace of dimension equal to $\operatorname{rank} \mu_A$ for which the form $\mu_A\colon \bigwedge^3 V \to \Bbbk$ factors through $\bigwedge^3 W$, and let $\bar{\mu}$ be the restriction of μ to W. By construction, this is a 3-form whose rank equals that of μ, that is, $\operatorname{rank} \bar{\mu} = \dim W$.

Let B be the PD_3 algebra corresponding to $\bar{\mu}$. Evidently, $B^1 = W$ and $\mu_B = \bar{\mu}$ is irreducible. It is now readily seen that $A \cong B\#C$, where C is the PD_3 algebra with $C^1 = A^1/B^1$ and $\mu_C = 0$.

By a previous observation, $\mathscr{R}^1_t(C) = C^1$ for $t < r$ and $\mathscr{R}^1_r(C) = \{0\}$. Formula (6.1) now follows from Proposition 5.4. □

Remark 6.3. Suppose $A = H^{\bullet}(M, \Bbbk)$ is the cohomology algebra of a closed, orientable 3-manifold M. Write $M = N\#P$, where P is the connected sum of the factors in the prime decomposition of M having the $\Bbbk$-homology of either S^3 or $S^1 \times S^2$ and N is the connected sum of all the other factors. Setting $B = H^{\bullet}(N, \Bbbk)$ and $C = H^{\bullet}(P, \Bbbk)$, we recover the decomposition $A \cong B\#C$ from the above result.

As an immediate consequence of Theorem 6.2, we have the following corollary.

Corollary 6.4. *If* A *is a* PD_3 *algebra, then* $\mathscr{R}^1_k(A) = A^1$ *for all* $k < \operatorname{corank} \mu_A$.

6.3. **Nullity and isotropic subspaces.** Before proceeding, we need a few more classical definitions, suitably adapted to our setup (see for instance [16, 31]).

Let $\mu\colon \bigwedge^3 V \to \Bbbk$ be a 3-form. A linear subspace $U \subset V$ is 2-*singular* with respect to μ if $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$. (If $\dim U = 2$, we simply say U is a *singular plane*.) The *nullity* of μ, denoted $\operatorname{null}(\mu)$, is the maximum dimension of a 2-singular subspace $U \subset V$. Clearly, V contains a μ-singular plane if and only if $\operatorname{null}(\mu) \geq 2$.

The following (very simple) lemma clarifies the relationship between singularity and isotropicity in the context of PD_3 algebras.

Lemma 6.5. *Let* A *be a* PD_3 *algebra. A linear subspace* $U \subset A^1$ *is* 2*-singular (with respect to* μ_A*) if and only if* U *is isotropic.*

Proof. If $U \subset A^1$ is a 2-singular subspace, then $\mu_A(a \wedge b \wedge c) = \varepsilon(abc) = 0$ for all $a, b \in U$ and $c \in A^1$. Since the bilinear form $A^2 \otimes_{\Bbbk} A^1 \to \Bbbk$, $\gamma \otimes c \mapsto \varepsilon(\gamma c)$ is non-degenerate, this implies $ab = 0$ for all $a, b \in U$, that is, U is isotropic.

Conversely, if $U \subset A^1$ is an isotropic subspace, then $ab = 0$ for all $a, b \in U$. Thus, $\mu_A(a \wedge b \wedge c) = \varepsilon(abc) = 0$ for all $a, b \in U$ and $c \in A^1$, that is, U is 2-singular. □

The next result gives a lower bound on the dimension of the degree-1 resonance varieties.

Theorem 6.6. *Let A be a* PD_3 *algebra over an algebraically closed field* $\Bbbk$ *(of characteristic different from* 2*), and let* $\nu = \mathrm{null}(\mu_A)$ *be the nullity of the associated alternating* 3*-form. If* $b_1(A) \geq 4$*, then*

$$\dim \mathscr{R}^1_{\nu-1}(A) \geq \nu \geq 2.$$

In particular, $\dim \mathscr{R}^1_1(A) \geq \nu$*.*

Proof. Since $\dim_{\Bbbk} A^1 \geq 4$ and $\Bbbk$ is algebraically closed, a result of Sikora [31, Cor. 20] implies that $\mathrm{null}(\mu_A) \geq 2$.

To prove the other inequality, pick a linear subspace $U \subset A^1$ of dimension ν such that $\mu_A(a \wedge b \wedge c) = \varepsilon(abc) = 0$ for all $a, b \in U$ and $c \in A^1$. By Lemma 6.5, the subspace U is isotropic. Also, by what we just established, $\dim U \geq 2$. Therefore, by Lemma 2.2, $U \subseteq \mathscr{R}^1_{\nu-1}(A)$. Hence, $\dim U \leq \dim \mathscr{R}^1_{\nu-1}(A)$, and we are done. ◻

6.4. **Resonance varieties of** PD_3 **algebras over** $\mathbb{R}$**.** Motivated by his study of cut numbers of 3-manifolds, Sikora made in [31] the following conjecture: If $\mu\colon \bigwedge^3 V \to \Bbbk$ is a 3-form with $\dim V \geq 4$ and if $\mathrm{char}(\Bbbk) \neq 2$, then the nullity of μ is at least 2 (i.e., V contains a singular plane). He noted that the conjecture holds if either $n := \dim V$ is even or equal to 5, or, as mentioned above, if $\Bbbk = \overline{\Bbbk}$. Nevertheless, work of Draisma and Shaw [15, 16] implies that the conjecture does not hold for $\Bbbk = \mathbb{R}$ and $n = 7$. The following result explains the reason, in terms of resonance varieties.

Theorem 6.7. *Let A be a* PD_3 *algebra defined over* $\mathbb{R}$*. Then* $\mathscr{R}^1_1(A) \neq \{0\}$*, except when* μ_A *is one of the forms* I, III*, or* X_b *from Appendix A.*

Proof. Set $n = b_1(A)$. If $n \leq 2$ everything is clear, so let's assume that $n > 2$. We may also assume that μ_A is irreducible, for otherwise, by Corollary 5.5, $\mathscr{R}^1_1(A) = A^1$, and there is nothing to prove.

Suppose now that $\mathscr{R}^1_1(A) = \{0\}$, i.e., $\mathscr{R}^1_1(A)$ contains no singular plane. Then, by Lemmas 2.2 and 6.5, A^1 contains no singular plane. Hence, as shown in [16, Theorem 2], the formula $(x \times y) \cdot z = \mu_A(x, y, z)$ defines a cross-product on $A^1 = \mathbb{R}^n$. In turn, this cross-product yields a division algebra structure on $\mathbb{R}^{n+1}$, and so, by a celebrated result of Bott–Milnor and Kervaire, we must have $n = 3$ or 7. An inspection of the tables from Appendix A shows that μ_A must be equivalent to either III (the associated cross-product on $\mathbb{R}^3$ arises from quaternionic multiplication in $\mathbb{R}^4$) or X_b (as noted in [16], the corresponding cross-product on $\mathbb{R}^7$ arises from octonionic multiplication in $\mathbb{R}^8$). This completes the proof. ◻

The above proof highlights the fact (already alluded to in Remark 2.4) that real resonance varieties may carry more refined information than their complex counterparts. We make this observation more explicit in the following example.

Example 6.8. Let A and A' be the real PD_3 algebras corresponding to the trivectors X_a and X_b. Then $A\otimes_{\mathbb{R}}\mathbb{C}\cong A'\otimes_{\mathbb{R}}\mathbb{C}$, since $\mu_A\sim\mu_{A'}$ over $\mathbb{C}$. On the other hand, $A\not\cong A'$ over $\mathbb{R}$, since $\mu_A\nsim\mu_{A'}$ over $\mathbb{R}$, but also because $\mathscr{R}^1_1(A)\neq\{0\}$, yet $\mathscr{R}^1_1(A')=\{0\}$.

Note that both $\mathscr{R}^1_1(A\otimes_{\mathbb{R}}\mathbb{C})$ and $\mathscr{R}^1_1(A'\otimes_{\mathbb{R}}\mathbb{C})$ are projectively smooth conics, and thus are projectively equivalent over $\mathbb{C}$. Nevertheless, $\mathscr{R}^1_1(A'\otimes_{\mathbb{R}}\mathbb{C})=\{x\in\mathbb{C}^7\mid\sum x_i^2=0\}$ has only one real point ($x=0$), whereas $\mathscr{R}^1_1(A\otimes_{\mathbb{R}}\mathbb{C})=\{x\in\mathbb{C}^7\mid x_1x_4+x_2x_5+x_3x_6=x_7^2\}$ contains, for instance, the real (isotropic) subspace $\{x_4=x_5=x_6=x_7=0\}$.

7. Pfaffians ideals and resonance

In this section we express the resonance varieties of a PD_3 algebra A in terms of the Pfaffians of the skew-symmetric matrix associated to the boundary map δ^1_A, and determine those varieties in bottom depth.

7.1. **The cochain complex $\mathbf{L}(A)$.**

Once again, let A be a PD_3 algebra over a field $\Bbbk$ of characteristic not equal to 2. Fix a basis $\{e_1,\dots,e_n\}$ for A^1, identify the ring $S=\mathrm{Sym}(A_1)$ with $\Bbbk[x_1,\dots,x_n]$, and consider the cochain complex $\mathbf{L}(A)=(A\otimes_{\Bbbk}S,\delta_A)$ defined by the BGG correspondence,

$$A^0\otimes_{\Bbbk}S\xrightarrow{\delta^0_A}A^1\otimes_{\Bbbk}S\xrightarrow{\delta^1_A}A^2\otimes_{\Bbbk}S\xrightarrow{\delta^2_A}A^3\otimes_{\Bbbk}S\ . \tag{7.1}$$

Recall from §3.1 that the differentials in $\mathbf{L}(A)$ are the S-linear maps given by $\delta^q(u)=\sum_{j=1}^n e_ju\otimes x_j$ for $u\in A^q$. In the bases for $A^0,\dots,A^3$ chosen in §4.3, we have that

$$\begin{aligned}\delta^0_A(1)&=\sum_{j=1}^n e_j\otimes x_j\,,\\ \delta^1_A(e_i)&=\sum_{j=1}^n e_je_i\otimes x_j=\sum_{j=1}^n\sum_{k=1}^n\mu_{jik}e_k^{\vee}\otimes x_j\,,\\ \delta^2_A(e_i^{\vee})&=\sum_{j=1}^n e_je_i^{\vee}\otimes x_j=\omega\otimes x_i\,.\end{aligned}\tag{7.2}$$

Observe that the first and third maps have matrices $\delta^0_A=\begin{pmatrix}x_1&\cdots&x_n\end{pmatrix}$ and $\delta^2_A=(\delta^0_A)^{\top}$.The most interesting to us is the skew-symmetric matrix associated to the boundary map δ^1_A.

Example 7.1. Let $\mu_A=(e^1\wedge e^2+e^3\wedge e^4)\wedge e^5$ be the trivector 5_1 from Appendix A. Then

$$\delta^1_A=\begin{pmatrix}0&x_5&0&0&-x_2\\-x_5&0&0&0&x_1\\0&0&0&x_5&-x_4\\0&0&-x_5&0&x_3\\x_2&-x_1&x_4&-x_3&0\end{pmatrix}.$$

Remark 7.2. The matrices δ^1_A also appear in recent work of De Poi, Faenzi, Mezzetti, and Ranestad [6], as well as Cardinali and Giuzzi [4], though in both cases the geometric origin and the motivation for studying them is very much different from ours.

7.2. **Pfaffians and resonance.** By (3.4), each resonance variety $\mathscr{R}^1_k(A)$ is the vanishing locus of the codimension k minors of the skew-symmetric matrix δ^1_A. More generally, let θ be a skew-symmetric matrix of size $n \times n$ with entries in the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$. Define the resonance varieties of θ as

$$\mathscr{R}_k(\theta) = V(I_{n-k}(\theta)), \tag{7.3}$$

for $0 \le k \le n-1$, and set $\mathscr{R}_n(\theta) = \{0\}$. Put another way, the resonance varieties of a skew-symmetric matrix θ are the degeneracy loci of such a matrix. The next result expresses these loci in terms of the Pfaffians of θ.

Theorem 7.3. *Let* $\mathrm{Pf}_{2r}(\theta)$ *be the ideal of* $2r \times 2r$ *Pfaffians of an* $n \times n$ *skew-symmetric matrix* θ *with entries in* S. *Then:*

$$\begin{aligned} &\mathscr{R}_{2k}(\theta) = \mathscr{R}_{2k+1}(\theta) = V(\mathrm{Pf}_{n-2k}(\theta)), && \textit{if } n \textit{ is even},\\ &\mathscr{R}_{2k-1}(\theta) = \mathscr{R}_{2k}(\theta) = V(\mathrm{Pf}_{n-2k+1}(\theta)), && \textit{if } n \textit{ is odd}. \end{aligned} \tag{7.4}$$

Proof. As shown by Buchsbaum and Eisenbud [3, Cor. 2.6], the following inclusions hold, for each $r \ge 1$:

$$I_{2r}(\theta) \subseteq \mathrm{Pf}_{2r}(\theta) \subseteq \sqrt{I_{2r}(\theta)}, \text{ and } I_{2r-1}(\theta) \subseteq \mathrm{Pf}_{2r}(\theta). \tag{7.5}$$

Consequently, $V(I_{2r-1}(\theta)) = V(I_{2r}(\theta)) = V(\mathrm{Pf}_{2r}(\theta))$, and the claim follows. $\square$

Note that the ideal $\mathrm{Pf}_n(\theta)$ is principal, generated by $\mathrm{pf}(\theta)$, the maximal Pfaffian of θ, which equals 0 if n is odd. Thus, if n is even and θ is non-singular, then $\mathscr{R}_1(\theta) = \mathscr{R}_0(\theta) = V(\mathrm{pf}(\theta))$ is a hypersurface, while if θ is singular, then $\mathscr{R}_1(\theta) = \Bbbk^n$. On the other hand, if n is odd, then $\mathscr{R}_1(\theta) = \mathscr{R}_2(\theta) = V(\mathrm{Pf}_{n-1}(\theta))$.

Remark 7.4. We shall view the scheme structure for $\boldsymbol{\mathcal{R}}_k(\theta)$ as being defined by the Pfaffian ideals from (7.4).

Let us return now to the case when A is a PD_3 algebra and $\theta = \delta^1_A$ is the boundary map from (7.1). In that case, the matrix δ^1_A is singular, since $\delta^1_A \circ \delta^0_A = 0$. Therefore, we have the following chain of inclusions for the varieties $\mathscr{R}^1_k = \mathscr{R}^1_k(A)$:

$$\begin{aligned} &A^1 = \mathscr{R}^1_0 = \mathscr{R}^1_1 \supseteq \mathscr{R}^1_2 = \mathscr{R}^1_3 \supseteq \mathscr{R}^1_4 = \cdots && \text{if } b_1(A) \text{ is even},\\ &A^1 = \mathscr{R}^1_0 \supseteq \mathscr{R}^1_1 = \mathscr{R}^1_2 \supseteq \mathscr{R}^1_3 = \mathscr{R}^1_4 \supseteq \cdots && \text{if } b_1(A) \text{ is odd}. \end{aligned} \tag{7.6}$$

7.3. **Bottom-depth resonance.** We conclude this section with a vanishing result for the bottom resonance varieties of a PD_3 algebra whose associated 3-form is irreducible.

Theorem 7.5. *Let* A *be a* PD_3 *algebra. If* μ_A *has maximal rank* $n \ge 3$, *then*

$$\mathscr{R}^1_{n-2}(A) = \mathscr{R}^1_{n-1}(A) = \mathscr{R}^1_n(A) = \{0\}. \tag{7.7}$$

Proof. Clearly, $\mathscr{R}^1_n(A) = \{0\}$. Let $\delta^1 = \delta^1_A$ be the differential from (7.2). By (7.4) and (3.4), we have that

$$\mathscr{R}^1_{n-2}(A) = \mathscr{R}^1_{n-1}(A) = V(I_1(\delta^1)).$$

To complete the proof, it suffices to show that $\sqrt{I_1(\delta^1)} = \mathfrak{m}$, where $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ is the maximal ideal at 0. By (7.1) all entries of the matrix δ^1 belong to $\mathfrak{m}$, and so $\sqrt{I_1(\delta^1)} \subseteq \mathfrak{m}$. Since, by assumption, the form μ_A has rank n, each variable x_i occurs in some entry of δ^1_A, and thus equality holds. □

Combining now Theorems 6.2 and 7.5, we obtain the following immediate corollary.

Corollary 7.6. *Let A be a PD_3 algebra, and decompose it as $A = B \# C$, where μ_B is irreducible and $\mu_C = 0$. If $n = \dim A^1$ is at least 3, then $\mathscr{R}^1_{n-2}(A) = \mathscr{R}^1_{n-1}(A) = C^1$.*

8. Top-depth resonance of PD_3 algebras

In this section we study the geometry of the top-depth resonance varieties of a PD_3 algebra, with special emphasis on the case when the associated 3-form satisfies certain genericity conditions.

8.1. **Determinants and Pfaffians.** Let A be a PD_3 algebra over $\Bbbk$. As before, identify $S = \mathrm{Sym}(A_1)$ with $\Bbbk[x_1, \dots, x_n]$, where $n = b_1(A)$, and let $\delta^1 = \delta^1_A : A^1 \otimes_{\Bbbk} S \to A^2 \otimes_{\Bbbk} S$ be the first differential in the cochain complex $\mathbf{L}(A)$. In the previously chosen bases for A^1 and A^2, the matrix of δ^1 is skew-symmetric. Furthermore, δ^1 is singular, since the vector $(x_1, \dots, x_n)$ is in its kernel. Hence, both its determinant $\det(\delta^1)$ and its Pfaffian $\mathrm{pf}(\delta^1)$ vanish.

In [37, Ch. III, Lemmas 1.2 and 1.3.1], Turaev shows how to remedy this situation, so as to obtain well-defined determinant and Pfaffian polynomials for the form $\mu = \mu_A$ by looking at codimension 1 minors of the associated matrix δ^1.

Lemma 8.1 ([37])**.** *Suppose $n \geq 3$. There is then a polynomial $\mathrm{Det}(\mu) \in S$ such that, if $\delta^1(i; j)$ is the sub-matrix obtained from δ^1 by deleting the i-th row and j-th column, then*

$$\det \delta^1(i; j) = (-1)^{i+j} x_i x_j \,\mathrm{Det}(\mu).$$

Moreover, if n is even, then $\mathrm{Det}(\mu) = 0$, while if n is odd, then $\mathrm{Det}(\mu) = \mathrm{Pf}(\mu)^2$, where $\mathrm{pf}(\delta^1(i; i)) = (-1)^{i+1} x_i \,\mathrm{Pf}(\mu)$.

Remark 8.2. If n is odd, then $\mathrm{Det}(\mu)$ is a homogeneous polynomial of degree $n - 3$, while $\mathrm{Pf}(\mu)$ is a homogeneous polynomial of degree $(n - 3)/2$.

Let us note the following immediate corollary to Lemma 8.1.

Corollary 8.3. *With notation as above, let $\mathfrak{m}$ be the maximal ideal of S at 0. Then*

$$I_{n-1}(\delta^1) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \mathfrak{m}^2 \cdot (\mathrm{Pf}(\mu)^2) & \text{if } n \text{ is odd}. \end{cases}$$

We illustrate these notions with a simple example.

Example 8.4. Let $A = H^{\bullet}(\Sigma_g \times S^1, \Bbbk)$, where Σ_g is a Riemann surface of genus $g \geq 1$. The corresponding 3-form on $A^1 = \Bbbk^{2g+1}$ is $\mu = \sum_{i=1}^{g} a_i b_i c$, while $\operatorname{Pf}(\mu) = x_{2g+1}^{g-1}$. See also Example 7.1 for the case $g = 2$. ◇

8.2. **Generic forms.** The alternating 3-forms from Example 8.4 fit into the more general class of 'generic' 3-forms, a class introduced and studied by Berceanu and Papadima in [1]. For our purposes, it will be enough to consider the case when $n = 2g + 1$, for some $g \geq 1$.

We say that a 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ is *BP-generic* if there is an element $v \in V$ such that the 2-form $\gamma_v \in V^* \wedge V^*$ defined by

$$\gamma_v(a \wedge b) = \mu_A(a \wedge b \wedge v) \quad \text{for } a, b \in V \tag{8.1}$$

has rank $2g$, that is, $\gamma_v^g \neq 0$ in $\bigwedge^{2g} V^*$. Equivalently, in a suitable basis for V, we may write

$$\mu = \sum_{i=1}^{g} a_i \wedge b_i \wedge v + \sum w_{ijk}\, z_i \wedge z_j \wedge z_k, \tag{8.2}$$

where each z_i belongs to the span of $a_1, b_1, \dots, a_g, b_g$ in V, and the coefficients w_{ijk} are in $\Bbbk$.

The following lemma, which was first suggested by S. Papadima, was recorded in [12, Remark 5.2] (see also [11, Remark 4.5]). For completeness, we supply a proof, in this slightly more general context.

Lemma 8.5. *Assume that n is odd and greater than* 1. *Then $\mathscr{R}^1_1(A) \neq A^1$ if and only if μ_A is BP-generic.*

Proof. Suppose there is a class $c \in A^1$ such that $c \notin \mathscr{R}^1_1(A)$. Then, for any class $a \in A^1$ which is not a multiple of c, we have that $ac \neq 0$. Letting $b = (ac)^\vee \in A^1$, we infer that $\mu_A(a \wedge b \wedge c)$ is non-zero. It follows that the 2-form γ_c from (8.1) defines a symplectic form on a complementary subspace to the vector $c \in A^1$, thereby showing that μ_A is BP-generic. Backtracking through this argument proves the reverse implication. □

8.3. **The top resonance variety of a PD_3 algebra.** We are now in a position to describe fairly explicitly the first resonance variety of a 3-dimensional Poincaré duality algebra.

Theorem 8.6. *Let A be a PD_3 algebra over a field $\Bbbk$. Set $n = \dim A^1$ and let $\mu = \mu_A$ be the associated* 3*-form. Then*

$$\mathscr{R}^1_1(A) = \begin{cases} \emptyset & \text{if } n = 0; \\ \{0\} & \text{if } n = 1 \text{ or } n = 3 \text{ and } \mu \text{ has rank } 3; \\ V(\operatorname{Pf}(\mu)) & \text{if } n \text{ is odd, } n > 3, \text{ and } \mu \text{ is BP-generic}; \\ A^1 & \text{otherwise.} \end{cases} \tag{8.3}$$

Proof. If $n \leq 2$, then $\mu = 0$, and the conclusion is immediate. So suppose $n \geq 3$, and let $\delta^1 = \delta^1_A$ be the skew-symmetric matrix associated to μ, as in (7.1). Recall from (3.4) that $\mathscr{R}^1_1(A) = V(I_{n-1}(\delta^1))$.

If n is even, then, by Corollary 8.3, $I_{n-1}(\delta^1) = 0$, and so $\mathscr{R}^1_1(A) = A^1$.

If n is odd, then again by Corollary 8.3, $I_{n-1}(\delta^1) = \mathfrak{m}^2 \cdot (\operatorname{Pf}(\mu)^2)$. On the other hand, by Lemma 8.5, $I_{n-1}(\delta^1)$ is non-zero if and only if μ is BP-generic. In this case, either $n = 3$ and so $\operatorname{Pf}(\mu) = 1$ and $\mathscr{R}^1_1(A) = \{0\}$, or $n > 3$ and $\mathscr{R}^1_1(A) = V(\operatorname{Pf}(\mu))$ is a hypersurface of degree $(n-3)/2$. This completes the proof. ◻

As a corollary, we recover a closely related result, proved by Draisma and Shaw in [15, Thm. 3.2] by very different methods.

Corollary 8.7 ([15])**.** *Let V be a vector space of odd dimension $n \geq 5$ over a field $\Bbbk$ and let $\mu \in \bigwedge^3 V^*$. Then the union of all μ-singular planes is either all of V or a hypersurface defined by a homogeneous polynomial in $\Bbbk[V]$ of degree $(n-3)/2$.*

Proof. Let A be the PD_3 algebra corresponding to μ. By Lemmas 2.2 and 6.5, the union of all μ-singular planes in $A^1 = V$ coincides with $\mathscr{R}^1_1(A)$. Suppose that n is odd, $n \geq 5$, and assume $\mathscr{R}^1_1(A) \neq A^1$ (by Lemma 8.5, this means that μ is BP-generic). It follows from Theorem 8.6 that $\mathscr{R}^1_1(A) = V(\operatorname{Pf}(\mu))$. By Remark 8.2, $\operatorname{Pf}(\mu)$ is a homogeneous polynomial of degree $(n-3)/2$, and we are done. ◻

8.4. **Another genericity condition.** For a trivector $\mu \in \bigwedge^3 V^*$, there is another genericity condition studied by De Poi, Faenzi, Mezzetti, and Ranestad in [6]. This condition requires that, for any non-zero vector $v \in V$, the bilinear form γ_v from (8.1) have rank greater than 2 (this is condition (GC3) from Definition 2.9 in *loc. cit.*, a condition which implies that μ is irreducible).

In the presence of the aforementioned genericity condition, a more precise geometric description of the two top resonance schemes of the corresponding PD_3 algebra is given in [6, Prop. 4.4]. We summarize this result in our terminology, as follows.

Theorem 8.8 ([6])**.** *Let A be a PD_3 algebra over $\mathbb{C}$, and suppose μ_A is generic in the above sense. Writing $n = \dim A^1$, the following hold.*

1. *If n is odd, then $\boldsymbol{\mathcal{R}}^1_1(A)$ is a hypersurface of degree $(n-3)/2$ which is smooth if $n \leq 7$, and singular in codimension 5 if $n \geq 9$.*
2. *If n is even, then $\boldsymbol{\mathcal{R}}^1_2(A)$ has codimension 3 and degree $\frac{1}{4}\binom{n-2}{3} + 1$; it is smooth if $n \leq 10$, and singular in codimension 7 if $n \geq 12$.*

Appendix A. Resonance varieties of 3-forms of low rank

The following tables list the irreducible 3-forms $\mu = \mu_A$ of rank $n \leq 8$, and the corresponding resonance varieties, $\mathscr{R}_k = \mathscr{R}^1_k(A)$. The ground field $\Bbbk$ is either $\mathbb{C}$ or $\mathbb{R}$, as indicated. For simplicity, we will denote a trivector $e^i \wedge e^j \wedge e^k$ as ijk. We use the classification of 3-forms of rank at most 8 of Gurevich [22], with further elaborations from [5, 13, 14]. For $n = 6$ and 7, we record the way complex orbits split into real orbits, based on the tables of Djoković [13]. The computation of the resonance varieties was done using the package Macaulay2 [20].

$\mathbb{C}$	μ	$\mathscr{R}_1$	$\mathscr{R}_2$	$\mathscr{R}_3$
I	0	$\emptyset$	$\emptyset$	$\emptyset$
II	123	0	0	0
III	125 + 345	$\{x_5 = 0\}$	$\{x_5 = 0\}$	0

$\mathbb{C}$	$\mathbb{R}$	μ	$\mathscr{R}_1$	$\mathscr{R}_2 = \mathscr{R}_3$	$\mathscr{R}_4$
IV		135 + 234 + 126	$\Bbbk^6$	$\{x_1 = x_2 = x_3 = 0\}$	0
V	a	123 + 456	$\Bbbk^6$	$\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}$	0
	b	−135 + 146 + 236 + 245	$\Bbbk^6$	$V(x_1^2 + x_2^2, x_3^2 + x_4^2, x_5^2 + x_6^2, x_4x_5 - x_3x_6, x_3x_5 + x_4x_6, x_2x_5 - x_1x_6, x_1x_5 + x_2x_6, x_2x_3 - x_1x_4, x_1x_3 + x_2x_4)$	0

$\mathbb{C}$	$\mathbb{R}$	μ	$\mathscr{R}_1 = \mathscr{R}_2$	$\mathscr{R}_3 = \mathscr{R}_4$	$\mathscr{R}_5$
VI		123 + 145 + 167	$\{x_1 = 0\}$	$\{x_1 = 0\}$	0
VII		125 + 136 + 147 + 234	$\{x_1 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\}$	0
VIII	a	134 + 256 + 127	$\{x_1 = 0\} \cup \{x_2 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_2 = x_5 = x_6 = 0\}$	0
	b	−135 + 146 + 236 + 245 + 127	$\{x_1^2 + x_2^2 = 0\}$	$V(x_1, x_2, x_3^2 + x_4^2, x_5^2 + x_6^2, x_3x_5 + x_4x_6, x_4x_5 - x_3x_6)$	0
IX	a	125 + 346 + 137 + 247	$\{x_1x_4 + x_2x_5 = 0\}$	$V(x_7^2 - x_3x_6, x_1, x_2, x_4, x_5)$	0
	b	−135 + 146 + 236 + 245 + 127 + 347	$\{x_1x_3 + x_2x_4 = 0\}$	$V(x_7^2 - x_5x_6, x_1, x_2, x_3, x_4)$	0
X	a	123 + 456 + 147 + 257 + 367	$\{x_1x_4 + x_2x_5 + x_3x_6 = x_7^2\}$	0	0
	b	−135 + 146 + 236 + 245 + 127 + 347 + 567	$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 = 0\}$	0	0

$\mathbb{C}$	μ	$\mathscr{R}_1$	$\mathscr{R}_2 = \mathscr{R}_3$	$\mathscr{R}_4 = \mathscr{R}_5$	$\mathscr{R}_6$
XI	147+257+367+358	$\mathbb{C}^8$	$\{x_7 = 0\}$	$\{x_3 = x_5 = x_7 = x_8 = 0\} \cup \{x_1 = x_3 = x_4 = x_5 = x_7 = 0\}$	0
XII	456 + 147 + 257 + 367 + 358	$\mathbb{C}^8$	$\{x_5 = x_7 = 0\}$	$\{x_3 = x_4 = x_5 = x_7 = x_1x_8 + x_6^2 = 0\}$	0
XIII	123+456+147+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = 0\}$	$\{x_1 = x_3 = x_4 = x_5 = x_2x_6 + x_7x_8 = 0\}$	0
XIV	123 + 456 + 147 + 257 + 358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = x_5 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = 0\}$	0
XV	123 + 456 + 147 + 257 + 367 + 358	$\mathbb{C}^8$	$\{x_3 = x_5 = x_1x_4 - x_7^2 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$	0
XVI	147 + 268 + 358	$\mathbb{C}^8$	$\{x_1 = x_4 = x_7 = 0\} \cup \{x_8 = 0\}$	$\{x_1 = x_4 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_5 = x_6 = x_8 = 0\}$	0
XVII	147+257+268+358	$\mathbb{C}^8$	$\{x_7 = x_8 = 0\} \cup \{x_2 = x_5 = x_8 = 0\} \cup \{x_1 = x_4 = x_7 = 0\}$	$\{x_1 = x_2 = x_4 = x_5 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_5 = x_6 = x_7 = x_8 = 0\}$	0
XVIII	456 + 147 + 257 + 268 + 358	$\mathbb{C}^8$	$\{x_5 = x_8 = x_4x_6 - x_2x_7 = 0\} \cup \{x_4 = x_7 = x_5^2 - x_1x_8 = 0\}$	$\{x_1 = x_2 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\}$	0
XIX	147 + 257 + 367 + 268 + 358	$\mathbb{C}^8$	$\{x_2 - x_3 = x_5 - x_6 = x_7 - x_8 = 0\} \cup \{x_2 + x_3 = x_5 + x_6 = x_7 + x_8 = 0\} \cup \{x_7 = x_8 = 0\} \cup \{x_1 = x_4 = x_7 = 0\}$	$\{x_1 = x_4 = x_7 = x_8 = x_2 - x_3 = x_5 - x_6 = 0\} \cup \{x_1 = x_4 = x_7 = x_8 = x_2 + x_3 = x_5 + x_6 = 0\} \cup \{x_2 = x_3 = x_5 = x_6 = x_7 = x_8 = 0\}$	0
XX	456 + 147 + 257 + 367 + 268 + 358	$\mathbb{C}^8$	$\{x_5 - x_6 = x_7 - x_8 = x_4x_6 - x_2x_7 + x_3x_7 = 0\} \cup \{x_5 + x_6 = x_7 + x_8 = x_4x_6 - x_2x_7 - x_3x_7 = 0\} \cup \{x_7 = x_4 = x_5^2 - x_6^2 - x_1x_8 = 0\}$	$\{x_1 = x_4 = x_5 = x_6 = x_7 = x_8 = x_2 - x_3 = 0\} \cup \{x_1 = x_4 = x_5 = x_6 = x_7 = x_8 = x_2 + x_3 = 0\} \cup \{x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\}$	0
XXI	123 + 456 + 147 + 268 + 358	$\mathbb{C}^8$	$\{x_3x_5 + x_2x_6 + x_7x_8 = x_1 = x_4 = 0\} \cup \{x_1x_4 + x_8^2 = x_1x_3 - x_6x_8 = x_1x_2 + x_5x_8 = x_4x_6 + x_3x_8 = x_4x_5 - x_2x_8 = x_3x_5 + x_2x_6 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = 0\}$	0
XXII	123 + 456 + 147 + 257 + 268 + 358	$\mathbb{C}^8$	$\{f_1 = \cdots = f_{20} = 0\}$	0	0
XXIII	123 + 456 + 147 + 257+367+268+358	$\mathbb{C}^8$	$\{g_1 = \cdots = g_{20} = 0\}$	0	0

Note: In XXII and XXIII, the polynomials f_i and g_i are homogeneous of degree 3. The varieties cut out by each of these two sets of polynomials have codimension 3.

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