HOMOLOGY ISOMORPHISMS BETWEEN ALGEBRAIC GROUPS MADE DISCRETE

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1. Statement of Results

Theorem 1. *Consider a split exact sequence of discrete groups*

$$(*) \qquad \{1\} \to G \to \Gamma \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \Gamma/G \to \{1\}.$$

Suppose there exists a normal series

$$(**) \qquad G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n \triangleright G_{n+1} = \{1\}$$

such that:

(1) *G_i/G_{i+1} is a rational vector space for $i = 0, \ldots, n$;*
(2) *G_i/G_{i+1} is contained in the center of G/G_{i+1} for $i = 0, \ldots, n$;*
(3) *There exists an element in the center of Γ/G that induces a diagonalizable endomorphism of each G_i/G_{i+1} with all eigenvalues rational and greater than 1.*

Then, the map π induces an isomorphism

$$\pi_* : H_*(B\Gamma, \mathbb{Z}) \longrightarrow H_*(B(\Gamma/G), \mathbb{Z}).$$

The proof will be given in section 3.

There is a natural context covered by Theorem 1. For Γ a Lie group, we will write Γ^δ for Γ made discrete and $H_*(\Gamma^\delta)$ for $H_*(B\Gamma)$. The next theorem identifies the homology of certain Lie groups made discrete with the homology of their reductive parts.

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Theorem 2. *Let Γ be a connected affine algebraic group over a field K of characteristic* 0*. Let G be the unipotent radical of Γ and* $(**)$ *the descending central series of G. Assume* (3) *holds. Then $H_*(\Gamma^\delta, \mathbb{Z}) \to H_*((\Gamma/G)^\delta, \mathbb{Z})$ is an isomorphism.*

Proof. In this case, the split exact sequence $(*)$ is the decomposition of an affine algebraic group into the the semi-direct product of a maximal reductive subgroup and its maximal unipotent subgroup [5]. The algebraic groups G_i/G_{i+1} are connected, unipotent and abelian, and thus vector spaces over K, so (1) holds; (2) is satisfied by definition. Whence the conclusion.

2. Examples

The simplest example to which Theorem 2 applies is the following. Let $\mathrm{T}(n,K)$ be the group of upper triangular matrices in $\mathrm{GL}(n,K)$, and $\mathrm{D}(n,K)$ (resp. $\mathrm{U}(n,K)$) the subgroup consisting of diagonal matrices (resp. upper triangular matrices with all diagonal entries 1). Then $\mathrm{T}(n,K)$ is an algebraic group, with decomposition $\mathrm{U}(n,K) \rtimes \mathrm{D}(n,K)$. The descending central series for $U = \mathrm{U}(n,K)$ is

$$U \triangleright U_{1,2} \triangleright U_{2,3} \triangleright \cdots \triangleright U_{2,n} \triangleright U_{1,n} = \{1\},$$

where $U_{i,j} = \{\, x \in U \mid x_{k,l} = 0 \,\text{for}\, (k,l) \le (i,j)\,\}, 1 \le i < j \le n$, and the successor of (i,j) is $s(i,j) = (i-1,j)$ if $i > 1$ and $(j, j+1)$ if $i = 1$. A diagonal matrix with entries $\lambda_1, \dots, \lambda_n\, (\lambda_k \neq 0)$ acts on $U_{i,j}/U_{s(i,j)} = K$ by multiplication by $\lambda_i \lambda_j^{-1}$. Choosing rational numbers $\lambda_1 > \cdots > \lambda_n > 1$ gives an element of $\mathrm{D}(n,K)$ for which (3) holds. This proves:

Corollary 3. *There is an isomorphism $H_*(\mathrm{T}(n,K)^\delta, \mathbb{Z}) \to H_*(\mathrm{D}(n,K)^\delta, \mathbb{Z})$.*

The next example is $J_k(n,K)$, the group, under composition, of k-jets of invertible formal series in n variables over K^n which fix 0. Let $J_{k,i}(n,K) = \ker(J_k(n,K) \twoheadrightarrow J_i(n,K)), 1 \le i \le k$, be the subgroup of k-jets which vanish to order i at 0, and $\mathrm{GL}(n,K) = J_1(n,K)$ be the subgroup of linear jets. Then $J_k(n,K)$ is an algebraic group, with decomposition $J_{k,1}(n,K) \rtimes \mathrm{GL}(n,K)$. The descending central series for $J_{k,1}$ is

$$J_{k,1} \triangleright J_{k,2} \triangleright \cdots \triangleright J_{k,k-1} \triangleright J_{k,k} = \{1\}.$$

Each quotient $J_{k,i}/J_{k,i+1}$ is isomorphic to $J_{i+1,i}$, which is a direct sum of copies of K, indexed by the monomials of degree $i+1$ in n variables. A diagonal matrix with all diagonal entries $\lambda\, (\lambda \neq 0)$ acts on $J_{k,i}/J_{k,i+1}$ by multiplication by λ^{-i}. Choosing a rational number $\lambda, 0 < \lambda < 1$, yields:

Corollary 4. *There is an isomorphism $H_p(J_k(n,K)^\delta, \mathbb{Z}) \to H_p(\mathrm{GL}(n,K)^\delta, \mathbb{Z})$, for all $p \ge 0$.*

The homology theory of discrete jet groups in one variable was first treated in [3], where the isomorphism was established for $p \le 2$, and conjectured to hold for all p. In [2], Dartnell considered jets in n variables, and gave the first complete proof of Corollary 4. Independently, in a preliminary version of this paper, we discovered how to apply the vanishing criterion from [4] to verify the conjecture for $n = 1$. Subsequently, we formulated the results herein.

Corollary 4 also holds for jet subgroups whose linearization contains diagonal matrices of the required sort. For example, let $J_k^+(n,\mathbb{R})$ denote the group of k-jets at 0 of orientation-preserving local diffeomorphisms of $\mathbb{R}^n$ fixing 0. Then, the linearization map induces an isomorphism $H_p(J_k^+(n,\mathbb{R})^\delta,\mathbb{Z}) \to H_p(\mathrm{GL}^+(n,\mathbb{R})^\delta,\mathbb{Z})$. For $n = 1$, the homology can be easily computed. Indeed, $\mathrm{GL}^+(1,\mathbb{R}) \cong \mathbb{R}$ and $H_p(\mathbb{R}^\delta,\mathbb{Z})$ is the p'th exterior power of $\mathbb{R}$ (see the proof of Lemma 9 below).

3. Proof of Theorem 1

Let V be an abelian group which carries the structure of a rational vector space, that is, the $\mathbb{Z}$ scalar multiplication extends to one of $\mathbb{Q}$. Let α be an endomorphism of the group V. Then it is readily seen that α is $\mathbb{Q}$-linear. Given a rational number r, let $V(\alpha,r)$ be the collection of all elements $v \in V$ such that $(\alpha - r)^k v = 0$, for some integer $k > 0$. If $V(\alpha,r) \neq \{0\}$, then r is called an *eigenvalue* for the action of α on V; the non-zero elements of $V(\alpha,r)$ are called the *generalized eigenvectors* corresponding to r. Let $\mathcal{E}(V,\alpha)$ denote the set of eigenvalues for the action of α on V. The action of α on V is said to be *almost diagonalizable* if V is the direct sum of the generalized eigenspaces of α: $V = \bigoplus_{r\in\mathbb{Q}} V(\alpha,r)$.

Given an endomorphism α of the rational vector space V, construct an action of the polynomial ring $\mathbb{Q}[t]$ on V by letting t act via α. The ring $\mathbb{Q}[t]$ is a principal ideal domain; a prime ideal has the form $(f(t))$ for an irreducible polynomial $f(t)$ over $\mathbb{Q}$. It follows from the definitions that the action of α on V is almost diagonalizable if and only if V is a torsion module over $\mathbb{Q}[t]$ with the property that the $\mathfrak{p}$-primary components of V vanish except for prime ideals $\mathfrak{p} \subset \mathbb{Q}[t]$ of the form $(t-r), r \in \mathbb{Q}$. The submodule $V(\alpha,r)$ is the primary summand of V for the prime $(t-r)$. The next two lemmas follow from standard properties of modules over principal ideal domains.

Lemma 5. *If the action of the endomorphism α of the rational vector space V is almost diagonalizable, then for $r \in \mathbb{Q}$ the map $(\alpha - r) : V \to V$ is an isomorphism, unless $r \in \mathcal{E}(V,\alpha)$.*

Lemma 6. *Let α be an endomorphism of the short exact sequence*

$$0 \to U \to V \to W \to 0$$

of rational vector spaces: α amounts to an endomorphism α_V of V which restricts to an endomorphism α_U of U and induces a quotient endomorphism α_W of W. Then:

(i) *The action of α_V on V is almost diagonalizable if and only if the actions of α_U on U and α_W on W are almost diagonalizable;*
(ii) *If the action of α_V on V is almost diagonalizable, then $\mathcal{E}(V,\alpha_V) = \mathcal{E}(U,\alpha_U) \cup \mathcal{E}(W,\alpha_W)$.*

Remark 7. Let $0 \to U \to A \to V \to 0$ be a short exact sequence of abelian groups, with U and V rational vector spaces. Since the $\mathbb{Z}$-module U is divisible, and thus injective, the sequence splits, i.e. $A \cong U \oplus V$. Hence A carries the structure of a rational vector space.

The next lemma is a little more special. Before stating it, let us point out that, if U and V are rational vector spaces, then $U \otimes V \cong U \otimes_{\mathbb{Q}} V$; in particular, $U \otimes V$ carries the structure of a rational vector space.

Lemma 8. *Let α be an endomorphism of U and β an endomorphism of V, where U and V are rational vector spaces. Suppose that the actions of α on U and β on V are almost diagonalizable. Then the action of $\alpha \otimes \beta$ on $U \otimes V$ is almost diagonalizable and*

$$\mathcal{E}(U \otimes V, \alpha \otimes \beta) = \{\, rs \mid r \in \mathcal{E}(U, \alpha) \text{ and } s \in \mathcal{E}(V, \beta)\,\}.$$

Proof. Since tensor product commutes with direct sums, we can assume that $U = U(\alpha, r)$ and $V = V(\beta, s)$ for $r, s \in \mathbb{Q}$. It is then enough to show that for any $u \in U$ and $v \in V$ there exists a positive integer k such that $(\alpha \otimes \beta - rs)^k(u \otimes v) = 0$. By assumption there exist positive integers i and j such that $(\alpha - r)^i u = 0$ and $(\beta - s)^j v = 0$. The existence of the desired integer k follows from the binomial theorem and the formula $\alpha \otimes \beta - rs = (\alpha - r) \otimes \beta + r \otimes (\beta - s)$.

We now consider the action on the homology level.

Lemma 9. *Let V be a rational vector space and α an endomorphism of V such that the action of α is almost diagonalizable. Let α_p $(p \geq 1)$ be the endomorphism of $H_p(BV, \mathbb{Z})$ induced by α. Then α_p is almost diagonalizable, and each element $\mathcal{E}(H_p(BV, \mathbb{Z}), \alpha_p)$ can be written as a product of p elements of $\mathcal{E}(V, \alpha)$.*

Proof. The group $H_p(BV, \mathbb{Z})$ is naturally isomorphic to the rational vector space $\wedge^p V$, the p'th exterior power of V, see [1]. In a natural way this is a summand of the p'th tensor power of V. The result thus follows from Lemma 8.

Lemma 10. *Let G be a group admitting a normal series $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n \triangleright G_{n+1} = \{1\}$ so that G_i/G_{i+1} is a rational vector space and is contained in the center of G/G_{i+1}. Suppose α is an automorphism of the series with the property that the induced homology automorphisms $\alpha_* : H_*(B(G_i/G_{i+1}), \mathbb{Z}) \to H_*(B(G_i/G_{i+1}), \mathbb{Z})$ are almost diagonalizable. Then $\alpha_* : H_*(BG, \mathbb{Z}) \to H_*(BG, \mathbb{Z})$ is almost diagonalizable, and each element of $\mathcal{E}(H_*(BG, \mathbb{Z}), \alpha_*)$ can be written as a product of elements of $\mathcal{E}(H_*(B(G_i/G_{i+1}), \mathbb{Z}), \alpha_*)$.*

Proof. By induction on n.

For $n = 1$ we are given an automorphism α of the extension

$$\{1\} \to G_1 \to G_0 \to G_0/G_1 \to \{1\}.$$

The automorphism α acts on the Lyndon-Hochschild-Serre spectral sequence

$$\{E^2_{p,q} = H_p(B(G_0/G_1), \mathbb{Z}) \otimes H_q(BG_1, \mathbb{Z})\} \Longrightarrow H_{p+q}(BG_0, \mathbb{Z})$$

(where the tensor product formula for E^2 comes from the fact that G_1 is in the center of G_0.) By Lemma 8 the action of α on $E^2_{p,q}$ is almost diagonalizable and each element of $\mathcal{E}(E^2_{p,q}, \alpha_*)$ can be written as a product rs where r and s are rational numbers in $\mathcal{E}(H_p(B(G_0/G_1), \mathbb{Z}), \alpha_*)$ and $\mathcal{E}(H_q(BG_1, \mathbb{Z}), \alpha_*)$, respectively.

The proof of step $n = 1$ is completed by repeated applications of Lemma 6 (and Remark 7) to deduce first that the action of α on E^∞ has the desired properties, and then that these properties are inherited by the action of α on $H_*(BG_0, \mathbb{Z})$.

The group G/G_n admits a normal series $G/G_n \triangleright G_1/G_n \triangleright \cdots \triangleright G_{n-1}/G_n \triangleright \{1\}$ satisfying the hypothesis. By induction the automorphism $\alpha_* : H_*(B(G/G_n), \mathbb{Z}) \to H_*(B(G/G_n), \mathbb{Z})$ has the desired properties. The lemma follows by applying step $n = 1$ of the induction to the extension $\{1\} \to G_n \to G \to G/G_n \to \{1\}$.

The following lemma, which is the vanishing theorem of [4], will enable us to finish the proof of Theorem 1.

Lemma 11 ([4]). *Let G be a discrete group and M a module over $\mathbb{Z}[G]$. If there exists an element ε in the center of $\mathbb{Z}[G]$ of augmentation zero such that multiplication by ε gives an isomorphism $M \to M$, then $H_*(BG, M) = 0$.*

Now consider the Lyndon-Hochschild-Serre spectral sequence associated to the extension $(*)$

$$\{E^2_{p,q} = H_p(B(\Gamma/G), H_q(BG, \mathbb{Z}))\} \Longrightarrow H_{p+q}(B\Gamma, \mathbb{Z}).$$

Note that the $E^2_{p,0} = H_p(B(\Gamma/G), \mathbb{Z})$. Let $q > 0$. We are given an element ρ in the center of Γ/G so that conjugation by $\sigma(\rho)$ induces a diagonalizable action on each G_i/G_{i+1} with all eigenvalues greater than 1. (In fact, all we need is that the action on each quotient is almost diagonalizable and all possible products of eigenvalues are not equal to 1). It follows from Lemma 10 that the induced homology automorphism $\rho_* : H_q(BG, \mathbb{Z}) \to H_q(BG, \mathbb{Z})$ is almost diagonalizable and all its eigenvalues are greater than 1. By Lemma 5, then, multiplication by the element $\varepsilon = \rho - 1$ of $\mathbb{Z}[\Gamma/G]$ gives an automorphism ε_* of $H_q(BG, \mathbb{Z})$. It follows from Lemma 11 that in the above spectral sequence $E^2_{p,q} = 0$ for $q > 0$. The desired result follows immediately.

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