INFINITESIMAL FINITENESS OBSTRUCTIONS

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Abstract. Does a space enjoying good finiteness properties admit an algebraic model with commensurable finiteness properties? In this note, we provide a rational homotopy obstruction for this to happen. As an application, we show that the maximal metabelian quotient of a very large, finitely generated group is not finitely presented. Using the theory of 1-minimal models, we also show that a finitely generated group π admits a connected 1-model with finite-dimensional degree 1 piece if and only if the Malcev Lie algebra $\mathfrak{m}(\pi)$ is the lower central series completion of a finitely presented Lie algebra.

Contents

1. Introduction and statement of results

1.1. **Finite cdga models.** A recurring theme in topology is to determine the geometric and homological finiteness properties of spaces and groups. A prototypical such question is to determine whether a path-connected space X is homotopy equivalent to a CW-complex with finite q-skeleton, for some $1 \leqslant q \leqslant \infty$, in which case we say X is q-*finite*. Another question is to decide whether a finitely generated group π admits a finite presentation, or, more generally, a classifying space $K(\pi, 1)$ with finite q-skeleton.

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A fruitful approach to this type of question is to compare the finiteness properties of the spaces or groups under consideration to the corresponding finiteness properties of algebraic models for such spaces and groups. To formulate our motivating question, we need some terminology.

Let A be commutative differential graded algebra (for short, a cdga) over a field $\Bbbk$ of characteristic 0. By analogy with the aforementioned topological notion, we say that A is *q-finite* if it is connected (i.e., $A^0 = \Bbbk \cdot 1$) and $\sum_{i\leqslant q} \dim A^i < \infty$. Furthermore, we say that two cdgas A and B have the same (homotopy) q-type (written $A \simeq_q B$) if there is a zig-zag of cdga maps connecting A and B, with each such map inducing isomorphisms in homology up to degree q and a monomorphism in degree $q+1$.

We say that a cdga A is a *q-model* for a space X if it has the same q-type as Sullivan's cdga of piecewise polynomial, complex-valued forms on X. The basic question that we shall address in this paper is the following.

Question 1.1. When does a q-finite space X admit a q-finite q-model A?

An important motivation for this question comes from work of Dimca–Papadima [10] and Budur–Wang [6], who discovered some deep connections between the finiteness properties of algebraic models for spaces and the structure of the corresponding cohomology jump loci.

Observe that X is 1-finite if and only if the group $\pi = \pi_1(X)$ is finitely generated; moreover, if X is 2-finite, then π is finitely presented. Further motivation for considering Question 1.1 comes from an effort to understand whether the maximal metabelian quotient π/π'' of a finitely presented group π is also finitely presentable.

1.2. **Finiteness obstructions.** Our first result (which will be proved in §2.4), provides an infinitesimal obstruction to a positive answer to the above question.

Theorem 1.2. *Let X be a space which admits a q-finite q-model. If $\mathscr{M}_q(X)$ is the Sullivan q-minimal model* cdga *of X, then* $\dim H^i(\mathscr{M}_q(X)) < \infty$, *for all* $i \leqslant q+1$.

In [7], Budur and Wang found a completely different finiteness obstruction, involving the structure of the cohomology jump loci for rank 1 local systems on X. Namely, if X is as above and q-finite, then all the irreducible components passing through the origin of those jump loci (in degree at most q) are algebraic subtori of the character group of $\pi_1(X)$. As shown in Example 4.6, our infinitesimal obstruction may be subtler than this jump loci test.

As an application of Theorem 1.2, we produce a large class of finitely generated groups whose maximal metabelian quotients have no good finiteness properties, either at the level of presentation complexes, or at the level of 1-models. First, some quick definitions. A group G is said to be *very large* if it has a free, non-cyclic quotient; the group G is merely *large* if it has a finite-index subgroup which is very large. Our result (which will be proved in §3.4), may be stated as follows.

Theorem 1.3. *Let G be a metabelian group of the form $G = \pi/\pi''$, where π is a finitely generated, very large group. Then:*

(1) *G is not finitely presentable.*
(2) *G does not admit a 1-finite 1-model.*

Consequently, if π is a group as in Theorem 1.3, then the derived subgroup G' is not finitely generated. We observe in Example 4.10 that the condition that the group π be very large cannot be relaxed to it only being large.

1.3. **Malcev and holonomy Lie algebras.** In the last part of this paper, we turn to studying the finiteness properties of Lie algebras. We start in §5 by analyzing the holonomy Lie algebra, $\mathfrak{h}(A)$, associated to a 1-finite cdga A. Using the Chevalley–Eilenberg cochain functor $\mathscr{C}$, we define a *functorial* 1-minimal cdga, $\widehat{\mathscr{C}}(\mathfrak{h}(A))$, and a natural cdga transformation, $f_A \colon \widehat{\mathscr{C}}(\mathfrak{h}(A)) \to A$. Our main technical result (which we prove in §5.5), reads as follows.

Theorem 1.4. *The classifying map f_A is a functorial 1-minimal model map for A.*

As an application, we prove in Corollary §5.7 the following: If a finitely generated group π admits a 1-finite 1-model A, then the Malcev Lie algebra $\mathfrak{m}(\pi)$ is isomorphic to the lower central series (LCS) completion of the holonomy Lie algebra $\mathfrak{h}(A)$. This recovers a result from [3], which in turn generalizes a result from [4].

Finally, we use Theorem 1.4 to prove in §6 the following theorem, which provides a complete answer to Question 1.1 in the case when $q = 1$.

Theorem 1.5. *A space with finitely generated fundamental group π admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(\pi)$ is the LCS completion of a finitely presented Lie algebra.*

2. Algebraic models and finiteness obstructions

2.1. **Differential graded algebras.** We shall fix throughout a ground field $\Bbbk$ of characteristic 0. A commutative, differential graded algebra (cdga) over $\Bbbk$ is a positively-graded $\Bbbk$-vector space, $A = \bigoplus_{i\geqslant 0} A^i$, endowed with a graded-commutative multiplication map $\cdot \colon A^i \otimes A^j \to A^{i+j}$, and a differential $d \colon A^i \to A^{i+1}$ satisfying $d(a\cdot b) = da\cdot b + (-1)^i a\cdot db$, for every $a \in A^i$ and $b \in A^j$. We let $A = (A^\bullet, d)$ denote such an object. Let $Z^i(A) = \ker(d : A^i \to A^{i+1})$, $B^i(A) = \operatorname{im}(d : A^{i-1} \to A^i)$, and $H^i(A) = Z^i(A)/B^i(A)$. The cohomology of the underlying cochain complex, $H^\bullet(A)$, is a commutative, graded algebra (cga); we let $b_i(A) = \dim H^i(A)$ be its Betti numbers.

Fix an integer $q \geqslant 1$. (We will also allow $q = \infty$, although we will usually omit q from terminology and notation in that case.) We say that two cdgas A and B have *the same (homotopy) q-type* if there is a zig-zag of cdga maps from A to B,

$$A \xleftarrow{\varphi_1} A_1 \xrightarrow{\varphi_2} \cdots \longleftarrow A_{k-1} \xrightarrow{\varphi_k} B\,, \tag{1}$$

with each map φ_j being a *q-equivalence*, i.e., inducing isomorphisms in homology up to degree q and a monomorphism in degree $q+1$. (If $q=\infty$, the maps φ_j are also called quasi-isomorphisms.) Clearly, if $A \simeq_q B$, then $b_i(A) = b_i(B)$ for all $i \leqslant q$.

Let $\bigwedge V$ be the free graded-commutative algebra generated by the graded vector space $V = \bigoplus_{i>0} V^i$. Following [39, 25], we say that a cdga is *q-minimal* if it is of the form $(\bigwedge V, d)$, where the differential structure is the inductive limit of a sequence of Hirsch extensions of increasing degrees, and $V^i = 0$ for $i > q$, if $q \neq \infty$.

A *q-minimal model map* for a cdga A is a q-equivalence $(\bigwedge V, d) \to A$ with $(\bigwedge V, d)$ q-minimal. Every cdga A with connected homology admits such a map. Moreover, the isomorphism type of the cdga $(\bigwedge V, d)$ is uniquely determined; this cdga is called *the q-minimal model of* A, and is denoted by $\mathscr{M}_q(A)$. It is readily seen that two cdga's with connected cohomology have the same q-type if and only if their q-minimal models are isomorphic.

2.2. **Algebraic models for spaces and groups.** Given a topological space X, we let $\Omega^{\bullet}(X)$ be Sullivan's algebra [39] of piecewise polynomial, complex-valued forms on X. This is a functorially defined cdga with the property that $H^{\bullet}(\Omega(X)) \cong H^{\bullet}(X, \mathbb{C})$, as graded rings. When X is a smooth manifold, $\Omega(X) \simeq \Omega_{\mathrm{dR}}(X)$, de Rham's algebra of smooth $\mathbb{C}$-forms on X. Furthermore, if X is a simplicial complex, then $\Omega(X) \simeq \Omega_{\mathrm{s}}(X)$, the algebra of piecewise polynomial, $\mathbb{C}$-forms on the simplices of X.

We say that a cdga (A, d) is a q-model for X if $A \simeq_q \Omega(X)$. By considering the classifying space $X = K(\pi, 1)$ of a group π, and replacing X by π in both terminology and notation, we may speak about q-minimal models, q-types and finiteness properties of groups, in the sense from Question 1.1. If A is q-finite and $A \simeq_q \Omega(X)$, then clearly $b_i(X) = b_i(A) < \infty$, for all $i \leqslant q$ (or all i, if $q = \infty$). Moreover, if $\pi = \pi_1(X)$ is the fundamental group of a path-connected space X, then any classifying map $X \to K(\pi, 1)$ induces an isomorphism between the corresponding 1-minimal models, $\mathscr{M}_1(X) \cong \mathscr{M}_1(\pi)$.

A continuous map $f\colon X \to Y$ is said to be a *rational homotopy equivalence* if the induced map $f^*\colon H^{\bullet}(Y, \mathbb{Q}) \to H^{\bullet}(X, \mathbb{Q})$ is an isomorphism. Clearly, such a map induces an equivalence $\Omega(Y) \simeq \Omega(X)$. Consequently, the existence of a q-finite q-model for a space X is an invariant of rational homotopy type (and thus, of homotopy type).

Unless otherwise specified, all spaces we consider here will be path-connected, and will have the homotopy type of a CW-complex; for short, we will call such objects *CW-spaces*. Oftentimes, the geometry of a CW-space forces the existence of a finite model for it. Examples of spaces having finite models include quasi-projective manifolds, compact solvmanifolds, Kähler manifolds, Sasakian manifolds, and principal bundles with compact, connected structural group over finite CW-complexes having finite models; see for instance [10, 14, 32] and references therein.

2.3. **Nilpotent spaces and formal spaces.** A CW-space X is said to be *nilpotent* if the fundamental group $\pi = \pi_1(X)$ is nilpotent and acts unipotently on $\pi_n(X)$ for all $n > 1$.

Nilpotent spaces provide a class of examples for which the answer to Question 1.1 is unobstructed.

Theorem 2.1 ([39]). *Let X be a nilpotent CW-space.*

1. *If all the Betti numbers of X are finite, then X admits a q-finite q-model, for all $1 \leqslant q < \infty$.*
2. *Moreover, if $\dim H_\bullet(X, \Bbbk) < \infty$, then X admits a finite model.*

Proof. Sullivan proved in [39] that the minimal model of a nilpotent CW-space with finite Betti numbers is of the form $\mathscr{M}(X) = (\bigwedge V, d)$, where V is a graded vector space of finite type. Hence, $\mathscr{M}(X)$ is of finite type, as a graded vector space, from which claim (1) follows.

Assume now that $H^{>n}(X) = 0$, for some $n > 0$. Pick a vector space decomposition, $\mathscr{M}^n(X) = Z^n(\mathscr{M}(X)) \oplus C^n$. Plainly, the direct sum $J = \mathscr{M}^{\geqslant n+1}(X) \oplus C^n$ is an acyclic differential graded ideal of $\mathscr{M}(X)$. By construction, $\Omega(X) \simeq \mathscr{M}(X)/J$, and the cdga $\mathscr{M}(X)/J$ is finite; thus, claim (2) is also verified. □

A space X is said to be *q-formal* if $\Omega(X) \simeq_q (H^\bullet(X, \mathbb{C}), d = 0)$. For this interesting class of spaces, the answer to Question 1.1 is again unobstructed: clearly, if a space X is q-formal and q-finite, then it has the q-finite q-model $(H^\bullet(X, \mathbb{C}), d = 0)$.

2.4. **On the Betti numbers of minimal models.** We now go beyond nilpotent rational homotopy theory and formal spaces. We fix $1 \leqslant q < \infty$.

Lemma 2.2. *Given a q-finite cdga A, there is a natural equivalence $A \simeq_q A[q]$, where $A[q]$ is a finite cdga, and $A[q]^{>q+1} = 0$.*

Proof. The construction is in two steps. First, we replace A by the cdga $A/A^{>q+1}$. Since clearly the natural projection, $A \twoheadrightarrow A/A^{>q+1}$, is a q-equivalence, we may suppose in the second step that $A[q]^{>q+1} = 0$. Next, we put

$$A[q] := \bigoplus_{i \leqslant q} A^i \oplus \Big(dA^q + \sum_{i,j \leqslant q}^{i+j=q+1} A^i \cdot A^j \Big). \tag{2}$$

Plainly, $A[q]$ is a sub-cdga of A. By construction, the natural inclusion, $A[q] \hookrightarrow A$, is a q-equivalence, and $A[q]$ is finite. □

Theorem 2.3. *Assume that either $\Omega(X) \simeq_q A$ with A q-finite, or X is $(q+1)$-finite. Then $b_i(\mathscr{M}_q(X)) < \infty$, for all $i \leqslant q+1$.*

Proof. In the first case, we may assume by Lemma 2.2 that the cdga A is actually finite. Since $\Omega(X) \simeq_q A$, the differential graded algebra $\mathscr{M}_q(X)$ is the q-minimal model of A. The claim follows from the discussion in §2.1.

In the second case, the claim follows directly from the existence of a q-equivalence $\mathscr{M}_q(X) \to \Omega(X)$. □

Corollary 2.4. *Let π be a finitely generated group. Assume that either π has a 1-finite 1-model, or π is finitely presented. Then $b_2(\mathscr{M}_1(\pi)) < \infty$.*

Proof. Observe that π is finitely generated (resp., finitely presented) if and only if π admits a classifying space $X = K(\pi, 1)$ which is 1-finite (resp., 2-finite). The claim follows at once from the above theorem, by setting $q = 1$. ◻

2.5. **Equivariant algebraic models.** Let Φ be a finite group acting freely on a space Y, and let $X = Y/\Phi$ be the orbit space. Our next goal is to compare the algebraic models associated to Y and X, and understand how the answer to Question 1.1 for one space affects the answer for the other one.

We start by setting up the category Φ-cdga (over $\Bbbk$): the objects are cdgas A endowed with a compatible Φ-action, while the morphisms are Φ-equivariant cdga maps $A \to B$. Given a Φ-cdga A, we let A^{Φ} be the sub-cdga of elements fixed by Φ; there is then a canonical cdga map $A^{\Phi} \to A$. By definition, a q-equivalence $A \simeq_q B$ in Φ-cdga ($1 \leqslant q \leqslant \infty$) is a zigzag of Φ-equivariant q-equivalences in cdga.

Lemma 2.5. *If $A \simeq_q B$ in Φ-cdga, then $A^{\Phi} \simeq_q B^{\Phi}$ in cdga.*

Proof. It is readily seen that the fixed points functor $-^{\Phi}$ commutes with homology. Thus, this functor takes equivariant q-equivalences to q-equivalences. ◻

As is well-known, every CW-complex X has the homotopy type of a simplicial complex K; moreover, if X has finite q-skeleton, so does K; see [19, Theorem 2C.5]. Fix such a triangulation of X, and lift it to the cover Y. The corresponding simplicial Sullivan algebras are then related, as follows:

$$\Omega_{\rm s}(X) = \Omega_{\rm s}(Y)^{\Phi}. \tag{3}$$

Using now Lemma 2.5, we obtain the following result.

Proposition 2.6. *Let X be a CW-space, and let $Y \to X$ be a finite Galois cover, with group of deck transformations Φ. Let A be a Φ-cdga over $\mathbb{C}$.*

1. *Suppose $\Omega(Y) \simeq_q A$ in Φ-cdga, for some $1 \leqslant q \leqslant \infty$. Then $\Omega(X) \simeq_q A^{\Phi}$ in cdga.*
2. *If, moreover, A is q-finite, then A^{Φ} is q-finite.*

As a consequence, if Y admits an equivariant q-finite q-model, then X admits a q-finite q-model. The Φ-equivariant equivalence hypothesis from Proposition 2.6(1) cannot be completely dropped. Nevertheless, we venture the following conjecture.

Conjecture 2.7. Let X be a connected CW-space, and let $Y \to X$ be a finite Galois cover with deck group Φ. Suppose that Y has finite Betti numbers. Let A be a Φ-cdga, and assume that there is a zig-zag of quasi-isomorphisms connecting $\Omega(Y)$ to A in cdga, such that the induced isomorphism between $H^{\bullet}(Y, \mathbb{C})$ and $H^{\bullet}(A)$ is Φ-equivariant. Then A^{Φ} is a model for X.

In the formal case this conjecture holds, and leads to the following result (see also [2, §2.2]).

Proposition 2.8. *Suppose Φ is a finite group acting simplicially on a formal simplicial complex Y with finite Betti numbers. Then the orbit space $X = Y/\Phi$ is again formal.*

Proof. We deduce from [26, Corollary 2.9] and the remark following it that the space Y is formal if and only if $\Omega(Y) \simeq (H^{\bullet}(Y), d = 0)$ in Φ-cdga. This equivalence is based on the Halperin–Stasheff approach to formality from [18].

Using the natural equivalence $\Omega \simeq \Omega_{\rm s}$, we now infer from Lemma 2.5 that $\Omega_{\rm s}(Y)^{\Phi} \simeq (H^{\bullet}(Y)^{\Phi}, d = 0)$ in cdga. Observing that equality (3) holds in this broader setup (where the Φ-action is not necessarily free), we conclude that $\Omega_{\rm s}(X) \simeq (H^{\bullet}(X), d = 0)$ in cdga. Since $\Omega(X) \simeq \Omega_{\rm s}(X)$, we are done. ◻

Example 2.9. Recall that a quasi-projective variety is a Zariski open subset of a projective variety. We will say that a space X is a *quasi-projective manifold* if it is a connected, smooth, complex quasi-projective variety. As shown by Morgan [25], every manifold of this sort admits a finite model $A(\overline{X}, D)$; such a 'Gysin' model depends on a smooth compactification $\overline{X}$ for which the complement $D = \overline{X} \backslash X$ is a normal crossings divisor.

Suppose now that $Y \to X$ is a finite cover of a quasi-projective manifold X. Then Y is also a quasi-projective manifold, as shown for instance in [15, Lemma 4.1]. Suppose further that $Y \to X$ is a finite Galois cover, with deck group Φ. As shown in [16, 21], one may find a compactification $\overline{Y}$ such that the corresponding Gysin model $A = A(\overline{Y}, D)$ is a Φ-equivariant model for Y. With this choice, all the hypotheses of Proposition 2.6 are verified, for $q = \infty$.

3. Malcev Lie algebras, derived series, and Hall–Reutenauer bases

We devote this section to a proof of Theorem 1.3.

3.1. Malcev Lie algebras.

We start by recalling some notions from [33, Appendix A] (see also [28, 37]). A *Malcev Lie algebra* is a Lie algebra $\mathfrak{m}$ over a field $\Bbbk$ of characteristic 0, endowed with a decreasing, complete $\Bbbk$-vector space filtration $F = \{F_i\}_{i \geqslant 1}$ such that $F_1 = \mathfrak{m}$ and $[F_i, F_j] \subset F_{i+j}$, for all i, j, and with the property that the associated graded Lie algebra, $\operatorname{gr}(\mathfrak{m}) = \bigoplus_{i \geqslant 1} F_i/F_{i+1}$, is generated in degree 1.

For example, the completion $\widehat{L}$ of a Lie algebra L with respect to the lower central series (LCS) filtration $\Gamma = \{\Gamma_i(L)\}_{i \geqslant 1}$, endowed with the canonical completion filtration F, is a Malcev Lie algebra. (Note that the LCS filtration on L coincides with the degree filtration, in the case when L is positively graded and the degree 1 component L^1 generates L as a Lie algebra.)

In [33], Quillen associates to every group π, in a functorial way, a Malcev Lie algebra, denoted by $\mathfrak{m}(\pi)$. This object, called the *Malcev completion* of π, captures the properties of the torsion-free nilpotent quotients of π. Here is a concrete way to describe it. The group algebra $\Bbbk\pi$ has a natural Hopf algebra structure, with comultiplication given by $\Delta(g) = g \otimes g$, and counit the augmentation map. Let I be the augmentation ideal of $\Bbbk\pi$. One verifies that the Hopf algebra structure on $\Bbbk\pi$ extends to the I-adic completion, $\widehat{\Bbbk\pi} = \varprojlim_r \Bbbk\pi/I^r$. Finally, $\mathfrak{m}(\pi)$ coincides with the Lie algebra of primitive elements in $\widehat{\Bbbk\pi}$, endowed with the inverse limit filtration.

For a finitely generated group π, there is a natural duality between $\mathfrak{m}(\pi)$ and $\mathscr{M}_1(\pi)$ [39]: the cdga $\mathscr{M}_1(\pi)$ is the inductive limit $\varinjlim_i \mathscr{C}(\mathfrak{m}/F_i)$, where $\mathscr{C}$ is the Chevalley–Eilenberg cdga cochain functor applied to finite-dimensional Lie algebras [9].

3.2. **Filtered formality and finiteness properties.** We recall from [37] that a group π is said to be *filtered formal* if $\mathfrak{m}(\pi) \cong \widehat{L}$ as filtered Lie algebras, where the Lie algebra L is graded and generated in degree 1, of the form $L = \mathbb{L}/J$, with $\mathbb{L}$ a finitely generated, free, graded Lie algebra and $J \subseteq \mathbb{L}^{\geqslant 2}$ a graded Lie ideal generated in degrees 2 and higher.

We denote by $(\cdot)^{(k)}$ the k-th term of the derived series of groups and Lie algebras. It follows from [28, Theorem 3.5] and [37, Theorem 1.7] that the groups $\mathbb{F}_n/\mathbb{F}_n^{(k)}$ are filtered formal, with corresponding Lie algebras $L = \mathbb{L}_n/\mathbb{L}_n^{(k)}$, where $\mathbb{L}_n$ denotes the free Lie algebra on n generators and $\mathbb{F}_n$ is the free group on n generators.

Here is our first result tying certain finiteness properties of algebraic objects associated to a group π, under a filtered formality assumption.

Proposition 3.1. *Let π be a finitely generated, filtered formal group, so that $\mathfrak{m}(\pi) \cong \widehat{L}$, where $L = \mathbb{L}/J$ is a graded Lie algebra generated in degree 1 and $J \subseteq \mathbb{L}^{\geqslant 2}$. If $b_2(\mathscr{M}_1(\pi)) < \infty$, then* $\dim(J/[\mathbb{L}, J]) < \infty$.

Proof. By the aforementioned duality between $\mathfrak{m}(\pi)$ and $\mathscr{M}_1(\pi)$, the following holds: $b_k(\mathscr{M}_1(\pi)) < \infty$ if and only if the inverse limit $\varprojlim_i H_k(\mathfrak{m}/F_i)$ is a finite-dimensional $\Bbbk$-vector space, where $H_k(-)$ stands for Lie algebra homology, and $F = \{F_i\}_{i\geqslant 1}$ is the canonical inverse limit filtration on $\mathfrak{m} = \mathfrak{m}(\pi)$.

Our filtered formality assumption on the group π yields Lie algebra isomorphisms

$$\mathfrak{m}/F_i \cong \mathbb{L}/(\mathbb{L}^{\geqslant i} + J), \tag{4}$$

for all $i \geqslant 1$. Moreover, the isomorphism $\widehat{L} \cong \mathfrak{m}$ induces identifications $L/\Gamma_i(L) \cong \mathfrak{m}/F_i$, which yield compatible Lie algebra maps $L \to \mathfrak{m}/F_i$. Passing to homology, we obtain a natural homomorphism to the inverse limit,

$$\varphi\colon H_\bullet(L) \longrightarrow \varprojlim_i H_\bullet(\mathfrak{m}/F_i)\ . \tag{5}$$

Let us focus now on degree $\bullet = 2$, where a well-known formula of Hopf (see [20]) gives identifications

$$H_2(L) = J/[\mathbb{L}, J] \quad \text{and} \quad H_2(\mathfrak{m}/F_i) = (\mathbb{L}^{\geqslant i} + J)/(\mathbb{L}^{\geqslant i+1} + [\mathbb{L}, J]). \tag{6}$$

Since all vector spaces in sight are graded and the map φ respects degrees, we infer that the map φ_2 is injective. Putting things together completes the proof. ◻

3.3. **Hall–Reutenauer bases.** Once again, let $\mathbb{L}_n$ be the free Lie algebra on the (totally ordered) alphabet $\{1,\dots,n\}$, and let $\mathbb{L}''_n$ be its second derived Lie subalgebra. We analyze below a certain quotient of $\mathbb{L}''_n$; the *Hall–Reutenauer bases* constructed in [34, Theorem 5.7, p. 112] will be particularly well-adapted to our purposes.

Proposition 3.2. *For $n \geqslant 2$, the graded vector space $\mathbb{L}''_n/[\mathbb{L}_n, \mathbb{L}''_n]$ is infinite-dimensional.*

Proof. Given elements $h_1,\dots,h_k \in \mathbb{L}_n$, we denote by $[h_1,\dots,h_k] \in \mathbb{L}_n$ the iterated bracket $[\dots[[h_1,h_2],\dots,h_k]$. We set $\mathcal{H}_0 := \{1,\dots,n\}$, and define inductively

$$\mathcal{H}_i := \{[h_1,\dots,h_k]\}\,, \tag{7}$$

where $k \geqslant 2$, $h_1,\dots,h_k \in \mathcal{H}_{i-1}$, and $h_1 < h_2 \geqslant \cdots \geqslant h_k$. We pick a total order on the trees indexing the elements of $\mathcal{H}_i$, compatible with degrees in $\mathbb{L}_n$ (that is, $h < h'$ if $\deg(h) < \deg(h')$), and decree that $h > h'$, for $h \in \mathcal{H}_{i-1}$ and $h' \in \mathcal{H}_i$. As shown in [34] the Lie monomials from $\bigcup_{i\geqslant \ell}\mathcal{H}_i$, form a basis for the vector space $\mathbb{L}_n^{(\ell)}$, for all ℓ.

Let $\mathbb{L} = \mathbb{L}_{\mathcal{A}}$ be the free Lie algebra on the alphabet $\mathcal{A}$, and $J \subseteq \mathbb{L}$ an ideal. A straightforward induction on degree establishes an equality of vector spaces, $[\mathbb{L}, J] = [\mathcal{A}, J]$. By a split surjectivity argument, it will be enough to check that

$$\dim \frac{\mathbb{L}''}{[\mathcal{A}, \mathbb{L}'']} = \infty\,, \tag{8}$$

where $\mathcal{A} = \{x, y\}$ with $x < y$. Note that the Lie algebra $\mathbb{L} = \bigoplus_{i,j}\mathbb{L}_{i,j}$ is naturally bigraded, where i is the x-degree and j is the y-degree.

Clearly, $\mathcal{H}_0 = \{x, y\}$. It follows from (7) that

$$\mathcal{H}_1 = \{[xy^p x^q] \mid p \geqslant 1,\, q \geqslant 0,\, p+q \geqslant 1\}\,, \tag{9}$$

where $[xy^p x^q]$ denotes $[x, \overbrace{y,\dots,y}^{p}, \overbrace{x,\dots,x}^{q}]$. In particular, $x\text{-deg}(h) \geqslant 1$, for all $h \in \mathcal{H}_1$. Induction on $i \geqslant 1$ based on definition (7) shows that

$$x\text{-deg}(h) \geqslant 2^{i-1}, \text{ for all } h \in \mathcal{H}_i\,. \tag{10}$$

In particular, all elements of the Hall–Reutenauer basis of $\mathbb{L}''$ have x-degree $\geqslant 2$.

Consequently, if the vector space $\mathbb{L}''/[\mathcal{A}, \mathbb{L}'']$ is trivial in degree $i+2$, then necessarily ad_y induces a surjection $\mathbb{L}''_{2,i-1} \twoheadrightarrow \mathbb{L}''_{2,i}$. Hence,

$$\dim \mathbb{L}''_{2,i} \leqslant \dim \mathbb{L}''_{2,i-1}\,. \tag{11}$$

Next, we compute dimensions in (11). We infer from (10), together with (7) and (9), that the following elements form a basis of $\mathbb{L}''_{2,i}$:

$$\{[[xy^{p+1}], [xy^{q+1}]] \mid 0 \leqslant p < q \text{ and } p+q = i-2\}\,. \tag{12}$$

Therefore,

$$\dim \mathbb{L}''_{2,i} = \begin{cases} k & \text{if } i = 2k+1, \\ k-1 & \text{if } i = 2k. \end{cases} \tag{13}$$

Consequently, the graded vector space $\mathbb{L}''/[\mathcal{A}, \mathbb{L}'']$ must be non-zero in all odd degrees 5 and higher, since otherwise (13) would contradict (11). This verifies claim (8), thereby completing our proof. □

3.4. **Proof of Theorem 1.3.** Let π be a finitely generated group. By assumption, there is an epimorphism $\varphi\colon \pi \twoheadrightarrow \mathbb{F}_n$, for some $n \geqslant 2$. Since the group $\mathbb{F}_n$ is free, the map φ admits a splitting. Hence, the induced homomorphism, $\bar{\varphi}\colon \pi/\pi'' \twoheadrightarrow \mathbb{F}_n/\mathbb{F}_n''$, is again a split epimorphism. By the homotopy functoriality of the 1-minimal model construction, the map $\bar{\varphi}$ induces a cdga map,

$$\bar{\varphi}^*\colon \mathscr{M}_1(\mathbb{F}_n/\mathbb{F}_n'') \longrightarrow \mathscr{M}_1(\pi/\pi'')\ , \tag{14}$$

which is a split injection up to homotopy.

Suppose now that the conclusion of the theorem does not hold, i.e., suppose that the group π/π'' is finitely presentable, or that it admits a 1-finite 1-model. It then follows from Corollary 2.4 that $b_2(\mathscr{M}_1(\pi/\pi'')) < \infty$. Since, as we saw above, the map $\bar{\varphi}^*$ is split injective (up to homotopy), and since homology is a homotopy functor, we conclude that

$$b_2(\mathscr{M}_1(\mathbb{F}_n/\mathbb{F}_n'')) < \infty. \tag{15}$$

Now, since the group $\mathbb{F}_n/\mathbb{F}_n''$ is filtered formal and (15) holds, Proposition 3.1 implies that $\dim(\mathbb{L}_n''/[\mathbb{L}_n, \mathbb{L}_n'']) < \infty$. But this contradicts Proposition 3.2, and so we are done. □

4. Cohomology jump loci, finiteness properties, and largeness

4.1. **Cohomology jump loci.** Let X be a path-connected space with fundamental group $\pi = \pi_1(X)$. The cohomology jump loci with coefficients in rank 1 complex local systems on X are powerful homotopy-type invariants of the space, that have been the subject of intense investigation in recent years, see for instance [12, 30, 10, 6]. These loci sit inside the complex algebraic group $\widehat{\pi} := \operatorname{Hom}(\pi, \mathbb{C}^\times)$, and are defined for all $i, r \geqslant 0$ by

$$\mathscr{V}^i_r(X) = \{\rho \in \widehat{\pi} \mid \dim H^i(X, V_\rho) \geqslant r\}\,, \tag{16}$$

where V_ρ is the associated local system. When the space X is q-finite, the *characteristic varieties* $\mathscr{V}^i_r(X)$ are Zariski closed subsets of the character group $\widehat{\pi}$, for all $i \leqslant q$ and $r \geqslant 0$, see [31]. It is easily seen that the the sets $\mathscr{V}^i_r(X)$ with $i \leqslant 1$ and $r \geqslant 0$ depend only on the group $\pi = \pi_1(X)$.

Now let π be a finitely generated group, and define $\mathscr{V}^i_r(\pi) := \mathscr{V}^i_r(K(\pi, 1))$ for $i, r \geqslant 0$. It is known that the sets $\mathscr{V}^i_r(\pi)$ with $i \leqslant 1$ and $r \geqslant 0$ depend only on the maximal metabelian quotient π/π'' (see e.g. [11, 23]); more precisely, the following equality holds,

$$\mathscr{V}^i_r(\pi) = \mathscr{V}^i_r(\pi/\pi''). \tag{17}$$

The characteristic varieties have several useful naturality properties. For instance, suppose $\varphi\colon \pi \twoheadrightarrow G$ is an epimorphism. Then the induced morphism on character groups, $\varphi^*\colon \widehat{G} \to \widehat{\pi}$, is injective and sends $\mathscr{V}^1_r(G)$ into $\mathscr{V}^1_r(\pi)$ for all $r \geqslant 0$. Likewise, suppose that $H < \pi$ is a finite-index subgroup. Then, as noted in [27, Lemma 3.6], the inclusion

$\alpha\colon H \to \pi$ induces a morphism $\hat{\alpha}\colon \hat{\pi} \to \widehat{H}$ with finite kernel, which sends $\mathcal{V}^i_r(\pi)$ to $\mathcal{V}^i_r(H)$ for all $i, r \geqslant 0$.

For the free groups $\mathbb{F}_n$ of rank $n \geqslant 2$, we have that $\mathcal{V}^1_r(\mathbb{F}_n) = (\mathbb{C}^\times)^n$ for $r \leqslant n-1$ and $\mathcal{V}^1_n(\mathbb{F}_n) = \{1\}$. In general, though, the jump loci of a group can be arbitrarily complicated.

Example 4.1. Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be an integral Laurent polynomial with $f(1) = 0$. Then, as shown in [38], there is a finitely presented group π with $\pi_{\mathrm{ab}} = \mathbb{Z}^n$ such that $\mathcal{V}^1_1(\pi)$ coincides with the variety $\mathbf{V}(f) := \{t \in (\mathbb{C}^\times)^n \mid f(t) = 0\}$.

4.2. **A finiteness obstruction.** Let A be a connected cdga. Clearly, $H^1(A) = Z^1(A)$. For every $\omega \in H^1(A)$, the operator $d_\omega := d + \omega \cdot$ is a differential on $A^\bullet$. The *resonance varieties* $\mathscr{R}^i_r(A)$ are defined, for all $i, r \geqslant 0$, as the infinitesimal jump loci

$$\mathscr{R}^i_r(A) = \{\omega \in H^1(A) \mid \dim H^i(A, d_\omega) \geqslant r\}\,. \tag{18}$$

When the cdga A is q-finite, these sets are Zariski closed subsets of the affine space $H^1(A)$, for all $i \leqslant q$ and $r \geqslant 0$, The following key result is a particular case of [10, Theorem B].

Theorem 4.2 ([10])**.** *Let X be a q-finite space which admits a q-finite q-model A. There is then a local analytic isomorphism between the germ at 1 of $\pi_1(X)\hat{}$ and the germ at 0 of $H^1(A)$, that identifies the germ at 1 of $\mathcal{V}^i_r(X)$ with the germ at 0 of $\mathscr{R}^i_r(A)$, for all $i \leqslant q$ and $r \geqslant 0$.*

Recent work of Budur and Wang [7], building on the aforementioned theorem, provides a very strong finiteness obstruction for spaces, based on the geometry of their characteristic varieties.

Theorem 4.3. *If X is q-finite and has a q-finite q-model, then each irreducible component of $\mathcal{V}^i_r(X)$ passing through the identity $1 \in \hat{\pi}$ is an algebraic subtorus of the character group of $\pi = \pi_1(X)$, for all $i \leqslant q$ and $r \geqslant 0$.*

This theorem follows from [7, Corollary 2.2] and Theorem 4.2, and refines [10, Theorem C(2)]; see also [7, Theorem 1.3(1)]. The Budur–Wang jump loci finiteness obstruction from Theorem 4.3 may be used to give a negative answer to Question 1.1, even for spaces X which are finite CW-complexes.

Example 4.4. Let f be an integral Laurent polynomial in $n \geqslant 2$ variables, and assume its zero set in $(\mathbb{C}^\times)^n$ contains the origin 1, is irreducible but is not an algebraic subtorus; for instance, take $f(t) = \sum_{i=1}^n t_i - n$. Letting π be a finitely presented group with $\mathcal{V}^1_1(\pi) = \mathbf{V}(f)$ as in Example 4.1, we infer from Theorem 4.3 that the finite presentation complex of π admits no 1-finite 1-model.

Conversely, the existence of a 1-finite 1-model for a finitely generated group π does not necessarily imply that π is finitely presented.

Example 4.5. Let Y be a finite, connected CW-complex which is non-simply connected yet has $b_1(Y) = 0$, and let π be the Bestvina–Brady group associated to a flag triangulation of Y. It is proved in [29, §10] that π is finitely generated and 1-formal, but not finitely presented.

As the next family of examples illustrates, our infinitesimal finiteness obstruction from Theorem 2.3 is stronger than the Budur–Wang obstruction from Theorem 4.3, even in the case when $q = 1$.

Example 4.6. Consider the free metabelian group $\pi = \mathbb{F}_n/\mathbb{F}_n''$ with $n \geqslant 2$. The free group $\mathbb{F}_n = \pi_1(\bigvee^n S^1)$ has a finite, formal classifying space; thus, Theorem 4.3 applies to $\mathbb{F}_n$. It follows that the varieties $\mathcal{V}^i_r(\pi) \cong \mathcal{V}^i_r(\mathbb{F}_n)$ pass the Budur–Wang test for $i \leqslant 1$ and $r \geqslant 0$.

On the other hand, as we saw in the proof of Theorem 1.3, we have that $b_2(\mathscr{M}_1(\pi)) = \infty$, and so the group π admits no 1-finite 1-model.

4.3. **Largeness.** Following Gromov [17], a finitely generated group π is said to be *large* if there is a finite-index subgroup $H < \pi$ such that H surjects onto a free, non-cyclic group. In this definition, there is no loss of generality in assuming $H \lhd \pi$ is a normal subgroup. In general, it is a difficult problem to decide whether a group is large. In [22], Koberda gives the following largeness test for groups admitting a finite presentation.

Theorem 4.7 ([22])**.** *A finitely presented group π is large if and only if there exists a finite-index subgroup $K < \pi$ such that $\mathcal{V}^1_1(K)$ has infinitely many torsion points.*

As noted in [23], the finite presentation assumption is crucial for this test. For instance, taking again $\pi = \mathbb{F}_n/\mathbb{F}_n''$ with $n \geqslant 2$, the variety $\mathcal{V}^1_1(\pi) = (\mathbb{C}^\times)^n$ has infinitely many torsion points, though the group π is solvable, and thus not large.

Also in [23], the authors introduce the notion of RFRp group (for p a prime). The class of groups which are RFRp for all primes p includes all groups of the form $\pi_1(C)$, for C a smooth complex curve with $\chi(C) < 0$, and all right-angled Artin groups. Using the criterion from Theorem 4.7, they obtain the following generalization of an old result of Baumslag and Strebel.

Theorem 4.8 ([23])**.** *Let π be a finitely generated group which is non-abelian and RFRp for infinitely many primes p. Then π/π'' is not finitely presented.*

Using a similar method, we may give an alternate proof of Theorem 1.3(1).

Proposition 4.9. *Let π be a finitely generated, very large group. Then π/π'' is not finitely presented.*

Proof. By assumption, there is an epimorphism $\varphi\colon \pi \twoheadrightarrow \mathbb{F}_n$ to a free group of rank $n \geqslant 2$. The induced morphism, $\hat{\varphi}\colon \widehat{\mathbb{F}_n} \hookrightarrow \widehat{\pi}$, embeds $\mathcal{V}^1_1(\mathbb{F}_n) = (\mathbb{C}^\times)^n$ into $\mathcal{V}^1_1(\pi)$. Thus, this variety, which coincides with $\mathcal{V}^1_1(\pi/\pi'')$, contains infinitely many torsion points. If the group π/π'' were finitely presented, then, by Theorem 4.7, it would be large. But this is impossible, since π/π'' is solvable, and subgroups of solvable groups are again solvable. ∎

It is now natural to ask whether the above result holds in the more general setting of large groups. More precisely: Let π be a finitely generated, large group; is it true that the maximal metabelian quotient π/π'' is not finitely presented? As the next family of examples shows, the answer to this question is negative.

Example 4.10. Let $\pi = \pi_1 * \pi_2$ be the free product of two non-trivial finite groups, of orders m_1 and m_2, with $m_1 m_2 > 4$. By [36, Proposition 4, p. 6], the kernel of the natural surjection, $\pi_1 * \pi_2 \twoheadrightarrow \pi_1 \times \pi_2$, is a free group $\mathbb{F}$ of rank $(m_1-1)(m_2-1) > 1$. Hence, π is a large group. On the other hand, π' contains $\mathbb{F}$ as a subgroup of finite index. Therefore, π' is finitely generated; hence, the abelian group π'/π'' is finitely generated, as well. Since π/π' is also a finitely generated abelian group, we infer that π/π'' is finitely presented.

4.4. **Large quasi-projective groups.** We now turn to the question of deciding whether a quasi-projective group (i.e., a group that can be realized as the fundamental group of a quasi-projective manifold) is large. It turns out that a complete answer to this question can be given in terms of D. Arapura's theory of "admissible" maps to curves.

A map $f\colon X \to C$ from a quasi-projective manifold X to a smooth complex curve C is said to be *admissible* if it is regular, surjective, and has connected generic fiber. It is easy to see that the homomorphism on fundamental groups induced by such a map, $f_\sharp\colon \pi_1(X) \to \pi_1(C)$, is surjective. We denote by $\mathcal{E}(X)$ the family of admissible maps to curves with negative Euler characteristic, modulo automorphisms of the target.

Deep work of Arapura [1] characterizes those positive-dimensional, irreducible components of the characteristic variety $\mathcal{V}^1_1(X)$ which contain the origin of the character group $\pi_1(X)\hat{}$: all such components are connected, affine subtori, which arise by pull-back of the character torus $\pi_1(C)\hat{}$ along the homomorphism $f_\sharp\colon \pi_1(X) \to \pi_1(C)$ induced by some map $f \in \mathcal{E}(X)$.

Suppose now that C is a smooth complex curve with $\chi(C) < 0$. It is readily seen that the fundamental group $\pi = \pi_1(C)$ surjects onto a free, non-abelian group; consequently, π is large. More generally, we have the following characterization of large, quasi-projective groups.

Proposition 4.11. *Let X be a quasi-projective manifold. Then:*

(1) *$\pi_1(X)$ is large if and only if there is a finite cover $Y \to X$ such that $\mathcal{E}(Y) \neq \emptyset$.*
(2) *$\pi_1(X)$ is very large if and only if $\mathcal{E}(X) \neq \emptyset$.*

Proof. We only prove part (1); the proof of part (2) is entirely similar.

Assume first that the group $\pi = \pi_1(X)$ is large. There is then a finite-index subgroup $G < \pi$ and an epimorphism $\varphi\colon G \twoheadrightarrow F$, where F is a free group of rank $n \geqslant 2$. Let $Y \to X$ be the finite cover corresponding to G; by the discussion in Example 2.9, Y is again a quasi-projective manifold. Arguing as in the proof of [1, Corollary 1.9], we see that the induced morphism on character groups, $\varphi^*\colon \widehat{F} \hookrightarrow \widehat{G}$, takes $\mathcal{V}^1_1(F) \cong (\mathbb{C}^\times)^n$ to a positive-dimensional subvariety of $\mathcal{V}^1_1(Y)$ passing through 1. The irreducible component of $\mathcal{V}^1_1(Y)$ containing this subvariety, then, corresponds to an admissible map in $\mathcal{E}(Y)$.

Conversely, suppose that there is a finite cover $Y \to X$ supporting an admissible map $f\colon Y \to C$, where $\chi(C) < 0$. Composing the induced homomorphism $f_\sharp\colon \pi_1(Y) \twoheadrightarrow \pi_1(C)$ with a surjection $\pi_1(C) \twoheadrightarrow \mathbb{F}_n$ ($n \geqslant 2$), we obtain an epimorphism $\pi_1(Y) \twoheadrightarrow \mathbb{F}_n$, thereby showing that $\pi_1(X)$ is large. ◻

As noted in Example 2.9, a quasi-projective manifold X has the homotopy type of a finite CW-complex admitting a finite model. Among other things, this very useful property provides a concrete test for the fundamental group of X being very large, i.e., admitting an epimorphism onto a free, non-cyclic group. (Clearly, this property implies that $b_1(X) > 0$.)

As noted in Proposition 4.11(2), the group $\pi_1(X)$ is very large if and only if $\mathscr{E}(X) \neq \emptyset$. Assume that $b_1(X) > 0$. Again by Arapura theory [1], this geometric test for very largeness has the following interpretation in terms of cohomology jump loci: $\mathscr{E}(X) = \emptyset$ if and only if the analytic germ at 1 of $\mathscr{V}^1_1(X)$ equals $\{1\}$.

4.5. **Resonance and largeness.** To conclude this section, we rephrase the last condition in terms of resonance varieties. Let A be a Gysin model for X, or any one of the more general Orlik–Solomon models constructed by Dupont [13]. In either case, let us note that all resonance varieties of A have *positive weights*, i.e., they are invariant with respect to a $\mathbb{C}^\times$-action on $H^1(A)$ with positive weights.

Proposition 4.12. *Let X be a quasi-projective manifold with $b_1(X) > 0$. Let A be any Orlik–Solomon model of X. Then $\pi_1(X)$ is very large if and only if $\mathscr{R}^1_1(A) \neq \{0\}$.*

Proof. As we already know, $\pi_1(X)$ is not very large if and only if the analytic germ at 1 of $\mathscr{V}^1_1(X)$ equals $\{1\}$, or equivalently, by Theorem 4.2, the germ at 0 of $\mathscr{R}^1_1(A)$ equals $\{0\}$. Since $\mathscr{R}^1_1(A)$ has positive weights, all its irreducible components pass through 0. The argument from the first paragraph of [10, §9.23] shows then that the aforementioned local equality is equivalent to the global equality $\mathscr{R}^1_1(A) = \{0\}$. ◻

Remark 4.13. In practice, the geometric test for very largeness from Proposition 4.11 may be very difficult to implement, while the alternative test involving characteristic varieties requires an explicit presentation for the fundamental group. On the other hand, once an Orlik–Solomon model A has been constructed, the property that $\mathscr{R}^1_1(A) = \{0\}$ can be tested concretely, using standard tools from computational commutative algebra. The practical value of the infinitesimal test from Proposition 4.12 is well illustrated by the next class of examples.

Example 4.14. Let Σ_g be a compact, connected Riemann surface of genus g, and let $X = F_\Gamma(\Sigma_g)$ be the partial configuration space associated to a finite simple graph Γ. More concretely, if n is the number of vertices of Γ, then $F_\Gamma(\Sigma_g)$ is the complement in Σ_g^n of the union of the diagonals $z_i = z_j$, indexed by the edges of Γ.

No presentation is available for the fundamental group $\pi_{\Gamma,g} = \pi_1(F_\Gamma(\Sigma_g))$, for arbitrary graph Γ and genus g. On the other hand, as noted in [3], the quasi-projective manifold

$F_\Gamma(\Sigma_g)$ admits an easily computable Orlik–Solomon model A. Computing the resonance variety $\mathscr{R}^1_1(A)$ leads to a complete, explicit description of $\mathscr{E}(F_\Gamma(\Sigma_g))$; such a description is given in [3, Theorem 1.1], for all $g \geqslant 0$ and for all finite graphs Γ. In particular, $\mathscr{E}(F_\Gamma(\Sigma_g)) = \emptyset$, that is, $\pi_{\Gamma,g}$ is not very large, if and only if either $g = 1$ and Γ has no edges, or $g = 0$ and Γ contains no complete subgraph on 4 vertices.

5. A functorial 1-minimal model map

We devote this section to the proof of Theorem 1.4, and derive a topological interpretation.

5.1. **Holonomy Lie algebras.** Given a 1-finite cdga A, let $A_i = \mathrm{Hom}(A^i, \Bbbk)$ be the dual vector space. Let $\mu^*\colon A_2 \to A_1 \wedge A_1$ be the dual to the multiplication map $\mu\colon A^1 \wedge A^1 \to A^2$, and let $d^*\colon A_2 \to A_1$ be the dual of the differential $d\colon A^1 \to A^2$.

Definition 5.1 ([24])**.** The *holonomy Lie algebra* of a 1-finite cdga $A = (A^\bullet, d)$ is the quotient of the free Lie algebra on the $\Bbbk$-vector space A_1 by the ideal generated by the image of $\partial_A = d^* + \mu^*$:

$$\mathfrak{h}(A) = \mathbb{L}(A_1)/(\mathrm{im}(\partial_A)). \tag{19}$$

Clearly, this construction is functorial. Observe that the Lie algebra $\mathfrak{h}(A)$ depends only on the cdga $A[1]$ constructed in Lemma 2.2, i.e., on the sub-cdga

$$\Bbbk \cdot 1 \oplus A^1 \oplus (d(A^1) + \mu(A^1 \wedge A^1)) \tag{20}$$

of the truncation $A^{\leqslant 2}$. In particular, $\mathfrak{h}(A)$ is finitely presented.

In the case when $d = 0$, the above definition coincides with the classical holonomy Lie algebra of K.T. Chen [8]. In this situation, the Lie algebra $\mathfrak{h}(A)$ inherits a natural grading from the free Lie algebra $\mathbb{L}(A_1)$, compatible with the Lie bracket. Consequently, $\mathfrak{h}(A)$ is a finitely-presented, graded Lie algebra, with generators in degree 1 and relations in degree 2.

In general, though, the ideal generated by $\mathrm{im}(\partial_A)$ is not homogeneous, and the Lie algebra $\mathfrak{h}(A)$ is not graded. Here is a concrete example, extracted from [10, Example 5.8] and [24, Example 4.4].

Example 5.2. Let A be the exterior algebra on generators x, y in degree 1, endowed with the differential given by $dy = 0$ and $dx = x \wedge y$, and let $\mathfrak{sol}_2$ be the Borel subalgebra of $\mathfrak{sl}_2$. Then $\mathfrak{h}(A) \cong \mathfrak{sol}_2$ does not admit a graded Lie algebra structure.

5.2. **Holonomy and flat connections.** Next, we recall from [10] and [24] another (bi-functorial) construction which will be important to us. For a cdga A and a Lie algebra $\mathfrak{g}$, let $\mathscr{F}(A, \mathfrak{g})$ be the set of $\mathfrak{g}$-valued *flat connections* on A, i.e., the set of those elements $\omega \in A^1 \otimes \mathfrak{g}$ satisfying the Maurer–Cartan equation,

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0. \tag{21}$$

Suppose now that A is 1-finite. By [24, Proposition 4.5], the natural isomorphism $A^1 \otimes \mathfrak{g} \xrightarrow{\cong} \operatorname{Hom}(A_1, \mathfrak{g})$ induces a natural identification,

$$\mathscr{F}(A, \mathfrak{g}) \xrightarrow{\cong} \operatorname{Hom}_{\mathrm{Lie}}(\mathfrak{h}(A), \mathfrak{g}) \ . \tag{22}$$

Assume also that $\mathfrak{g}$ is finite-dimensional. We let then $\mathscr{C}(\mathfrak{g}) = \big(\bigwedge \mathfrak{g}^*, d\big)$ be the Chevalley–Eilenberg complex of $\mathfrak{g}$, that is, the cdga whose underlying graded algebra is the exterior algebra on $\mathfrak{g}^*$, and whose differential is the extension by the graded Leibnitz rule of the dual of the signed Lie bracket, $d = -\beta^*$, on the algebra generators, see e.g. [20]. We then have a natural isomorphism $A^1 \otimes \mathfrak{g} \xrightarrow{\cong} \operatorname{Hom}(\mathfrak{g}^*, A^1)$, which, by [10, Lemma 3.4], induces a natural identification,

$$\mathscr{F}(A, \mathfrak{g}) \xrightarrow{\cong} \operatorname{Hom}_{\mathsf{cdga}}(\mathscr{C}(\mathfrak{g}), A) \ . \tag{23}$$

We will also consider the cochain functor $\widehat{\mathscr{C}}$, defined on finitely generated Lie algebras and taking values in 1-minimal cdga's. This functor associates to a finitely generated Lie algebra $\mathfrak{h}$ the inductive limit of cdga's

$$\widehat{\mathscr{C}}(\mathfrak{h}) = \varinjlim_n \mathscr{C}(\mathfrak{h}/\Gamma_n). \tag{24}$$

The 1-minimality property of $\widehat{\mathscr{C}}(\mathfrak{h})$ is a consequence of the well-known fact that $\mathscr{C}$ sends finite-dimensional central Lie extensions to Hirsch extensions of cdga's.

5.3. **A classifying map.** As before, let A be a 1-finite cdga, with holonomy Lie algebra $\mathfrak{h} = \mathfrak{h}(A)$. By (22), the identity map of $\mathfrak{h}$ may be identified with the 'canonical' flat connection,

$$\omega = \sum_i x_i^* \otimes x_i \in \mathscr{F}(A, \mathfrak{h}(A)), \tag{25}$$

where $\{x_i\}$ is a basis for A_1 and $\{x_i^*\}$ is the dual basis for A^1. This gives rise to a compatible family of flat connections, $\{\omega_n \in \mathscr{F}(A, \mathfrak{h}/\Gamma_n)\}_{n\geqslant 1}$. Using the correspondence (23), we obtain a compatible family of cdga maps, $f_n\colon \mathscr{C}(\mathfrak{h}/\Gamma_n) \to A$. Passing to the limit, we arrive at a natural cdga map,

$$f\colon \widehat{\mathscr{C}}(\mathfrak{h}(A)) \longrightarrow A \ . \tag{26}$$

Our goal in this section is to show that this map is as an infinitesimal analog of the classifying map $X \to K(\pi_1(X), 1)$. More precisely, we will prove the following theorem.

Theorem 5.3. *For any 1-finite cdga A, the classifying map $f\colon \widehat{\mathscr{C}}(\mathfrak{h}(A)) \to A$ is a 1-minimal model map.*

To prove this theorem, we have to show that $H^1(f)$ is an isomorphism and $H^2(f)$ is a monomorphism. It will be convenient to assume that A is finite and $A^{>2} = 0$. By naturality and the discussion from §5.1, there is no loss of generality in doing so.

We will use duality and the dual Chevalley–Eilenberg complex, which computes the untwisted Lie algebra homology $H_\bullet(\mathfrak{g})$ of an arbitrary Lie algebra $\mathfrak{g}$. This chain complex has the form $\{\partial_n\colon \bigwedge^n\mathfrak{g} \to \bigwedge^{n-1}\mathfrak{g}\}_{n\geqslant 0}$, with differentials

$$\partial_n(x_1\wedge\cdots\wedge x_n) = \sum_{i<j}(-1)^{i+j}\beta(x_i\wedge x_j)x_1\wedge\cdots\wedge\widehat{x}_i\wedge\cdots\wedge\widehat{x}_j\wedge\cdots\wedge x_n\,. \tag{27}$$

This chain complex is functorial: given a morphism of Lie algebras, $\varphi\colon \mathfrak{g}\to\mathfrak{h}$, the following diagram commutes,

$$\begin{array}{ccc} \bigwedge^n\mathfrak{g} & \xrightarrow{\partial_n} & \bigwedge^{n-1}\mathfrak{g} \\ \big\downarrow{\scriptstyle \bigwedge^n\varphi} & & \big\downarrow{\scriptstyle \bigwedge^{n-1}\varphi} \\ \bigwedge^n\mathfrak{h} & \xrightarrow{\partial_n} & \bigwedge^{n-1}\mathfrak{h} \end{array} \tag{28}$$

for each $n\geqslant 1$. Furthermore, both vertical arrows are surjective when φ is an epimorphism.

We will come back to the proof of Theorem 5.3 in §5.5, after proving a couple of technical lemmas.

5.4. **Two stability properties.** By definition, the generating vector space V of a 1-minimal cdga $(\bigwedge V, d)$ has an increasing, exhaustive filtration $\{V^n\}_{n\geqslant 1}$ starting at $V^1=0$, with the property that each $(\bigwedge V^n, d)$ is a sub-cdga and each inclusion $(\bigwedge V^n, d)\hookrightarrow(\bigwedge V^{n+1}, d)$ is a Hirsch extension. We call such a filtration a *defining filtration* for $(\bigwedge V, d)$. Defining filtrations having the following two stability properties will be important for our approach:

- (I) For all $m>n>1$, the natural inclusion $V^n\hookrightarrow V^m$ induces an isomorphism $H^1\big(\bigwedge V^n, d\big)\xrightarrow{\cong} H^1\big(\bigwedge V^m, d\big)$.
- (II) For all $m>n$, the kernel of the map $H^2(\bigwedge V^n, d)\to H^2(\bigwedge V^m, d)$ coincides with the kernel of the map $H^2(\bigwedge V^n, d)\to H^2(\bigwedge V^{n+1}, d)$.

Let $\mathfrak{h}$ be a finitely generated Lie algebra.

Lemma 5.4. *The defining filtration $V^n=(\mathfrak{h}/\Gamma_n)^*$ on the 1-minimal cdga $\widehat{\mathscr{C}}(\mathfrak{h})$ satisfies properties (I) and (II).*

Proof. We will translate the stability properties for the defining filtration $\{V^n\}_{n\geqslant 1}$ on $\widehat{\mathscr{C}}(\mathfrak{h})$ in terms of Lie algebra homology, and use the commuting diagrams (28) in degrees $n\leqslant 3$, for the two canonical Lie projections, $p\colon \mathfrak{h}/\Gamma_m\twoheadrightarrow\mathfrak{h}/\Gamma_{n+1}$ and $q\colon \mathfrak{h}/\Gamma_{n+1}\twoheadrightarrow\mathfrak{h}/\Gamma_n$.

Property (I) is equivalent to the claim that the natural map $(\mathfrak{h}/\Gamma_m)_{\mathrm{ab}}\to(\mathfrak{h}/\Gamma_n)_{\mathrm{ab}}$ is an isomorphism, a claim which is obvious for $m>n>1$. Property (II) is equivalent to having an inclusion,

$$\operatorname{im}\big(H_2(\mathfrak{h}/\Gamma_{n+1})\to H_2(\mathfrak{h}/\Gamma_n)\big)\subseteq\operatorname{im}\big(H_2(\mathfrak{h}/\Gamma_m)\to H_2(\mathfrak{h}/\Gamma_n)\big). \tag{29}$$

We will verify this inclusion in a stronger form, with $\mathfrak{h}$ replacing $\mathfrak{h}/\Gamma_m$ in (29). To this end, we start with an element $e_n \in \bigwedge^2(\mathfrak{h}/\Gamma_n)$ with $\partial_2(e_n) = 0$. If $[e_n] \in \operatorname{im} H_2(q)$, we may find $e_{n+1} \in \bigwedge^2(\mathfrak{h}/\Gamma_{n+1})$ with $\partial_2(e_{n+1}) = 0$ and $f_n \in \bigwedge^3(\mathfrak{h}/\Gamma_n)$ such that $\bigwedge^2(q)e_{n+1} = e_n + \partial_3 f_n$.

Now lift f_n to $f_{n+1} \in \bigwedge^3(\mathfrak{h}/\Gamma_{n+1})$ via $\bigwedge^3(q)$. Replacing e_{n+1} by $e_{n+1} - \partial_3 f_{n+1}$, we see that we may suppose that $f_n = 0$. Next, lift e_{n+1} to $e \in \bigwedge^2(\mathfrak{h})$ via $\bigwedge^2(p)$. Since $\partial_2(e_{n+1}) = 0$, we infer that $\partial_2 e \in \Gamma_{n+1}$. By the definition of the LCS filtration [35], $\Gamma_{n+1} = \beta(\mathfrak{h} \wedge \Gamma_n)$. This implies that $\partial_2 e = \partial_2 e'$, where $e' \in \mathfrak{h} \wedge \Gamma_n$, and therefore $\bigwedge^2(q \circ p)(e') = 0$. Therefore, $\partial_2(e - e') = 0$ and $\bigwedge^2(q \circ p)(e - e') = e_n$. Hence, $[e_n] \in \operatorname{im} H_2(q \circ p)$, which completes the proof. ∎

Next, we examine the requirement that

(30)
$$\ker\left(H^2(f_n)\colon H^2\mathscr{C}(\mathfrak{h}/\Gamma_n) \to H^2A\right) \subseteq \ker\left(H^2(q)\colon H^2\mathscr{C}(\mathfrak{h}/\Gamma_n) \to H^2\mathscr{C}(\mathfrak{h}/\Gamma_{n+1})\right),$$

where $H^2(q)$ is the map induced by the canonical Lie projection, $q\colon \mathfrak{h}/\Gamma_{n+1} \twoheadrightarrow \mathfrak{h}/\Gamma_n$. Recall that the cdga A is supposed to be finite with $A^{>2} = 0$. Dualizing, we infer that the existence of inclusion (30) is equivalent to the existence of the inclusion

(31) $$\operatorname{im}\left(H_2(q)\colon H_2(\mathfrak{h}/\Gamma_{n+1}) \to H_2(\mathfrak{h}/\Gamma_n)\right) \subseteq \operatorname{im}\left(H_2(f_n)\colon H_2A \to H_2(\mathfrak{h}/\Gamma_n)\right).$$

By construction, the dual of the restriction of the map $f_n\colon \mathscr{C}(\mathfrak{h}/\Gamma_n) \to A$ to the space of algebra generators of the source, $f_n^*\colon A_1 \to \mathfrak{h}/\Gamma_n$, coincides with the composition $p_n \circ \iota$. Here, ι is the canonical inclusion $A_1 = \mathbb{L}^1 \hookrightarrow \mathbb{L}$, where $\mathbb{L}$ denotes the free Lie algebra on A_1, and $p_n\colon \mathbb{L} \twoheadrightarrow \mathfrak{h}/\Gamma_n$ stands for the canonical Lie projection.

The following commuting diagram describes the map between chain complexes in low degrees induced by the cochain map $f_n\colon \mathscr{C}(\mathfrak{h}/\Gamma_n) \to A$,

(32)
$$\begin{array}{ccc}
0 & \longrightarrow & \bigwedge^3(\mathfrak{h}/\Gamma_n) \\
\downarrow & & \downarrow{\scriptstyle \partial_3} \\
A_2 & \xrightarrow{\bigwedge^2(p_n\iota)\mu^*} & \bigwedge^2(\mathfrak{h}/\Gamma_n) \\
\downarrow{\scriptstyle d^*} & & \downarrow{\scriptstyle \partial_2} \\
A_1 & \xrightarrow{p_n\iota} & \mathfrak{h}/\Gamma_n\,.
\end{array}$$

We denote by $K \subseteq A_2$ the kernel of d^*. It follows from diagram (32) that the map $\bigwedge^2(p_n\iota)\mu^*$ sends K into $\ker(\partial_2)$, and thus induces a map $[\bigwedge^2(p_n\iota)\mu^*]\colon K \to H_2(\mathfrak{h}/\Gamma_n)$. Again by duality, (31) is equivalent to the existence of the inclusion

(33) $$\operatorname{im}\left(H_2(q)\colon H_2(\mathfrak{h}/\Gamma_{n+1}) \to H_2(\mathfrak{h}/\Gamma_n)\right) \subseteq \operatorname{im}\left([\textstyle\bigwedge^2(p_n\iota)\mu^*]\colon K \to H_2(\mathfrak{h}/\Gamma_n)\right).$$

Consider the map from Definition (19), $\partial = \partial_A\colon A_2 \to \mathbb{L}^1 \oplus \mathbb{L}^2$, where we use the Lie bracket β to identify $\bigwedge^2 \mathbb{L}^1$ with $\mathbb{L}^2$. Note that the map $p_n\colon \mathbb{L} \to \mathfrak{h}/\Gamma_n$ is the composition of the canonical Lie projections $\mathfrak{h} \twoheadrightarrow \mathfrak{h}/\Gamma_n$ and $p\colon \mathbb{L} \twoheadrightarrow \mathfrak{h}$. We will repeatedly use the commuting diagrams (28) for $\varphi = p$, in degrees up to 3. Consider the linear map

$$\partial' := \bigwedge^2(p) \circ \mu^*\colon K \to \bigwedge^2(\mathfrak{h}). \tag{34}$$

We first claim that the composition $\partial_2 \circ \partial'$ is the zero map. Indeed, for $y \in K$ we have that

$$\beta \circ \bigwedge^2(p)\mu^* y = p \circ \beta\mu^* y = p \circ \partial y = 0. \tag{35}$$

Lemma 5.5. *If A is a finite* cdga *with $A^{>2} = 0$, then the induced map $[\partial']\colon K \to H_2(\mathfrak{h})$ is surjective.*

Proof. We pick finite bases, $\{x_i\}_{i\in I}$ for A_1 and $\{y_\lambda\}_{\lambda\in\Lambda}$ for A_2. By construction, $\mathfrak{h} = \mathbb{L}/\mathfrak{r}$, where $\mathfrak{r}$ is the Lie ideal generated by ∂y_λ. For a length q multi-index $J = (i_1, \dots, i_q) \in I^q$ and an element $r \in \mathbb{L}$, we abbreviate $\mathrm{ad}_{x_{i_1}} \circ \cdots \circ \mathrm{ad}_{x_{i_q}}(r) \in \mathbb{L}$ by $\mathrm{ad}_J(r)$. Clearly, $\mathfrak{r}$ is additively generated by $\mathrm{ad}_J(\partial y_\lambda)$ with $q \geqslant 0$.

We start with an element $e \in \bigwedge^2\mathfrak{h}$ with the property that $\partial_2 e = 0$. We pick a lift $f \in \bigwedge^2\mathbb{L}$, via $\bigwedge^2(p)$. Since $\beta e = 0$, we have that $\beta f \in \mathfrak{r} \cap \mathbb{L}^{\geqslant 2}$. Write $\beta f = \sum c_{J,\lambda}\, \mathrm{ad}_J(\partial y_\lambda)$. Since $\beta f \in \mathbb{L}^{\geqslant 2}$, clearly $\sum_{\lambda\in\Lambda} c_{\varnothing,\lambda} d^* y_\lambda = 0$. Hence, the element $y = \sum_{\lambda\in\Lambda} c_{\varnothing,\lambda} y_\lambda$ belongs to K.

If J has positive length, we set $\widehat{J}_1 := (i_2, \dots, i_q)$. Clearly, the element

$$f' := \sum_{q>0,\lambda\in\Lambda} c_{J,\lambda} x_{i_1} \wedge \mathrm{ad}_{\widehat{J}_1}(\partial y_\lambda) \tag{36}$$

belongs to $\mathbb{L}^1 \wedge \mathfrak{r}$, and

$$\beta f' = \sum_{\substack{q>0\\ \lambda\in\Lambda}} c_{J,\lambda}\, \mathrm{ad}_J(\partial y_\lambda)\,. \tag{37}$$

On the other hand, $\beta f = \beta f' + \beta\mu^* y$, since $y \in K$. Hence, $\partial_2(f - f' - \mu^* y) = 0$, and $\bigwedge^2(p)(f - f' - \mu^* y) = e - \partial' y$, by construction.

Using the fact that the free Lie algebra $\mathbb{L}$ has vanishing homology in degrees greater than 2 [20], we may find an element $v \in \bigwedge^3\mathbb{L}$ such that $f - f' - \mu^* y = \partial_3 v$. We infer that $e - \partial' y = \partial_3 u$, where $u = \bigwedge^3(p)v$. Hence, $[e] = [\partial' y]$, as asserted. □

5.5. **Proof of Theorem 5.3.** Once again, let $f\colon \widehat{\mathscr{C}}(\mathfrak{h}(A)) \to A$ be the cdga map from (26). We have to show that $H^1(f)$ is an isomorphism and $H^2(f)$ is a monomorphism. In both cases, we will use the fact that

$$H^i\big(\widehat{\mathscr{C}}(\mathfrak{h})\big) = \varinjlim_n H^i\left(\mathscr{C}(\mathfrak{h}/\Gamma_n)\right). \tag{38}$$

For $i = 1$, by Lemma 5.4(I) and duality, we need to verify that $H_1(f_2)$ is an isomorphism. For $n = 2$, note that in diagram (32) we have that $\partial_2 = 0$ and $p_2\iota$ is surjective. Hence, we are left with checking that $H_1(f_2)$ is an injection. So, we assume that $p_2\iota y = 0$, i.e., $y \in A_1$ belongs to $\mathfrak{r}$ modulo Γ_2 in the free Lie algebra $\mathbb{L}$. It follows that necessarily $y \in \operatorname{im}(d^*)$, and we are done.

For $i = 2$, note that property (30) readily implies the injectivity of $H^2(f)$. The discussion from the preceding subsection reduces our proof to checking property (33). To verify the last property, pick an arbitrary element $[e_n] \in \operatorname{im} H_2(q)$. By property (29) in strong form, $[e_n] = H_2(p'_n)[e]$ for some $[e] \in H_2(\mathfrak{h})$, where $p'_n \colon \mathfrak{h} \twoheadrightarrow \mathfrak{h}/\Gamma_n$ denotes the canonical Lie projection. By Lemma 5.5, $[e] \in \operatorname{im}[\partial']$. Putting things together, we conclude that $[e_n] \in \operatorname{im}([\bigwedge^2(p_n\iota)\mu^*]\colon K \to H_2(\mathfrak{h}/\Gamma_n))$, and we are done. $\square$

5.6. **Topological interpretation.** A 1-minimal cdga $(\bigwedge V, d)$ admits a *canonical* defining filtration, $\{W^n\}_{n\geqslant 1}$, inductively defined by $W^1 = 0$ and

$$W^{n+1} = d_{|V}^{-1}(\textstyle\bigwedge^2 W^n)\,. \tag{39}$$

It is easy to check by induction that the inclusion $V^n \subseteq W^n$ holds for all $n \geqslant 1$, and for any defining filtration $\{V^n\}_{n\geqslant 1}$.

Lemma 5.6. *A defining filtration for the* 1*-minimal* cdga $(\bigwedge V, d)$ *is canonical if and only if it has the stability properties* (I) *and* (II).

Proof. First observe that $H^1(\bigwedge V^n) = V^n \cap W^2$, for all $n \geqslant 1$.

Assume now that $V^n = W^n$, for all n. Then clearly $V^n \cap W^2 = V^m \cap W^2 = W^2$ for all $m > n > 1$, which proves (I). To check (II), let us start with an element $\alpha \in \bigwedge^2 W^n$ with the property that $\alpha = dv_m$ with $v_m \in W^m$. Clearly, $v_m \in W^{n+1}$, and we are done.

Conversely, let us assume that properties (I) and (II) hold, and let us prove by induction that $W^n \subseteq V^n$, for all $n \geqslant 2$. By (I), $V^n \cap W^2 = V^2 \cap W^2$ for all $n \geqslant 2$, which shows that $W^2 \subseteq V^2$. For the induction step, start with an arbitrary element $v_{n+1} \in W^{n+1}$. Then $v_{n+1} \in V^m$ for some $m > n$. We infer that $dv_{n+1} \in \bigwedge^2 V^n$, by the construction of the canonical filtration and the inductive hypothesis. Since the cohomology class $[dv_{n+1}] \in H^2(\bigwedge V^n)$ goes to zero in $H^2(\bigwedge V^m)$, property (II) implies that $dv_{n+1} = du_{n+1}$, for some $u_{n+1} \in V^{n+1}$. Therefore, $v_{n+1} - u_{n+1} \in W^2 \subseteq V^{n+1}$. Hence, $v_{n+1} \in V^{n+1}$, and we are done. $\square$

As an application, we recover in the next corollary a theorem of Berceanu, Măcinic, Papadima, and Popescu [3, Theorem 3.1], a theorem which itself generalizes a result of Bezrukavnikov [4, Lemma 3.1 and Proposition 4.0] (see also [5, §4.3] for a summary of [4]).

Corollary 5.7. *If A is a* 1*-finite* 1*-model of π, then the Malcev Lie algebra $\mathfrak{m}(\pi)$ is isomorphic to the* LCS *completion of the holonomy Lie algebra $\mathfrak{h}(A)$.*

Proof. We start by recalling Sullivan's duality result [39] for Malcev Lie algebras and 1-minimal models of finitely generated groups (see also [10, §6] and [37, §6]). By dualizing the canonical filtration of $\mathscr{M}_1(\pi)$, we obtain a tower of central extensions of finite-dimensional nilpotent Lie algebras,

$$\cdots \twoheadrightarrow \mathfrak{m}_{n+1} \twoheadrightarrow \mathfrak{m}_n \twoheadrightarrow \cdots \twoheadrightarrow \mathfrak{m}_1 = \{0\} \; . \tag{40}$$

Then $\mathfrak{m}(\pi)$ is isomorphic to the inverse limit of the tower (40), endowed with the inverse limit filtration.

Our assumptions imply that the group π and the cdga A have the same 1-minimal model. By Theorem 5.3 and Lemmas 5.4 and 5.6, the above Lie tower is isomorphic to the Lie tower

$$\cdots \twoheadrightarrow \mathfrak{h}(A)/\Gamma_{n+1} \twoheadrightarrow \mathfrak{h}(A)/\Gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \mathfrak{h}(A)/\Gamma_1 = \{0\} \; , \tag{41}$$

whose inverse limit is the LCS completion of $\mathfrak{h}(A)$. This completes the proof. □

6. A complete finiteness obstruction for finitely generated groups

For $q = 1$, the finiteness property from Question 1.1 depends only on the finitely generated group $\pi = \pi_1(X)$. More precisely, it depends only on its 1-minimal model $\mathscr{M}_1(\pi)$, or equivalently, on the Malcev Lie algebra $\mathfrak{m}(\pi)$. In this situation, we completely solve Question 1.1, as follows.

Theorem 6.1. *A space with finitely generated fundamental group π admits a 1-finite 1-model if and only if the Malcev Lie algebra $\mathfrak{m}(\pi)$ is the LCS completion of a finitely presented Lie algebra.*

The rest of this section will be devoted to a proof of this theorem. We start with some preparatory notation and a lemma. Let $\mathbb{L} = \mathbb{L}(x_i)_{i\in I}$ be a finitely generated free Lie algebra. For each $k \geqslant 1$ and each k-tuple $J = (i_1, \dots, i_k) \in I^k$, set $|J| = k$, and define

$$\beta(J) := \mathrm{ad}_{x_{i_1}} \circ \cdots \circ \mathrm{ad}_{x_{i_{k-1}}}(x_{i_k}) \in \mathbb{L}^k. \tag{42}$$

An easy induction shows that each graded piece $\mathbb{L}^k$ is generated by the Lie monomials $\beta(J)$ with $|J| = k$.

Lemma 6.2. *Let $L = \mathbb{L}/\mathfrak{r}$ be a finitely presented Lie algebra. Then L admits an equivalent finite presentation $\mathscr{P}$, having only "linear plus quadratic" relations.*

Proof. The fact that L is finitely presented allows us to find an integer $N \geqslant 2$ and constants c^J_λ, where $1 \leqslant |J| \leqslant N$ and $\lambda \in \Lambda$ with Λ finite, with the property that the Lie ideal $\mathfrak{r}$ is generated by the elements $r_\lambda := \sum_J c^J_\lambda \beta(J)$.

For each $n \geqslant 2$, let us define a finite Lie algebra presentation $\mathscr{P}_n$ as having generators y_J indexed by $1 \leqslant |J| \leqslant n$, and relators $y_J - [y_{i_1}, y_{\hat{J}_1}]$, indexed by $2 \leqslant |J| \leqslant n$, where $\widehat{J}_1 := (i_2, \dots, i_k)$. Furthermore, let us define a finite Lie algebra presentation $\mathscr{P}$ by adding to the relators from $\mathscr{P}_N$ the elements $\rho_\lambda := \sum_{1\leqslant |J|\leqslant N} c^J_\lambda y_J$ for $\lambda \in \Lambda$.

The Lie presentations $\mathscr{P}_n$ and $\mathscr{P}_{n-1}$ are related by the morphisms of presentations $\phi\colon \mathscr{P}_{n-1} \to \mathscr{P}_n$ and $\psi\colon \mathscr{P}_n \to \mathscr{P}_{n-1}$. The map ϕ sends the generator y_J to y_J for each J, while ψ sends the generator y_J to y_J for each J with $|J| < n$, and maps y_J to $[y_{i_1}, y_{\hat{J}_1}]$

if $|J| = n$. It is straightforward to check that the associated Lie algebra morphisms are inverse to each other.

We infer that the Lie algebra morphism $\kappa\colon \mathbb{L} \to L_N$ which sends x_i to y_i is an isomorphism, where L_N denotes the Lie algebra associated to $\mathscr{P}_N$. On the other hand, a straightforward induction shows that κ sends $\beta(J)$ to y_J, for all J with $1 \leqslant |J| \leqslant N$. Finally, for each $\lambda \in \Lambda$ the relator r_λ is identified by κ with the element ρ_λ. This completes the proof. ◻

Proof of Theorem 6.1. For the forward implication, let $\pi = \pi_1(X)$ be a finitely generated group, and assume π has a 1-finite 1-model A. By Lemma 2.2, we may assume that A is finite with $A^{>2} = 0$. Denote by $\mathfrak{h} = \mathfrak{h}(A)$ the holonomy Lie algebra of A. Clearly, the Lie algebra $\mathfrak{h}$ is finitely presentable. By Corollary 5.7, the Malcev Lie algebra $\mathfrak{m} = \mathfrak{m}(\pi)$ is the LCS completion of $\mathfrak{h}$, and we are done.

Conversely, suppose that $\mathfrak{m}(\pi)$ is the LCS completion of a finitely presented Lie algebra L as above. Note that all relators of the presentation $\mathscr{P}$ from Lemma 6.2 are elements of $\mathbb{L}^{\leqslant 2}$, by construction. By dualizing this presentation, we obtain a finite cdga A with $A^{>2} = 0$, whose differential $d\colon A^1 \to A^2$ is dual to the degree one part of the relator map, and with multiplication $\mu\colon A^1 \wedge A^1 \to A^2$ dual the degree two part of the relator map. By construction, the holonomy Lie algebra $\mathfrak{h} = \mathfrak{h}(A)$ is isomorphic to L. Moreover, Theorem 5.3 implies that $\widehat{\mathscr{C}}(\mathfrak{h}) \simeq_1 A$.

On the other hand, $\Omega(X) \simeq_1 \mathscr{M}_1(\pi)$, as noted before. It is therefore enough to check that the cdga's $\widehat{\mathscr{C}}(\mathfrak{h})$ and $\mathscr{M}_1(\pi)$ are isomorphic. Exploiting the fact that both cdga's are inductive limits of cdga towers of Hirsch extensions, this may be seen as follows. The dual of the tower for $\widehat{\mathscr{C}}(\mathfrak{h})$ is by construction the Lie tower $\{\mathfrak{h}/\Gamma_{n+1} \to \mathfrak{h}/\Gamma_n\}_{n\geqslant 1}$. Furthermore, our assumption that $\mathfrak{m}(\pi) \cong \widehat{L}$ implies that the dual of the tower for $\mathscr{M}_1(\pi)$ is isomorphic to $\{L/\Gamma_{n+1} \to L/\Gamma_n\}_{n\geqslant 1}$. Hence, $\widehat{\mathscr{C}}(\mathfrak{h}) \cong \mathscr{M}_1(\pi)$, thereby establishing our claim, and completing the proof. ◻

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