

CUP PRODUCTS, LOWER CENTRAL SERIES, AND HOLONOMY LIE ALGEBRAS

ALEXANDER I. SUCIU¹ AND HE WANG

ABSTRACT. We generalize basic results relating the associated graded Lie algebra and the holonomy Lie algebra of a group, from finitely presented, commutator-relators groups to arbitrary finitely presented groups. Using the notion of “echelon presentation,” we give an explicit formula for the cup-product in the cohomology of a finite 2-complex. This yields an algorithm for computing the corresponding holonomy Lie algebra, based on a Magnus expansion method. As an application, we discuss issues of graded-formality, filtered-formality, 1-formality, and mildness. We illustrate our approach with examples drawn from a variety of group-theoretic and topological contexts, such as link groups, one-relator groups, and fundamental groups of orientable Seifert fibered manifolds.

CONTENTS

1. Introduction	1
2. Expansions for finitely presented groups	4
3. Echelon presentations and cellular chain complexes	8
4. Group presentations and (co)homology	10
5. A presentation for the holonomy Lie algebra	12
6. Lower central series and the holonomy Lie algebra	16
7. Mildness and graded-formality	18
8. One-relator groups	21
9. Seifert fibered spaces	23
References	25

1. INTRODUCTION

Throughout this paper G will be a finitely generated group. Our main focus will be on the cup-product in the rational cohomology of the 2-complex associated to a presentation of G , and on several rational Lie algebras attached to such a group.

2010 *Mathematics Subject Classification.* Primary 20F40, 57M05. Secondary 16A27, 17B70, 20F14, 20J05.

Key words and phrases. Lower central series, derived series, holonomy Lie algebra, graded formality, mildness, Magnus expansion, cohomology ring, Chen Lie algebra, link group, one-relator group, Seifert manifold.

¹Supported in part by the National Security Agency (grant H98230-13-1-0225) and the Simons Foundation (collaboration grant for mathematicians 354156).

1.1. Magnus expansions and cup products. The notion of expansion of a group, which goes back to W. Magnus [21], has been generalized and used in many ways. For instance, a presentation for the Malcev Lie algebra of a finitely presented group was given by S. Papadima [30] and G. Massuyeau [25], while X.-S. Lin [20] studied expansions of fundamental groups of smooth manifolds. Recently, D. Bar-Natan [2] has generalized the notion of expansion and has introduced the Taylor expansion of an arbitrary ring. In turn, we explored in [41] various relationships between expansions and formality properties of groups.

We go back here to the classical Magnus expansion, and adapt it for our purposes. Let G be a group with a finite presentation $G = F/R = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$. There exists then a 2-complex $K = K_G$ associated to such a presentation. In the case when G is a commutator-relators group, i.e., when all relators r_i belong to the commutator subgroup $[F, F]$, R. Fenn and D. Sjerve computed in [10] the cup-product map

$$(1) \quad \mu_K: H^1(K; \mathbb{Z}) \wedge H^1(K; \mathbb{Z}) \longrightarrow H^2(K; \mathbb{Z}), \quad u \wedge v \mapsto u \cup v,$$

using the Magnus expansion $M: \mathbb{Z}F \rightarrow \mathbb{Z}\langle\langle \mathbf{x} \rangle\rangle$ from the group ring of the free group $F = \langle x_1, \dots, x_n \rangle$ to the power series ring in n non-commuting variables, which is the ring morphism defined by $M(x_i) = 1 + x_i$.

Our first objective in this work is to generalize this result of Fenn and Sjerve, from commutator-relators groups to arbitrary finitely presented groups. We will avoid possible torsion in the first homology of G by working over the field of rationals. To that end, we start by defining a Magnus-like expansion $\kappa = \kappa_G$ relative to such a group G as the composition

$$(2) \quad \mathbb{Q}F \xrightarrow{M} \widehat{T}(H_1(F; \mathbb{Q})) \xrightarrow{\widehat{T}(\varphi_*)} \widehat{T}(H_1(G; \mathbb{Q})),$$

where $\varphi: F \rightarrow G$ is the canonical projection and $\widehat{T}(V)$ is the completed tensor algebra of a vector space V . We then show in Proposition 3.3 that there exists a group G_e admitting a ‘row-echelon’ presentation, $G_e = \langle x_1, \dots, x_n \mid w_1, \dots, w_m \rangle$, and a map $f: K_{G_e} \rightarrow K_G$ inducing an isomorphism in cohomology.

Using the κ -expansion of G_e , we determine in Theorem 4.3 the cup-product map for K_{G_e} , from which we obtain in Theorem 4.4 a formula for computing the cup-product map μ_K , with \mathbb{Q} -coefficients. Let $b = b_1(G)$ be the first Betti number of G , and let $\{u_1, \dots, u_b\}$ and $\{\beta_{n-b+1}, \dots, \beta_m\}$ be bases for $H^1(K; \mathbb{Q})$ and $H^2(K; \mathbb{Q})$, transferred from suitable bases in the rational cohomology of K_{G_e} via the isomorphism $f^*: H^*(K_{G_e}; \mathbb{Q}) \rightarrow H^*(K_G; \mathbb{Q})$. Our result then reads as follows.

Theorem 1.1. *Let K be a presentation 2-complex for a finitely presented group G . In the bases described above, the cup-product map $\mu_K: H^1(K; \mathbb{Q}) \wedge H^1(K; \mathbb{Q}) \rightarrow H^2(K; \mathbb{Q})$ is given by*

$$u_i \cup u_j = \sum_{k=n-b+1}^m \kappa(w_k)_{i,j} \beta_k, \quad \text{for } 1 \leq i, j \leq b.$$

Let us note that the map μ_K depends on the chosen presentation for G , and may differ from the cup-product map μ_G in a classifying space for G . However, the two maps share the same kernel, which makes the algorithm for determining the map μ_K useful in other contexts, for instance, in computing the first resonance variety of G (see e.g. [27, 35]), or finding a presentation for the holonomy Lie algebra $\mathfrak{h}(G)$, a procedure that we discuss next.

1.2. Holonomy Lie algebras. The *holonomy Lie algebra* of a finitely generated group G , denoted by $\mathfrak{h}(G)$, is the quotient of the free Lie algebra on $H_1(G; \mathbb{Q})$ by the Lie ideal generated by the image of the dual of the cup-product map μ_G . It is easy to see that $\mathfrak{h}(G)$ is a graded Lie algebra over \mathbb{Q} which admits a quadratic presentation depending only on $\ker \mu_G$. Moreover, this construction is functorial. The holonomy Lie algebra was introduced by T. Kohno in [13], building on work of Chen [6], and has been further studied in a number of papers, including [24, 31, 41].

Our next objective is to find a presentation for the holonomy Lie algebra $\mathfrak{h}(G)$. We start by showing in Proposition 5.4 that there is a homomorphism from a finitely presented group G_f to G inducing an isomorphism on holonomy Lie algebras. Hence, without loss of generality, we may assume that G admits a finite presentation.

Let $G_e = \langle x_1, \dots, x_n \mid w_1, \dots, w_m \rangle$ be a group with row-echelon presentation, and let $\rho: G_e \rightarrow G$ be the homomorphism induced on fundamental groups by the aforementioned map, $f: K_{G_e} \rightarrow K_G$. It follows from Corollary 5.3 that the induced map, $\mathfrak{h}(\rho): \mathfrak{h}(G_e) \rightarrow \mathfrak{h}(G)$, is an isomorphism of graded Lie algebras. Using the computation of the cup-product map μ_{K_G} from Theorem 1.1, we describe in Theorem 5.5 an algorithm for finding a presentation for the holonomy Lie algebra $\mathfrak{h}(G)$. Furthermore, we obtain in Theorem 5.11 a presentation for the derived quotients of this Lie algebra, $\mathfrak{h}(G)/\mathfrak{h}(G)^{(i)}$. Our results may be summarized as follows.

Theorem 1.2. *Let G be a finitely presented group. The holonomy Lie algebra $\mathfrak{h}(G)$ is the quotient of the free \mathbb{Q} -Lie algebra with generators $\mathbf{y} = \{y_1, \dots, y_b\}$ in degree 1 by the ideal I generated by $\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)$, where κ_2 is the degree 2 part of the Magnus expansion of G_e . Furthermore, for each $i \geq 2$, the solvable quotient $\mathfrak{h}(G)/\mathfrak{h}(G)^{(i)}$ is isomorphic to $\text{lie}(\mathbf{y})/(I + \text{lie}^{(i)}(\mathbf{y}))$.*

In the special case when G admits a presentation with only commutator relators, presentations for these Lie algebras were given by Papadima and Suciu in [31]. For arbitrary finitely generated groups G , the metabelian quotient $\mathfrak{h}(G)/\mathfrak{h}(G)''$, also known as the holonomy Chen Lie algebra of G , is closely related to the first resonance variety of G , a geometric object which has been studied intensely from many points of view, see for instance [27, 34, 35, 43, 44] and references therein.

1.3. Lower central series, graded formality, and mildness. The Lie methods in group theory were introduced by W. Magnus in [22], and further developed by E. Witt [48], M. Hall [12], M. Lazard [18], and many more authors, see for instance [23]. The *associated graded Lie ring* of a group G is the graded Lie ring $\text{gr}(G; \mathbb{Z})$, whose graded pieces are the successive quotients of the lower central series of G , and whose Lie bracket is induced from the group commutator. The quintessential example is the associated graded Lie ring of the free group on n generators, F_n , which is isomorphic to the free Lie ring $\text{lie}(\mathbb{Z}^n)$. Much of the power of this method comes from the various connections between lower central series, nilpotent quotients, and group homology, as evidenced in the work of J. Stallings [40], W. Dwyer [8], and many others.

We concentrate here on the associated graded Lie algebra over the rationals, $\text{gr}(G) = \text{gr}(G; \mathbb{Z}) \otimes \mathbb{Q}$, of a finitely generated group G . As we recall in §6.2, there is a natural epimorphism $\Phi_G: \mathfrak{h}(G) \rightarrow \text{gr}(G)$. Thus, the holonomy Lie algebra $\mathfrak{h}(G)$ may be viewed as a quadratic approximation to the associated graded Lie algebra of G . We say that the group G is *graded-formal* if the map Φ_G is an isomorphism of graded Lie algebras. A much stronger requirement is that G be *1-formal*, a condition we recall in §2.1. For much more on these notions, we refer to [31, 32, 41].

In Propositions 6.2 and 6.3, we compare the holonomy Lie algebra of G with the holonomy Lie algebras of the nilpotent quotients $G/\Gamma_i G$ and the derived quotients $G/G^{(i)}$. In Corollary 6.6, we

use Theorem 1.2 and a result from [41] to give explicit presentations for the graded Lie algebras $\text{gr}(G/G^{(i)})$ in the case when G is a finitely presented, 1-formal group.

Some of the motivation for our study comes from the work of J. Labute [14, 15] and D. Anick [1], who gave presentations for the associated graded Lie algebra $\text{gr}(G)$ in the case when G is ‘mildly’ presented. We revisit this topic in §7, where we relate the notion of mild presentation to that of graded formality, and derive some consequences, especially in the context of link groups.

1.4. Further applications. We illustrate our approach with several classes of finitely presented groups, including 1-relator groups in §8, and fundamental groups of orientable Seifert fibered 3-manifolds with orientable base in §9. We give here presentations for the holonomy Lie algebra $\mathfrak{h}(G)$ and the Chen Lie algebra $\text{gr}(G/G'')$ of such groups G . We also compute the Hilbert series of these graded Lie algebras, and discuss the formality properties of these groups.

This work was motivated by a desire to generalize some of the results of Fenn–Sjerve [10] and Papadima–Suciu [31], from commutator-relators groups to arbitrary finitely generated groups. In [41], we studied the formality properties of finitely generated groups, focusing on the filtered-formality and 1-formality properties. In related work, we apply the techniques developed in this paper and in [41] to the study of several families of “pure-braid like” groups. For instance, we investigate in [43] the pure virtual braid groups, and we investigate in [44] the McCool groups, also known as the pure welded braid groups. A summary of these results, as well as further motivation and background can be found in [42].

2. EXPANSIONS FOR FINITELY PRESENTED GROUPS

In this section, we introduce and study a Magnus-type expansion relative to a finitely presented group. We start by reviewing some basic notions.

2.1. Completed group algebras and expansions. Let G be a finitely generated group. As is well-known (see for instance [36, 37]), the group-algebra $\mathbb{Q}G$ has a natural Hopf algebra structure, with comultiplication $\Delta: \mathbb{Q}G \rightarrow \mathbb{Q}G \otimes_{\mathbb{Q}} \mathbb{Q}G$ given by $\Delta(g) = g \otimes g$ for $g \in G$, and counit the augmentation map $\varepsilon: \mathbb{Q}G \rightarrow \mathbb{Q}$ given by $\varepsilon(g) = 1$. The powers of the augmentation ideal, $I = \ker \varepsilon$, form a descending, multiplicative filtration of $\mathbb{Q}G$. The associated graded algebra, $\text{gr}(\mathbb{Q}G) = \bigoplus_{k \geq 0} I^k / I^{k+1}$, comes endowed with the degree filtration, $\mathcal{F}_k = \bigoplus_{j \geq k} I^j / I^{j+1}$. The corresponding completion, $\widehat{\text{gr}}(\mathbb{Q}G)$, is again an algebra, endowed with the inverse limit filtration.

The I -adic completion of the group-algebra, $\widehat{\mathbb{Q}G} = \varprojlim_k \mathbb{Q}G / I^k$, also comes equipped with an inverse limit filtration. Applying the I -adic completion functor to the map Δ yields a comultiplication map $\widehat{\Delta}$, which makes $\widehat{\mathbb{Q}G}$ into a complete Hopf algebra, see [38, App. A].

An element x in a Hopf algebra is called *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$. The set $\mathfrak{m}(G)$ of all primitive elements in $\widehat{\mathbb{Q}G}$, with bracket $[x, y] = xy - yx$, is a complete, filtered Lie algebra, called the *Malcev Lie algebra* of G . The set of all primitive elements in $\text{gr}(\mathbb{Q}G)$ forms a graded Lie algebra, which is isomorphic to the associated graded Lie algebra

$$(3) \quad \text{gr}(G) := \bigoplus_{k \geq 1} (\Gamma_k G / \Gamma_{k+1} G) \otimes \mathbb{Q},$$

where $\{\Gamma_k G\}_{k \geq 1}$ is the *lower central series* of G , defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [\Gamma_k G, G]$ for $k \geq 1$. As shown by Quillen in [37], there is an isomorphism of graded Lie algebras, $\text{gr}(\mathfrak{m}(G)) \cong \text{gr}(G)$.

The group G is said to be *filtered-formal* if its Malcev Lie algebra is isomorphic (as a filtered Lie algebra) to the degree completion of its associated graded Lie algebra. The group G is said to be *1-formal* if its Malcev Lie algebra admits a quadratic presentation (see [32, 41] for more details). For instance, all finitely generated free groups and free abelian groups are 1-formal.

An *expansion* for a group G is a filtration-preserving algebra morphism $E: \mathbb{Q}G \rightarrow \widehat{\text{gr}}(\mathbb{Q}G)$ with the property that $\text{gr}(E) = \text{id}$ (see [20, 2, 41]). As shown in [41], a finitely generated group G is filtered-formal if and only if it has an expansion E which induces an isomorphism of complete Hopf algebras, $\widehat{E}: \widehat{\mathbb{Q}G} \rightarrow \widehat{\text{gr}}(\mathbb{Q}G)$.

2.2. The Magnus expansion for a free group. Let F be a finitely generated free group, with generating set $\mathbf{x} = \{x_1, \dots, x_n\}$, and let $\mathbb{Z}F$ be its group-ring. Then the degree completion of the associated graded ring, $\widehat{\text{gr}}(\mathbb{Z}F)$, can be identified with the completed tensor ring $\widehat{T}(F_{\text{ab}}) = \mathbb{Z}\langle\langle \mathbf{x} \rangle\rangle$, the power series ring over \mathbb{Z} in n non-commuting variables, by sending $[x_i - 1]$ to x_i . There is a well known expansion $M: \mathbb{Z}F \rightarrow \mathbb{Z}\langle\langle \mathbf{x} \rangle\rangle$, called the *Magnus expansion*, given by

$$(4) \quad M(x_i) = 1 + x_i \quad \text{and} \quad M(x_i^{-1}) = 1 - x_i + x_i^2 - x_i^3 + \dots$$

The *Fox derivatives* are the ring morphisms $\partial_i: \mathbb{Z}F \rightarrow \mathbb{Z}F$ defined by the rules $\partial_i(1) = 0$, $\partial_i(x_j) = \delta_{ij}$, and $\partial_i(uv) = \partial_i(u)\varepsilon(v) + u\partial_i(v)$ for $u, v \in \mathbb{Z}F$, where $\varepsilon: \mathbb{Z}F \rightarrow \mathbb{Z}$ is the augmentation map. The higher Fox derivatives $\partial_{i_1, \dots, i_k}$ are then defined inductively. We refer to [9, 10, 23, 27] for more details and references on these notions.

The Magnus expansion can be computed in terms of the Fox derivatives, as follows. Given an element $y \in F$, if we write $M(y) = 1 + \sum a_I x_I$, then $a_I = \varepsilon_I(y)$, where $I = (i_1, \dots, i_s)$, and $\varepsilon_I = \varepsilon \circ \partial_I$ is the composition of the augmentation map with the iterated Fox derivative $\partial_I: \mathbb{Z}F \rightarrow \mathbb{Z}F$. For each $k \geq 1$, let M_k be the composite

$$(5) \quad \mathbb{Z}F \xrightarrow{M} \widehat{T}(F_{\text{ab}}) \xrightarrow{\text{gr}_k} \text{gr}_k(\widehat{T}(F_{\text{ab}}))$$

M_k

For each y in F , we have that $M_1(y) = \sum_{i=1}^n \varepsilon_i(y)x_i$, while for each y in the commutator subgroup $[F, F]$, we have

$$(6) \quad M_2(y) = \sum_{i < j} \varepsilon_{i,j}(y)(x_i x_j - x_j x_i).$$

The tensor algebra $T(F_{\mathbb{Q}})$ on the \mathbb{Q} -vector space $F_{\mathbb{Q}} = F_{\text{ab}} \otimes \mathbb{Q}$ has a natural graded Hopf algebra structure, with comultiplication Δ and counit ε given by $\Delta(a) = a \otimes 1 + 1 \otimes a$ and $\varepsilon(a) = 0$ for $a \in F_{\mathbb{Q}}$. The set of primitive elements in $T(F_{\mathbb{Q}})$ is the free Lie algebra $\text{lie}(F_{\mathbb{Q}}) = \{v \in T(F_{\mathbb{Q}}) \mid \Delta(v) = v \otimes 1 + 1 \otimes v\}$, with Lie bracket $[v, w] = v \otimes w - w \otimes v$. Notice that, if $y \in [F, F]$, then $M_2(y)$ is a primitive element in the degree 2 piece of the Hopf algebra $T(F_{\mathbb{Q}}) = \text{gr}(\widehat{T}(F_{\mathbb{Q}}))$, which corresponds to the degree 2 element $\sum_{i < j} \varepsilon_{i,j}(y)[x_i, x_j]$ in the free Lie algebra $\text{lie}(F_{\mathbb{Q}})$.

2.3. The Magnus expansion relative to a finitely generated group. Given a finitely generated group G , there exists an epimorphism $\varphi: F \twoheadrightarrow G$ from a free group F of finite rank to G . Let $\varphi_{\text{ab}}: F_{\text{ab}} \twoheadrightarrow G_{\text{ab}}$ be the induced epimorphism between the respective abelianizations.

Definition 2.1. The *Magnus κ -expansion* for F relative to G , denoted by κ_G (or κ for short), is the composition

$$(7) \quad \mathbb{Z}F \xrightarrow{M} \widehat{T}(F_{\text{ab}}) \xrightarrow{\widehat{T}(\varphi_{\text{ab}})} \widehat{T}(G_{\text{ab}}),$$

κ

where M is the classical Magnus expansion for the free group F , and the morphism $\widehat{T}(\varphi_{\text{ab}}): \widehat{T}(F_{\text{ab}}) \rightarrow \widehat{T}(G_{\text{ab}})$ is induced by the abelianization map φ_{ab} .

In particular, if the group G is a commutator-relators group, i.e., if all the relators of G lie in the commutator subgroup $[F, F]$, then the projection φ_{ab} identifies F_{ab} with G_{ab} , and the Magnus expansion κ coincides with the classical Magnus expansion M .

More generally, let G be a group generated by $\mathbf{x} = \{x_1, \dots, x_n\}$, and let F be the free group generated by the same set. The rational Magnus κ -expansion, still denoted by κ_G or κ , is the composition

$$(8) \quad \mathbb{Q}F \xrightarrow{M} \widehat{T}(F_{\mathbb{Q}}) \xrightarrow{\widehat{T}(\pi)} \widehat{T}(G_{\mathbb{Q}}),$$

where $\pi = \varphi_{\text{ab}} \otimes \mathbb{Q} = H_1(\varphi, \mathbb{Q})$ is the induced epimorphism in homology from $F_{\mathbb{Q}} := H_1(F; \mathbb{Q})$ to $G_{\mathbb{Q}} := H_1(G; \mathbb{Q})$. Pick a basis $\mathbf{y} = \{y_1, \dots, y_b\}$ for $G_{\mathbb{Q}}$, and identify $\widehat{T}(G_{\mathbb{Q}})$ with $\mathbb{Q}\langle\langle \mathbf{y} \rangle\rangle$. Let $\kappa(r)_I$ be the coefficient of $y_I := y_{i_1} \cdots y_{i_s}$ in $\kappa(r)$, for $I = (i_1, \dots, i_s)$. Then we can write

$$(9) \quad \kappa(r) = 1 + \sum_I \kappa(r)_I y_I.$$

Lemma 2.2. *If $r \in \Gamma_k F$, then $\kappa(r)_I = 0$, for $|I| < k$. Furthermore, if $r \in \Gamma_2 F$, then $\kappa(r)_{i,j} = -\kappa(r)_{j,i}$.*

Proof. Since $M(r)_I = \varepsilon_I(r) = 0$ for $|I| < k$ (see for instance [27]), we have that $\kappa(r)_I = 0$ for $|I| < k$. To prove the second assertion, identify the completed symmetric algebras $\widehat{\text{Sym}}(F_{\mathbb{Q}})$ and $\widehat{\text{Sym}}(G_{\mathbb{Q}})$ with the power series rings $\mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$ and $\mathbb{Q}\langle\langle \mathbf{y} \rangle\rangle$, respectively, in the following commuting diagram of \mathbb{Q} -linear maps.

$$(10) \quad \begin{array}{ccc} \mathbb{Q}F & \xrightarrow{M} & \widehat{T}(F_{\mathbb{Q}}) & \xrightarrow{\alpha_1} & \widehat{\text{Sym}}(F_{\mathbb{Q}}) \\ & \searrow \kappa & \downarrow \widehat{T}(\pi) & & \downarrow \widehat{\text{Sym}}(\pi) \\ & & \widehat{T}(G_{\mathbb{Q}}) & \xrightarrow{\alpha_2} & \widehat{\text{Sym}}(G_{\mathbb{Q}}). \end{array}$$

When $r \in [F, F]$, we have that $\alpha_2 \circ \kappa(r) = \widehat{\text{Sym}}(\pi) \circ \alpha_1 \circ M(r) = 1$. Thus, $\kappa_i(r) = 0$ and $\kappa(r)_{i,j} + \kappa(r)_{j,i} = 0$. \square

Lemma 2.3. *If $u, v \in F$ satisfy $\kappa(u)_J = \kappa(v)_J = 0$ for all $|J| < s$, for some $s \geq 2$, then*

$$\kappa(uv)_I = \kappa(u)_I + \kappa(v)_I, \quad \text{for } |I| = s.$$

Moreover, the above formula is always true for $s = 1$.

Proof. We have that $\kappa(uv) = \kappa(u)\kappa(v)$ for $u, v \in F$. If $\kappa(u)_J = \kappa(v)_J = 0$ for all $|J| < s$, then $\kappa(u) = 1 + \sum_{|I|=s} \kappa(u)_I y_I$ up to higher-order terms, and similarly for $\kappa(v)$. Then

$$(11) \quad \kappa(uv) = \kappa(u)\kappa(v) = 1 + \sum_{|I|=s} (\kappa(u)_I + \kappa(v)_I) y_I + \text{higher-order terms}.$$

Therefore, $\kappa(uv)_i = \kappa(u)_i + \kappa(v)_i$, and so $\kappa(uv)_I = \kappa(u)_I + \kappa(v)_I$. \square

2.4. Truncating the Magnus expansions. Recall that we defined in (5) truncations M_k of the Magnus expansion M of a free group F . In a similar manner, we can also define the truncations of the Magnus expansion κ for any finitely generated group G .

Lemma 2.4. *For each $k \geq 1$, the following diagram commutes:*

$$(12) \quad \begin{array}{ccccc} \mathbb{Q}F & \xrightarrow{M} & \widehat{T}(F_{\mathbb{Q}}) & \xrightarrow{\text{gr}_k} & \text{gr}_k(\widehat{T}(F_{\mathbb{Q}})) = \bigotimes^k \mathbb{Q}^n \\ & \searrow \kappa & \downarrow \widehat{T}(\pi) & \downarrow \text{gr}_k(\widehat{T}(\pi)) & \downarrow \otimes^k \pi \\ & & \widehat{T}(G_{\mathbb{Q}}) & \xrightarrow{\text{gr}_k} & \text{gr}_k(\widehat{T}(G_{\mathbb{Q}})) = \bigotimes^k \mathbb{Q}^b. \end{array}$$

Proof. The triangle on the left of diagram (12) commutes, since it consists of ring morphisms, by the definition of the Magnus expansion for a group. The morphisms in the two squares are homomorphisms between \mathbb{Q} -vector spaces. The squares commute, since π is a linear map. \square

In diagram (12), let us denote the composition of κ and gr_k by κ_k . We then obtain the diagram

$$(13) \quad \begin{array}{ccccc} & & \xrightarrow{\kappa_k} & & \\ & \searrow & & \searrow & \\ \mathbb{Q}F & \xrightarrow{\kappa} & \widehat{T}(G_{\mathbb{Q}}) & \xrightarrow{\text{gr}_k} & \text{gr}_k(\widehat{T}(G_{\mathbb{Q}})). \end{array}$$

In particular, $\kappa_1(r) = \sum_{i=1}^b \kappa(r)_i y_i$ for $r \in F$. By Lemma 2.2, if $r \in [F, F]$, then

$$(14) \quad \kappa_2(r) = \sum_{1 \leq i < j \leq b} \kappa(r)_{i,j} (y_i y_j - y_j y_i).$$

Notice that $\kappa_2(r)$ is a primitive element in the Hopf algebra $T(G_{\mathbb{Q}})$, which corresponds to the element $\sum_{i < j} \kappa_{i,j}(r) [y_i, y_j]$ in the free Lie algebra $\text{lie}(G_{\mathbb{Q}})$.

The next lemma provides a close connection between the Magnus expansion κ and the classical Magnus expansion M .

Lemma 2.5. *Let $(a_{i,s})$ be the $b \times n$ matrix associated to the linear map $\pi: F_{\mathbb{Q}} \rightarrow G_{\mathbb{Q}}$, and let $r \in F$ be an arbitrary element. Then, for each $1 \leq i, j \leq b$, we have that*

$$\kappa(r)_i = \sum_{s=1}^n a_{i,s} \varepsilon_s(r) \quad \text{and} \quad \kappa(r)_{i,j} = \sum_{s,t=1}^n a_{i,s} a_{j,t} \varepsilon_{s,t}(r).$$

Proof. By assumption, $\pi(x_s) = \sum_{i=1}^b a_{i,s} y_i$. By Lemma 2.4 (for $k = 1$), we have

$$\kappa_1(r) = \pi \circ M_1(r) = \pi \left(\sum_{s=1}^n \varepsilon_s(r) x_s \right) = \sum_{s=1}^n \sum_{i=1}^b a_{i,s} \varepsilon_s(r) y_i,$$

which gives the claimed formula for $\kappa(r)_i$. By Lemma 2.4 (for $k = 2$), we have

$$\kappa_2(r) = \pi \otimes \pi \circ M_2(r) = \pi \otimes \pi \left(\sum_{s,t=1}^n \varepsilon_{s,t}(r) x_s \otimes x_t \right) = \sum_{s,t=1}^n \sum_{i,j=1}^b \varepsilon_{s,t}(r) a_{i,s} a_{j,t} y_i \otimes y_j,$$

which gives the claimed formula for $\kappa(r)_{i,j}$. \square

3. ECHELON PRESENTATIONS AND CELLULAR CHAIN COMPLEXES

In this section we associate to every finitely presented group G an “echelon approximation”, G_e , such that they have isomorphic cohomology on their respective 2-complexes.

3.1. Presentation 2-complex. We start with a brief review of the cellular chain complexes associated to a presentation 2-complex of a group, following the exposition from [4, 9, 10, 27, 33]. Let G be a group with a finite presentation $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{r} = \{r_1, \dots, r_m\}$. Then $G = F/R$, where F is the free group on generating set \mathbf{x} and R is the (free) subgroup of F normally generated by the set $\mathbf{r} \subset F$.

Let K_P be the 2-complex associated to this presentation of G , consisting of a 0-cell e^0 , one-cells $\{e_1^1, \dots, e_n^1\}$ corresponding to the generators, and two-cells $\{e_1^2, \dots, e_m^2\}$ corresponding to the relators. The 2-complex K_P depends on the presentation P for the group G . However, if the presentation is understood, we may also denote this 2-complex by K_G .

The (integral) cellular chain complex $C_* = C_*(K_P; \mathbb{Z})$ is of the form $C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$, where C_j are the free abelian groups on the specified bases. Furthermore, $d_1 = 0$, while the matrix of the boundary map $d_2: C_2(K_P; \mathbb{Z}) \rightarrow C_1(K_P; \mathbb{Z})$ is the $m \times n$ Jacobian matrix $J_P = (\varepsilon_i(r_k))$.

Next, let $p: \widetilde{K}_P \rightarrow K_P$ be the universal cover of the presentation 2-complex, and fix a lift \tilde{e}^0 of the basepoint e^0 . The cells e_j^i of K_P lift to cells \tilde{e}_j^i at the basepoint \tilde{e}^0 . Let $\widetilde{C}_* = C_*(\widetilde{K}_P; \mathbb{Z})$ be the (equivariant) cellular chain complex of the universal cover. This is a chain complex of free $\mathbb{Z}G$ -modules of the form $\widetilde{C}_2 \xrightarrow{\tilde{d}_2} \widetilde{C}_1 \xrightarrow{\tilde{d}_1} \widetilde{C}_0$, with $\widetilde{C}_0 = \mathbb{Z}G$, $\widetilde{C}_1 = (\mathbb{Z}G)^n$ generated by the set $\{\tilde{e}_1^1, \dots, \tilde{e}_n^1\}$, and $\widetilde{C}_2 = (\mathbb{Z}G)^m$ generated by the set $\{\tilde{e}_1^2, \dots, \tilde{e}_m^2\}$. The differentials in this chain complex are the $\mathbb{Z}G$ -linear maps given by

$$(15) \quad \tilde{d}_1(\tilde{e}_i^1) = x_i - 1, \quad \tilde{d}_2(\tilde{e}_j^2) = \sum_{k=1}^m \varphi(\partial_k r_j) \tilde{e}_k^1,$$

where $\varphi: F \rightarrow G$ is the presenting homomorphism for our group.

3.2. Echelon presentations. We now introduce a special type of group presentations which will play an important role in the sequel.

Definition 3.1. Let G be a group with a finite presentation $P = \langle \mathbf{x} \mid \mathbf{w} \rangle$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{w} = \{w_1, \dots, w_m\}$. We say P is an *echelon presentation* if the augmented Fox Jacobian matrix $(\varepsilon_i(w_k))$ is in row-echelon form.

Let K_G be the 2-complex associated to the above presentation for G . Suppose the pivot elements of the $m \times n$ matrix $(\varepsilon_i(w_k))$ are in position $\{i_1, \dots, i_d\}$, and let $b = n - d$. Since this matrix is in row-echelon form, the vector space $H_1(K_G; \mathbb{Q}) = \mathbb{Q}^b$ has basis $\mathbf{y} = \{y_1, \dots, y_b\}$, where $y_j = e_{i_d+j}^1$ for $1 \leq j \leq b$. Furthermore, the vector space $H_2(K_G; \mathbb{Q}) = \mathbb{Q}^{m-d}$ has basis $\{e_{d+1}^2, \dots, e_m^2\}$. We will choose as basis for $H^1(K_G; \mathbb{Q})$ the set $\{u_1, \dots, u_b\}$, where u_i is the Kronecker dual to y_i .

Remark 3.2. Suppose G admits a commutator-relators presentation of the form $P = F/R$, with $R \subset [F, F]$. Then the augmented Fox Jacobian matrix $(\varepsilon_i(r_k))$ is the zero matrix, and thus the presentation P is an echelon presentation. In this case, the integer (co)homology groups of K_G are torsion-free, and so the aforementioned choices of bases work for integer (co)homology, as well.

More generally, the next proposition shows that for any finitely presented group, we can construct a group with an echelon presentation such that the cohomology groups of the corresponding presentation 2-complexes are isomorphic.

Proposition 3.3. *Let G be a finitely presented group. There exists then a group G_e with echelon presentation, and a map $f: K_{G_e} \rightarrow K_G$ between the respective presentation 2-complexes such that the induced homomorphism on fundamental groups, $\rho = f_{\#}: G_e \rightarrow G$, is surjective, and the induced homomorphism in cohomology, $f^*: H^*(K_G; \mathbb{Z}) \rightarrow H^*(K_{G_e}; \mathbb{Z})$, is an isomorphism.*

Proof. Suppose G has presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$. As in the above discussion, the matrix of the boundary map $d_2^*: C^1(K_G; \mathbb{Z}) \rightarrow C^2(K_G; \mathbb{Z})$ is the transpose of the $m \times n$ Jacobian matrix $(\varepsilon_i(r_k))$. By Gaussian elimination over \mathbb{Z} , there exists a matrix $C = (c_{l,k}) \in \text{GL}(m; \mathbb{Z})$ such that $C \cdot d_2^*$ is in row-echelon form (also known as Hermite normal form). We define a new group,

$$(16) \quad G_e = \langle x_1, \dots, x_n \mid w_1, \dots, w_m \rangle,$$

by setting $w_k = r_1^{c_{1,k}} r_2^{c_{2,k}} \dots r_m^{c_{m,k}}$ for $1 \leq k \leq m$.

Let $h: K_{G_e}^{(1)} \rightarrow K_G^{(1)}$ be the homeomorphism between the 1-skeleta of the respective 2-complexes obtained by matching 1-cells. If $\psi_k: S^1 \rightarrow K_{G_e}^{(1)}$ denotes the attaching map of the 2-cell in K_{G_e} corresponding to the relator w_k , then by construction $h \circ \psi_k$ is null-homotopic in K_G . Thus, h extends to a cellular map $f: K_{G_e} \rightarrow K_G$. Clearly, the induced homomorphism $\rho = f_{\#}: G_e \rightarrow G$ is surjective. Furthermore, the map f induces a chain map between the respective cellular chain complexes, $f_*: C_*(K_{G_e}; \mathbb{Z}) \rightarrow C_*(K_G; \mathbb{Z})$, with f_2 given by the matrix C . It is now straightforward to see that the map f induces an isomorphism in homology, and thus, by the Universal Coefficients theorem, an isomorphism in cohomology, too. \square

Note that the group G_e constructed above depends on the given (finite) presentation for G , not just on the isomorphism type of G . On the other hand, if G is a commutator-relators group, then, by Remark 3.2, the group G_e is isomorphic to G .

3.3. A transferred basis. Once again, let G be a group admitting a finite presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{r} = \{r_1, \dots, r_m\}$, and let K_G be the corresponding presentation 2-complex. Using an echelon approximation for the given presentation, we describe now convenient bases for the \mathbb{Q} -vector spaces $H^1(K_G, \mathbb{Q})$ and $H^2(K_G, \mathbb{Q})$, which will be used extensively in the next two sections.

By Proposition 3.3, there exists a group G_e with echelon presentation $\langle \mathbf{x} \mid \mathbf{w} \rangle$, where $\mathbf{w} = \{w_1, \dots, w_m\}$, and a map $f: K_{G_e} \rightarrow K_G$ inducing an isomorphism in (co)homology. As in §3.2, we may choose a basis $\mathbf{y} = \{y_1, \dots, y_b\}$ for the \mathbb{Q} -vector space $H_1(K_G; \mathbb{Q}) \cong H_1(K_{G_e}; \mathbb{Q})$; let $\{u_1, \dots, u_b\}$ be the dual basis for $H^1(K_G; \mathbb{Q}) \cong H^1(K_{G_e}; \mathbb{Q})$. We also choose a basis $\{z_1, \dots, z_m\}$ for $C_2(K_G; \mathbb{Q})$ and a basis $\{e_1^2, \dots, e_m^2\}$ for $C_2(K_{G_e}; \mathbb{Q})$ corresponding to $\{1 \otimes \tilde{e}_1^2, \dots, 1 \otimes \tilde{e}_m^2\}$. Finally, if we set

$$(17) \quad \gamma_k := f_*(e_k^2) = \sum_{l=1}^m c_{l,k} z_l,$$

then $\{\gamma_1, \dots, \gamma_m\}$ is another basis for $C_2(K_G; \mathbb{Q})$. Furthermore, $\{e_{d+1}^2, \dots, e_m^2\}$ is a basis for $H_2(K_{G_e}; \mathbb{Q})$ and $\{\gamma_{d+1}, \dots, \gamma_m\}$ is a basis for $H_2(K_G; \mathbb{Q})$. Thus, $H^2(K_G; \mathbb{Q})$ has dual basis $\{\beta_{d+1}, \dots, \beta_m\}$.

4. GROUP PRESENTATIONS AND (CO)HOMOLOGY

We compute in this section the cup-product in the cohomology ring of the 2-complex of a finitely presented group in terms of the Magnus expansion associated to the presentation.

4.1. A chain transformation. We start by reviewing the classical bar construction. Let G be a discrete group, and let $B_*(G)$ be the normalized bar resolution (see e.g. [4, 10]), where $B_p(G)$ is the free left $\mathbb{Z}G$ -module on generators $[g_1| \dots |g_p]$, with $g_i \in G$ and $g_i \neq 1$, and $B_0(G) = \mathbb{Z}G$ is free on one generator, $[]$. The boundary operators are G -module homomorphisms, $\delta_p: B_p(G) \rightarrow B_{p-1}(G)$, defined by

$$(18) \quad \delta_p[g_1| \dots |g_p] = g_1[g_2| \dots |g_p] + \sum_{i=1}^{p-1} (-1)^i [g_1| \dots |g_i g_{i+1}| \dots |g_p] + (-1)^p [g_1| \dots |g_{p-1}].$$

In particular, $\delta_1[g] = (g-1)[]$ and $\delta_2[g_1|g_2] = g_1[g_2] - [g_1 g_2] + [g_1]$. Let $\varepsilon: B_0(G) \rightarrow \mathbb{Z}$ be the augmentation map. We then have a free resolution $B_*(G) \xrightarrow{\varepsilon} \mathbb{Z}$ of the trivial G -module \mathbb{Z} .

We view here \mathbb{Z} as a right $\mathbb{Z}G$ -module, with action induced by the augmentation map. An element of the cochain group $B^p(G) = \text{Hom}_{\mathbb{Z}G}(B_p(G), \mathbb{Z})$ may be viewed as a set function $u: G^p \rightarrow \mathbb{Z}$ satisfying the normalization condition $u(g_1, \dots, g_p) = 0$ if some $g_i = 1$. The cup-product of two 1-dimensional classes $u, u' \in H^1(G; \mathbb{Z}) \cong \text{Hom}(G, \mathbb{Z})$ is given by

$$(19) \quad u \cup u'[g_1|g_2] = u(g_1)u'(g_2).$$

For future use, we record a result due to Fenn and Sjerve ([10, Thm. 2.1 and p. 327]).

Lemma 4.1 ([10]). *There exists a chain transformation $T: C_*(\widetilde{K}_G) \rightarrow B_*(G)$ of augmented chain complexes,*

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\varepsilon} & C_0(\widetilde{K}_G) & \xleftarrow{\tilde{d}_1} & C_1(\widetilde{K}_G) & \xleftarrow{\tilde{d}_2} & C_2(\widetilde{K}_G) & \longleftarrow & 0 & \longleftarrow & \dots \\ & & \parallel & & \downarrow T_0 & & \downarrow T_1 & & \downarrow T_2 & & \downarrow & & \\ 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{\varepsilon} & B_0(G) & \xleftarrow{\delta_1} & B_1(G) & \xleftarrow{\delta_2} & B_2(G) & \longleftarrow & B_3(G) & \longleftarrow & \dots \end{array}$$

defined by $T_0(\lambda) := \lambda[]$,

$$(20) \quad T_1(\tilde{e}_i^1) = [x_i] \quad \text{and} \quad T_2(\tilde{e}_k^2) = \tau_1 T_1 \tilde{d}_2(\tilde{e}_k^2),$$

where $\tau_1: B_1(G) \rightarrow B_2(G)$ is the homomorphism defined by

$$(21) \quad \tau_1(g[g_1]) = [g|g_1],$$

for all $g, g_1 \in G$.

4.2. Cup products for echelon presentations. Now let G be a group with echelon presentation $G = \langle \mathbf{x} \mid \mathbf{w} \rangle$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{w} = \{w_1, \dots, w_m\}$, as in Definition 3.1. We let $B_*(G; \mathbb{Q}) = \mathbb{Q} \otimes B_*(G)$ and $B^*(G; \mathbb{Q}) = \mathbb{Q} \otimes B^*(G)$.

Lemma 4.2. *For each basis element $u_i \in H^1(K_G; \mathbb{Q}) \cong H^1(G; \mathbb{Q})$ as above, and each $r \in F$, we have that*

$$u_i([\varphi(r)]) = \sum_{s=1}^n \varepsilon_s(r) a_{i,s} = \kappa_i(r),$$

where $(a_{i,s})$ is the $b \times n$ matrix for the projection map $\pi: F_{\mathbb{Q}} \rightarrow G_{\mathbb{Q}}$.

Proof. If $r \in F$, then $\varphi(r) \in G$ and $[\varphi(r)] \in B_1(G)$. Hence,

$$(22) \quad u_i([\varphi(r)]) = \sum_{s=1}^n \varepsilon_s(r) u_i([x_s]) = \sum_{s=1}^n \varepsilon_s(r) a_{i,s} = \kappa_i(r).$$

Since $H^1(G; \mathbb{Q}) \cong B^1(G; \mathbb{Q}) \cong \text{Hom}(G, \mathbb{Q})$, we may view u_i as a group homomorphism. This yields the first equality in (22). Since $\pi(x_s) = \sum_{j=1}^b a_{i,s} y_j$ and $u_i = y_i^*$, the second equality follows. The last equality follows from Lemma 2.5. \square

Theorem 4.3. *Let G be a group with echelon presentation $G = \langle \mathbf{x} \mid \mathbf{w} \rangle$. The cup-product map $\mu_{K_G}: H^1(K_G; \mathbb{Q}) \wedge H^1(K_G; \mathbb{Q}) \rightarrow H^2(K_G; \mathbb{Q})$ is given by $(u_i \cup u_j, e_k^2) = \kappa(w_k)_{i,j}$, for $1 \leq i, j \leq b$ and $d+1 \leq k \leq m$, where κ is the Magnus expansion of G .*

Proof. Let us write the Fox derivative $\partial_t(w_k)$ as a finite sum,

$$(23) \quad \partial_t(w_k) = \sum_{x \in F} p_{tk}^x x,$$

for $1 \leq t \leq n$, and $1 \leq k \leq m$. We then have

$$(24) \quad \begin{aligned} T_2(\tilde{e}_k^2) &= \tau_1 T_1(\tilde{d}_2(\tilde{e}_k^2)) && \text{by (20)} \\ &= \tau_1 T_1(\varphi(\partial_1(w_k)), \dots, \varphi(\partial_n(w_k))) && \text{by (15)} \\ &= \tau_1 \left(\sum_{t=1}^n \varphi(\partial_t(w_k))[x_t] \right) && \text{by (20)} \\ &= \sum_{t=1}^n \sum_{x \in F} p_{tk}^x [\varphi(x)|x_t]. && \text{by (21)} \end{aligned}$$

The chain transformation $T: C_*(\widetilde{K}_G; \mathbb{Q}) \rightarrow B_*(G; \mathbb{Q})$ induces an isomorphism on first cohomology, $T^*: H^1(G; \mathbb{Q}) \rightarrow H^1(K_G; \mathbb{Q})$. Let us view u_i and u_j as elements in $H^1(G; \mathbb{Q})$. We then have

$$\begin{aligned} (u_i \cup u_j, 1 \otimes \tilde{e}_k^2) &= (u_i \cup u_j, 1 \otimes T_2(\tilde{e}_k^2)) \\ &= (u_i \cup u_j, \sum_{t=1}^n \sum_{x \in F} p_{tk}^x [\varphi(x)|x_t]) && \text{by (24)} \\ &= \sum_{t=1}^n \sum_{x \in F} p_{tk}^x u_i(\varphi(x)) u_j(x_t) && \text{by (19)} \\ &= \sum_{t=1}^n \sum_{x \in F} p_{tk}^x u_i(\varphi(x)) a_{j,t} && \text{by Lemma 4.2} \\ &= \sum_{t=1}^n \sum_{x \in F} p_{tk}^x \sum_{s=1}^n a_{i,s} \varepsilon_s(x) a_{j,t} && \text{by Lemma 4.2} \\ &= \sum_{t=1}^n \sum_{s=1}^n (a_{j,t} a_{i,s} \varepsilon_{s,t}(w_k)) && \text{by (23)} \\ &= \kappa(w_k)_{i,j}, && \text{by Lemma 2.5} \end{aligned}$$

and this completes the proof. \square

4.3. Cup products for finite presentations. Let G be a group with a finite presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$. By Proposition 3.3, there exists a group G_e with echelon presentation $\langle \mathbf{x} \mid \mathbf{w} \rangle$, and a map $f: K_{G_e} \rightarrow K_G$ inducing an isomorphism in cohomology. Using the bases for $H^*(K_G; \mathbb{Q})$ transferred from suitable bases for $H^*(K_{G_e}; \mathbb{Q})$ as in §3.3, we obtain an explicit formula for computing cup-products in the rational cohomology of the presentation 2-complex K_G .

Theorem 4.4. *In the aforementioned bases for $H^*(K_G; \mathbb{Q})$, the cup-product map $\mu_{K_G}: H^1(K_G; \mathbb{Q}) \wedge H^1(K_G; \mathbb{Q}) \rightarrow H^2(K_G; \mathbb{Q})$ is given by*

$$u_i \cup u_j = \sum_{k=d+1}^m \kappa(w_k)_{i,j} \beta_k.$$

That is, $(u_i \cup u_j, \gamma_k) = \kappa(w_k)_{i,j}$, for all $1 \leq i, j \leq b$.

Proof. As in the discussion from §3.3, the elements $\gamma_k = f_*(w_k)$ with $d < k \leq m$ form a basis for $H_2(K_G; \mathbb{Q})$. Hence,

$$\begin{aligned} (u_i \cup u_j, \gamma_k) &= (u_i \cup u_j, f_*(e_k^2)) = (f^*(u_i \cup u_j), e_k^2) \\ &= (u_i \cup u_j, e_k^2) && \text{since } f^*(u_i) = u_i \\ &= \kappa(w_k)_{i,j} && \text{by Theorem 4.3.} \end{aligned}$$

The claim follows. \square

Let us consider now in more detail the case when the group G is a commutator-relators group. In that case, as noted in §2.3, the Magnus expansion $\kappa = \kappa_G$ coincides with the classical Magnus expansion M . Furthermore, by Remark 3.2 both $H_*(K_G; \mathbb{Z})$ and $H^*(K_G; \mathbb{Z})$ are torsion-free, and the aforementioned rational bases are also integral bases for these free \mathbb{Z} -modules. Moreover, we may take $G_e = G$, and note that all the arguments from this section work over \mathbb{Z} in this case. Using these observations, and the fact that $M(r_k)_{i,j} = \varepsilon_{ij}(r_k)$, we recover as a corollary the following result of Fenn and Sjerve [10].

Corollary 4.5 ([10], Thm. 2.4). *For a commutator-relators group $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$, the cup-product map $\mu_K: H^1(K_G; \mathbb{Z}) \wedge H^1(K_G; \mathbb{Z}) \rightarrow H^2(K_G; \mathbb{Z})$ is given by $(u_i \cup u_j, e_k^2) = \varepsilon_{ij}(r_k)$, for $1 \leq i, j \leq n$ and $1 \leq k \leq m$.*

5. A PRESENTATION FOR THE HOLONOMY LIE ALGEBRA

In this section, we give presentations for the holonomy Lie algebra of a finitely presented group, and for the solvable quotients of this Lie algebra. In the process, we complete the proof of Theorem 1.2 from the Introduction.

5.1. The holonomy Lie algebra of a group. We start by reviewing the construction of the holonomy Lie algebra of a finitely generated group G , following [6, 13, 24, 31, 41]. Set

$$(25) \quad \mathfrak{h}(G) = \text{lie}(H_1(G; \mathbb{Q})) / \langle \text{im } \mu_G^\vee \rangle,$$

where μ_G^\vee is the dual to the cup-product map $\mu_G: H^1(G; \mathbb{Q}) \wedge H^1(G; \mathbb{Q}) \rightarrow H^2(G; \mathbb{Q})$. If $\varphi: G_1 \rightarrow G_2$ is a group homomorphism, then the induced homomorphism in cohomology, $\varphi^*: H^1(G_2; \mathbb{Q}) \rightarrow H^1(G_1; \mathbb{Q})$, yields a morphism of graded Lie algebras, $\mathfrak{h}(\varphi): \mathfrak{h}(G_1) \rightarrow \mathfrak{h}(G_2)$. Moreover, if φ is surjective, then $\mathfrak{h}(\varphi)$ is also surjective.

In the definition of the holonomy Lie algebra of G , we used the cohomology ring of a classifying space $K(G, 1)$. More generally, if X is a connected space with $b_1(X) < \infty$, we may define its holonomy Lie algebra as $\mathfrak{h}(X) = \text{lie}(H_1(X; \mathbb{Q})) / \langle \text{im } \mu_X^\vee \rangle$. As above, a continuous map $f: X \rightarrow Y$ induces a morphism $\mathfrak{h}(f): \mathfrak{h}(X) \rightarrow \mathfrak{h}(Y)$; moreover, if $f \simeq g$, then $\mathfrak{h}(f) = \mathfrak{h}(g)$. The proof of the next lemma is straightforward.

Lemma 5.1. *Let $f: X \rightarrow Y$ be a map between connected spaces with finite first Betti numbers. Suppose f induces isomorphisms in rational cohomology in degrees 1 and 2. Then the map $\mathfrak{h}(f): \mathfrak{h}(X) \rightarrow \mathfrak{h}(Y)$ is an isomorphism.*

In definition (25), we may replace the classifying space $K(G, 1)$ used to compute group cohomology by any other connected CW-complex with the same fundamental group. The next lemma, which slightly improves on a result from [31, 41], makes this more precise.

Lemma 5.2. *Let G be a finitely generated group, and let X be a connected CW-complex with $\pi_1(X) = G$. There is then a natural isomorphism $\mathfrak{h}(X) \xrightarrow{\cong} \mathfrak{h}(G)$.*

Proof. We may construct a classifying space for the group G by attaching cells of dimension 3 and higher to the space X . The inclusion map, $j = j_X: X \rightarrow K(G, 1)$, induces a map on cohomology rings, $j^*: H^*(K(G, 1); \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$, which is an isomorphism in degree 1 and an injection in degree 2. In particular, $b_1(X) = b_1(G) < \infty$. Furthermore, j^2 restricts to an isomorphism from $\text{im}(\mu_G)$ to $\text{im}(\mu_X)$. Taking duals, we obtain the following diagram.

$$(26) \quad \begin{array}{ccccc} & & \mu_X^\vee & & \\ & & \curvearrowright & & \\ H_2(X; \mathbb{Q}) & \twoheadrightarrow & \text{im}(\mu_X^\vee)^\subset & \longrightarrow & H_1(X; \mathbb{Q}) \otimes H_1(X; \mathbb{Q}) \\ & \downarrow j_* & \cong \downarrow j_* & & \cong \downarrow j_* \otimes j_* \\ H_2(G; \mathbb{Q}) & \twoheadrightarrow & \text{im}(\mu_G^\vee)^\subset & \longrightarrow & H_1(G; \mathbb{Q}) \otimes H_1(G; \mathbb{Q}) \\ & & \mu_G^\vee & & \end{array}$$

Hence, the isomorphism $j_* \otimes j_*$ identifies $\text{im}(\mu_X^\vee)$ with $\text{im}(\mu_G^\vee)$. Thus, the extension to free Lie algebras of the isomorphism $j_*: H_1(X; \mathbb{Q}) \rightarrow H_1(G; \mathbb{Q})$ factors through an isomorphism $\mathfrak{h}(j): \mathfrak{h}(X) \rightarrow \mathfrak{h}(G)$.

To show that this isomorphism is natural, let $f: X \rightarrow Y$ be a map of pointed, connected CW-complexes with finitely generated fundamental groups, and let $g: K(\pi_1(X), 1) \rightarrow K(\pi_1(Y), 1)$ be the map (unique up to homotopy) induced by the homomorphism $f_\#: \pi_1(X) \rightarrow \pi_1(Y)$. Then $g \circ j_X \simeq j_Y \circ f$, and thus $\mathfrak{h}(g) \circ \mathfrak{h}(j_X) = \mathfrak{h}(j_Y) \circ \mathfrak{h}(f)$. \square

Putting together the previous two lemmas, we obtain the following corollary.

Corollary 5.3. *Let G_1 and G_2 be two finitely generated groups, with presentation 2-complexes K_1 and K_2 . Let $f: K_1 \rightarrow K_2$ be a cellular map, and let $\varphi = f_\#: G_1 \rightarrow G_2$ be the induced homomorphism. If $f^*: H^*(K_2, \mathbb{Q}) \rightarrow H^*(K_1, \mathbb{Q})$ is an isomorphism, then $\mathfrak{h}(\varphi): \mathfrak{h}(G_1) \rightarrow \mathfrak{h}(G_2)$ is also an isomorphism.*

Next, we show that, if need be, the group G we started with may be replaced by a finitely presented group with the same holonomy Lie algebra.

Proposition 5.4. *Let G be a finitely generated group. There exists then a finitely presented group G_f and a homomorphism $G_f \rightarrow G$ inducing an isomorphism $\mathfrak{h}(G_f) \xrightarrow{\cong} \mathfrak{h}(G)$.*

Proof. Let X be a connected CW-complex with $\pi_1(X) = G$. Since G is finitely generated, we may assume X has finitely many 1-cells. The proof of Proposition 4.1 from [34] shows that there exists a connected, finite subcomplex Z of X such that the inclusion $Z \rightarrow X$ induces isomorphisms $H_1(Z; \mathbb{Q}) \cong H_1(X; \mathbb{Q})$ and $\text{im}(\mu_Z^\vee) \cong \text{im}(\mu_X^\vee)$. Consequently, $\mathfrak{h}(Z) \cong \mathfrak{h}(X)$. Letting $G_f = \pi_1(Z)$, the claim follows from Lemma 5.2. \square

5.2. Magnus expansion and holonomy. Let G be a group admitting a finite presentation, $P = \langle \mathbf{x} \mid \mathbf{r} \rangle$. As shown in Proposition 3.3, there exists a group G_e with echelon presentation $P_e = \langle \mathbf{x} \mid \mathbf{w} \rangle$, and a map $f: K_{G_e} \rightarrow K_G$ inducing an isomorphism in cohomology and an epimorphism $\rho: G_e \twoheadrightarrow G$ on fundamental groups. By Corollary 5.3, the induced map between the respective holonomy Lie algebras, $\mathfrak{h}(\rho): \mathfrak{h}(G_e) \xrightarrow{\cong} \mathfrak{h}(G)$, is an isomorphism.

So let us consider a group $G = F/R$ admitting an echelon presentation $P = \langle \mathbf{x} \mid \mathbf{w} \rangle$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and $\mathbf{w} = \{w_1, \dots, w_m\}$. We now give a more explicit presentation for the holonomy Lie algebra $\mathfrak{h}(G)$.

Let $\partial_i(w_k) \in \mathbb{Z}F$ be the Fox derivatives of the relations, and let $\varepsilon_i(w_k) = \varepsilon(\partial_i(w_k)) \in \mathbb{Z}$ be their augmentations. Recall from §4.2 that we can choose a basis $\mathbf{y} = \{y_1, \dots, y_b\}$ for $H_1(K_P; \mathbb{Q})$ and a basis $\{e_{d+1}^2, \dots, e_m^2\}$ for $H_2(K_P; \mathbb{Q})$, where d is the rank of Jacobian matrix $J_P = (\varepsilon_i(w_k))$, viewed as an $m \times n$ matrix over \mathbb{Q} . Let $\text{lie}(\mathbf{y})$ be the free Lie algebra over \mathbb{Q} generated by \mathbf{y} in degree 1. Recall that κ_2 is the degree 2 part of the Magnus expansion of G given explicitly in (14). Thus, we can identify $\kappa_2(w_k)$ with $\sum_{i < j} \kappa(w_k)_{i,j} [y_i, y_j]$ in $\text{lie}(\mathbf{y})$ for $d+1 \leq k \leq m$.

Theorem 5.5. *Let G be a group admitting an echelon presentation $P = \langle \mathbf{x} \mid \mathbf{w} \rangle$. Then there exists an isomorphism of graded Lie algebras*

$$\mathfrak{h}(G) \xrightarrow{\cong} \text{lie}(\mathbf{y}) / \text{ideal}(\kappa_2(w_{d+1}), \dots, \kappa_2(w_m)) .$$

Proof. Combining Theorem 4.3 with the fact that $(u_i \wedge u_j, \mu^\vee(e_k^2)) = (\mu(u_i \wedge u_j), e_k^2)$, we see that the dual cup-product map, $\mu^\vee: H_2(K_P; \mathbb{Q}) \rightarrow H_1(K_P; \mathbb{Q}) \wedge H_1(K_P; \mathbb{Q})$, is given by

$$(27) \quad \mu^\vee(e_k^2) = \sum_{1 \leq i < j \leq b} \kappa(w_k)_{i,j} (y_i \wedge y_j).$$

Hence, the following diagram commutes,

$$(28) \quad \begin{array}{ccc} H_2(K_P; \mathbb{Q}) & \xrightarrow{\mu^\vee} & H_1(K_P; \mathbb{Q}) \wedge H_1(K_P; \mathbb{Q}) \\ \downarrow & & \downarrow \\ C_2(K_P; \mathbb{Q}) & \xrightarrow{\kappa_2} & H_1(K_P; \mathbb{Q}) \otimes H_1(K_P; \mathbb{Q}) . \end{array}$$

Using now the identification of $\kappa_2(w_k)$ and $\sum_{i < j} \kappa(w_k)_{i,j} [y_i, y_j]$ as elements of $\text{lie}(\mathbf{y})$, the definition of the holonomy Lie algebra, and the fact that $\mathfrak{h}(G) \cong \mathfrak{h}(K_P)$, we arrive at the desired conclusion. \square

Corollary 5.6. *The universal enveloping algebra of $\mathfrak{h}(G)$ has presentation*

$$U(\mathfrak{h}(G)) = \mathbb{Q}\langle \mathbf{y} \rangle / \text{ideal}(\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)).$$

If $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ is a commutator-relators group, then the group $H_1(K_G; \mathbb{Z})$ is torsion-free, and thus the integer holonomy Lie ring $\mathfrak{h}(G; \mathbb{Z})$ can be defined as in (25), using integer (co)homology, instead, see [24, 31] for details. Furthermore, as in §2.2, for each $r \in [F, F]$, the primitive element $M_2(r) \in \widehat{T}_2(F_{\text{ab}})$ corresponds to the element $\sum_{i < j} \varepsilon_{i,j}(r)[x_i, x_j]$ from the degree 2 piece of the free Lie ring $\text{lie}_{\mathbb{Z}}(\mathbf{x})$. Using this observation and Corollary 4.5, we recover a result from [31].

Corollary 5.7 ([31], Prop. 7.2). *If $G = \langle x_1, \dots, x_n \mid \mathbf{r} \rangle$ is a commutator-relators group, then*

$$\mathfrak{h}(G; \mathbb{Z}) = \text{lie}_{\mathbb{Z}}(\mathbf{x}) / \text{ideal} \left\{ \sum_{1 \leq i < j \leq n} \varepsilon_{i,j}(r)[x_i, x_j] \mid r \in \mathbf{r} \right\}.$$

Proposition 5.8. *For every quadratic Lie algebra \mathfrak{g} over \mathbb{Q} , there exists a commutator-relators group G_c such that $\mathfrak{h}(G_c) = \mathfrak{g}$.*

Proof. We use an approach similar to the proof of [35, Prop. 6.2]. By assumption, we may write $\mathfrak{g} = \text{lie}(\mathbf{x})/\mathfrak{a}$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ and \mathfrak{a} is an ideal generated by elements of the form $\ell_k = \sum_{1 \leq i < j \leq n} c_{ijk}[x_i, x_j]$ for $1 \leq k \leq m$, and where the coefficients c_{ijk} are in \mathbb{Q} . Clearing denominators, we may assume all c_{ijk} are integers. We can then define words $r_k = \prod_{1 \leq i < j \leq n} [x_i, x_j]^{c_{ijk}}$ in the free group generated by \mathbf{x} , and set $G_c = \langle \mathbf{x} \mid r_1, \dots, r_m \rangle$. Clearly, $\varepsilon_{i,j}(r_k) = c_{ijk}$. The desired conclusion follows from Corollary 5.7. \square

Corollary 5.9. *For every finitely generated group G , there exists a commutator-relators group G_c such that $\mathfrak{h}(G_c) = \mathfrak{h}(G)$.*

Proof. From Proposition 5.4, the holonomy Lie algebra $\mathfrak{h}(G)$ has a quadratic presentation. Letting $\mathfrak{g} = \mathfrak{h}(G)$ and applying Proposition 5.8 yields the desired conclusion. \square

5.3. Solvable quotients of holonomy Lie algebras. The next lemma follows straight from the definitions, using the standard isomorphism theorems.

Lemma 5.10. *Let $\mathfrak{g} = \text{lie}(V)/\mathfrak{r}$ be a finitely generated Lie algebra. Then $\mathfrak{g}/\mathfrak{g}^{(i)} \cong \text{lie}(V)/(\mathfrak{r} + \text{lie}(V)^{(i)})$. Furthermore, if \mathfrak{r} is a homogeneous ideal, then this is an isomorphism of graded Lie algebras.*

The next result sharpens and extends the first part of Theorem 7.3 from [31].

Theorem 5.11. *Let $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ be a finitely presented group, and set $\mathfrak{h} = \mathfrak{h}(G)$. Let $\mathbf{y} = \{y_1, \dots, y_b\}$ be a basis of $H_1(G; \mathbb{Q})$. Then, for each $i \geq 2$,*

$$\mathfrak{h}/\mathfrak{h}^{(i)} \cong \text{lie}(\mathbf{y}) / (\text{ideal}(\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)) + \text{lie}^{(i)}(\mathbf{y})),$$

where $b = b_1(G)$ and w_k is defined in (16).

Proof. By Theorem 5.5, the holonomy Lie algebra \mathfrak{h} is isomorphic to the quotient of the free Lie algebra $\text{lie}(\mathbf{y})$ by the ideal generated by $\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)$. The claim follows from Lemma 5.10. \square

Using Corollary 5.7, we obtain the following consequence.

Corollary 5.12. *Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a commutator-relators group, and let $\mathfrak{h} = \mathfrak{h}(G)$. Then, for each $i \geq 2$, the Lie algebra $\mathfrak{h}/\mathfrak{h}^{(i)}$ is isomorphic to the quotient of the free Lie algebra $\text{lie}(\mathbf{x})$ by the sum of the ideals $M_2(r_1), \dots, M_2(r_m)$, and $\text{lie}^{(i)}(\mathbf{x})$.*

6. LOWER CENTRAL SERIES AND THE HOLONOMY LIE ALGEBRA

6.1. Lower central series. Let G be a finitely generated group, and let $\{\Gamma_k G\}_{k \geq 1}$ be its lower central series (LCS). The LCS quotients of G are finitely generated abelian groups. Taking the direct sum of these groups, we obtain a graded Lie ring over \mathbb{Z} ,

$$(29) \quad \text{gr}(G; \mathbb{Z}) = \bigoplus_{k \geq 1} \Gamma_k G / \Gamma_{k+1} G.$$

The Lie bracket $[x, y]$ on $\text{gr}(G; \mathbb{Z})$ is induced from the group commutator, $[x, y] = xyx^{-1}y^{-1}$. More precisely, if $x \in \Gamma_r G$ and $y \in \Gamma_s G$, then $[x + \Gamma_{r+1} G, y + \Gamma_{s+1} G] = xyx^{-1}y^{-1} + \Gamma_{r+s+1} G$. The Lie algebra $\text{gr}(G; \mathbb{Q}) = \text{gr}(G; \mathbb{Z}) \otimes \mathbb{Q}$ is called the *associated graded Lie algebra* (over \mathbb{Q}) of the group G . For simplicity, we will usually drop the \mathbb{Q} -coefficients, and simply write it as $\text{gr}(G)$.

The group G is said to be *nilpotent* of class $\leq k$ if $\Gamma_{k+1} G = \{1\}$. For each $k \geq 2$, the factor group $G/\Gamma_k G$ is the maximal $(k-1)$ -step nilpotent quotient of G . The canonical projection $G \rightarrow G/\Gamma_k G$ induces an epimorphism $\text{gr}(G) \rightarrow \text{gr}(G/\Gamma_k G)$, which is an isomorphism in degrees $s < k$. We refer to Lazard [18] and Magnus et al. [23] for more details..

6.2. A comparison map. Once again, let G be a finitely generated group. Although the next lemma is known, we provide a proof, both for the sake of completeness, and for later use.

Lemma 6.1 ([24, 31]). *There exists a natural epimorphism of graded \mathbb{Q} -Lie algebras,*

$$\Phi_G: \mathfrak{h}(G) \twoheadrightarrow \text{gr}(G),$$

inducing isomorphisms in degrees 1 and 2.

Proof. As first noted by Sullivan [45] in a particular case, and then proved by Lambe [17] in general, there is a natural exact sequence

$$(30) \quad 0 \longrightarrow (\Gamma_2 G / \Gamma_3 G \otimes \mathbb{Q})^* \xrightarrow{\beta} H^1(G; \mathbb{Q}) \wedge H^1(G; \mathbb{Q}) \xrightarrow{\mu_G} H^2(G; \mathbb{Q}),$$

where β is the dual of Lie bracket product. Consequently, $\text{im}(\mu_G^\vee) = \ker(\beta^\vee)$.

Recall now that the associated graded Lie algebra $\text{gr}(G)$ is generated by its degree 1 piece, $H_1(G; \mathbb{Q}) \cong \text{gr}_1(G)$. Hence, there is a natural epimorphism of graded \mathbb{Q} -Lie algebras,

$$(31) \quad \varphi_G: \text{lie}(H_1(G; \mathbb{Q})) \twoheadrightarrow \text{gr}(G),$$

restricting to the identity in degree 1, and to the Lie bracket map $[\cdot, \cdot]: \wedge^2 \text{gr}_1(G) \rightarrow \text{gr}_2(G)$ in degree 2. By the above observation, the kernel of this map coincides with the image of μ_G^\vee . Thus, φ_G factors through a morphism $\Phi_G: \mathfrak{h}(G) \rightarrow \text{gr}(G)$, which enjoys all the claimed properties. \square

6.3. Nilpotent and derived quotients. As a quick application, let us compare the holonomy Lie algebra of a group to the holonomy Lie algebras of its nilpotent quotients and derived quotients.

Proposition 6.2. *Let G be a finitely generated group. Then*

$$\mathfrak{h}(G/\Gamma_k G) = \begin{cases} \mathfrak{h}(G)/\mathfrak{h}(G)' & \text{for } k = 2, \\ \mathfrak{h}(G) & \text{for } k \geq 3. \end{cases}$$

In particular, the holonomy Lie algebra of G depends only on the second nilpotent quotient, $G/\Gamma_3 G$.

Proof. The case $k = 2$ is trivial, so let us assume $k \geq 3$. By a previous remark, the projection $G \rightarrow G/\Gamma_k G$ induces an isomorphism $\text{gr}_2(G) \rightarrow \text{gr}_2(G/\Gamma_k G)$. Furthermore, $H_1(G; \mathbb{Q}) \cong H_1(G/\Gamma_k G)$. Using now the dual of the exact sequence (30), we see that $\text{im}(\mu_G^\vee) = \text{im}(\mu_{G/\Gamma_k G}^\vee)$. The desired conclusion follows. \square

Proposition 6.3. *The holonomy Lie algebras of the derived quotients of G are given by*

$$\mathfrak{h}(G/G^{(i)}) = \begin{cases} \mathfrak{h}(G)/\mathfrak{h}(G)' & \text{for } i = 1, \\ \mathfrak{h}(G) & \text{for } i \geq 2. \end{cases}$$

Proof. For $i = 1$, the statement trivially holds, so we may as well assume $i \geq 2$. It is readily proved by induction that $G^{(i)} \subseteq \Gamma_{2^i}(G)$. Hence, the projections

$$(32) \quad G \twoheadrightarrow G/G^{(i)} \twoheadrightarrow G/\Gamma_{2^i}G$$

yield natural projections $\mathfrak{h}(G) \twoheadrightarrow \mathfrak{h}(G/G^{(i)}) \twoheadrightarrow \mathfrak{h}(G/\Gamma_{2^i}G) = \mathfrak{h}(G)$. By Proposition 6.2, the composition of these projections is an isomorphism of Lie algebras. Therefore, the surjection $\mathfrak{h}(G) \twoheadrightarrow \mathfrak{h}(G/G^{(i)})$ is an isomorphism. \square

An analogous result holds for associated graded Lie algebras, albeit in somewhat weaker form. The quotient map, $q_i: G \twoheadrightarrow G/G^{(i)}$, induces a surjective morphism between associated graded Lie algebras. Plainly, this morphism is the canonical identification in degree 1. In fact, more is true.

Lemma 6.4. *For each $i \geq 2$ and each $k \leq 2^i - 1$, the map $\text{gr}(q_i): \text{gr}_k(G) \twoheadrightarrow \text{gr}_k(G/G^{(i)})$ is an isomorphism.*

Proof. Taking associated graded Lie algebras in sequence (32) yields epimorphisms

$$(33) \quad \text{gr}(G) \twoheadrightarrow \text{gr}(G/G^{(i)}) \twoheadrightarrow \text{gr}(G/\Gamma_{2^i}G).$$

By a remark we made at the end of §6.1, the composition of these maps is an isomorphism in degrees $k < 2^i$. The conclusion follows at once. \square

The next result distills the statements of Theorem 9.3 and Corollary 9.5 from [41], in a form needed here; this result sharpens and extends Theorem 4.2 from [31].

Theorem 6.5 ([41]). *Let G be a finitely generated group. For each $i \geq 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded Lie algebras, $\text{gr}(G)/\text{gr}(G)^{(i)} \twoheadrightarrow \text{gr}(G/G^{(i)})$. Moreover, if G is a 1-formal group, then $\mathfrak{h}(G)/\mathfrak{h}(G)^{(i)} \cong \text{gr}(G/G^{(i)})$.*

Combining Theorem 6.5 with Theorem 5.11, we obtain the following corollary.

Corollary 6.6. *Let $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ be a finitely presented, 1-formal group. Let $\mathbf{y} = \{y_1, \dots, y_b\}$ be a basis of $H_1(G; \mathbb{Q})$. Then, for each $i \geq 2$,*

$$\text{gr}(G/G^{(i)}) \cong \text{lie}(\mathbf{y}) / (\text{ideal}(\kappa_2(w_{n-b+1}), \dots, \kappa_2(w_m)) + \text{lie}^{(i)}(\mathbf{y})),$$

where $b = b_1(G)$ and w_k is defined in (16).

6.4. Graded-formality. We conclude our discussion of associated graded Lie algebras and holonomy Lie algebras by recalling a notion that will be important in the sequel. Recall from Lemma 6.1 that, for any finitely generated group G , there is a canonical epimorphism of graded Lie algebras, $\Phi_G: \mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$. We say that the group G is *graded-formal* (over \mathbb{Q}) if the map Φ_G is an isomorphism.

This notion was considered in various ways by Chen [6], Kohno [13], Labute [15], and Hain [11]. It was also recently studied by Lee in [19], where it was called ‘graded 1-formality.’ Various relationships between graded-formality and other formality properties were studied in [41]. In particular, a finitely generated group G is 1-formal if and only if it is both graded-formal and filtered-formal. We give here two alternate characterizations of graded formality, which oftentimes are easier to verify.

Proposition 6.7. *A finitely generated group G is graded-formal if and only if one of the following two conditions is satisfied.*

- (1) *The Lie algebra $\text{gr}(G)$ is quadratic.*
- (2) *$\dim_{\mathbb{Q}} \mathfrak{h}_n(G) = \dim_{\mathbb{Q}} \text{gr}_n(G)$, for all $n \geq 1$.*

Proof. Clearly, if the group G is graded-formal, then both conditions are satisfied.

Assume now that (1) holds, that is, the graded Lie algebra $\text{gr}(G)$ admits a presentation of the form $\text{lie}(V)/\langle U \rangle$, where V is a finite-dimensional \mathbb{Q} -vector space concentrated in degree 1 and U is a \mathbb{Q} -vector subspace of $\text{lie}_2(V)$. In particular, $V = \text{gr}_1(G) = H_1(G; \mathbb{Q})$. From the exact sequence (30), we see that the image of μ_G^\vee coincides with the kernel of the Lie bracket map $[\cdot, \cdot]: \wedge^2 \text{gr}_1(G) \rightarrow \text{gr}_2(G)$, which can be identified with U . Hence, the surjection $\varphi_G: \text{lie}(V) \twoheadrightarrow \text{gr}(G)$ induces an isomorphism $\Phi_G: \mathfrak{h}(G) \xrightarrow{\cong} \text{gr}(G)$.

Finally, assume (2) holds. In general, the homomorphism $(\Phi_G)_n: \mathfrak{h}_n(G) \rightarrow \text{gr}_n(G)$ is an isomorphism for $n \leq 2$ and an epimorphism for $n \geq 3$. Our assumption, together with the fact that each \mathbb{Q} -vector space $\mathfrak{h}_n(G)$ is finite-dimensional implies that all homomorphisms $(\Phi_G)_n$ are isomorphisms. Therefore, the map $\Phi_G: \mathfrak{h}(G) \twoheadrightarrow \text{gr}(G)$ is an isomorphism of graded Lie algebras. \square

7. MILDNESS AND GRADED-FORMALITY

We start this section with the notion of mild (or inert) presentation of a group, due to J. Labute and D. Anick, and its relevance to the associated graded Lie algebra. We then continue with some applications to two important classes of finitely presented groups: one-relator groups and fundamental groups of link complements.

7.1. Mild presentations. Let F be a free group generated by $\mathbf{x} = \{x_1, \dots, x_n\}$. The *weight* of a word $r \in F$ is defined as $\omega(r) = \sup\{k \mid r \in \Gamma_k F\}$. Since F is residually nilpotent, $\omega(r)$ is finite. The image of r in $\text{gr}_{\omega(r)}(F)$ is called the *initial form* of r , and is denoted by $\text{in}(r)$.

Let $G = F/R$ be a quotient of F , with presentation $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$, where $\mathbf{r} = \{r_1, \dots, r_m\}$. Let $\text{in}(\mathbf{r})$ be the ideal of the free \mathbb{Q} -Lie algebra $\text{lie}(\mathbf{x})$ generated by $\{\text{in}(r_1), \dots, \text{in}(r_m)\}$. Clearly, this is a homogeneous ideal; thus, the quotient

$$(34) \quad L(G) := \text{lie}(\mathbf{x}) / \text{in}(\mathbf{r})$$

is a graded Lie algebra. As noted by Labute in [15], the ideal $\text{in}(\mathbf{r})$ is contained in $\text{gr}^{\widetilde{\Gamma}}(R)$, where $\widetilde{\Gamma}_k R = \Gamma_k F \cap R$ is the induced filtration on R . Hence, there exists an epimorphism $L(G) \twoheadrightarrow \text{gr}(G)$.

Proposition 7.1. *Let G be a commutator-relators group, and let $\mathfrak{h}(G)$ be its holonomy Lie algebra. Then the canonical projection $\Phi_G: \mathfrak{h}(G) \rightarrow \text{gr}(G)$ factors through an epimorphism $\mathfrak{h}(G) \rightarrow L(G)$.*

Proof. Let $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ be a commutator-relators presentation for our group. By Corollary 5.7, the holonomy Lie algebra $\mathfrak{h}(G)$ admits a presentation of the form $\text{lie}(\mathbf{x})/\mathfrak{a}$, where \mathfrak{a} is the ideal generated by the degree 2 part of $M(r) - 1$, for all $r \in \mathbf{r}$. On the other hand, $\text{in}(r)$ is the smallest degree homogeneous part of $M(r) - 1$. Hence, $\mathfrak{a} \subseteq \text{in}(\mathbf{r})$, and this complete the proof. \square

Following [1, 15], we say that a group G is a *mildly presented group* (over \mathbb{Q}) if it admits a presentation $G = \langle \mathbf{x} \mid \mathbf{r} \rangle$ such that the quotient $\text{in}(\mathbf{r})/[\text{in}(\mathbf{r}), \text{in}(\mathbf{r})]$, viewed as a $U(L(G))$ -module via the adjoint representation of $L(G)$, is a free module on the images of $\text{in}(r_1), \dots, \text{in}(r_m)$. As shown by Anick in [1], a presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ is mild if and only if

$$(35) \quad \text{Hilb}(U(L(G)), t) = \left(1 - nt + \sum_{i=1}^m t^{\omega(r_i)} \right)^{-1}.$$

Theorem 7.2 (Labute [14, 15]). *Let G be a finitely-presented group.*

- (1) *If G is mildly presented, then $\text{gr}(G) = L(G)$.*
- (2) *If G has a single relator r , then G is mildly presented. Moreover, the LCS ranks $\phi_k(G) := \text{rank gr}_k(G)$ are given by*

$$(36) \quad \phi_k(G) = \frac{1}{k} \sum_{d|k} \mu(k/d) \left[\sum_{0 \leq i \leq \lfloor d/e \rfloor} (-1)^i \frac{d}{d+i-ei} \binom{d+i-ei}{i} n^{d-ei} \right],$$

where μ is the Möbius function and $e = \omega(r)$.

Labute states this theorem over \mathbb{Z} , but his proof works for any commutative PID with unity. There is an example in [15] showing that the mildness condition is crucial for part (1) of the theorem to hold. We give now a much simpler example to illustrate this phenomenon.

Example 7.3. Let $G = \langle x_1, x_2, x_3 \mid x_3, x_3[x_1, x_2] \rangle$. Clearly, $G \cong \langle x_1, x_2 \mid [x_1, x_2] \rangle$, which is a mild presentation. However, the Lie algebra $\text{lie}(x_1, x_2, x_3)/\text{ideal}(x_3)$ is not isomorphic to $\text{gr}(G) = \text{lie}(x_1, x_2)/\text{ideal}([x_1, x_2])$. Hence, the first presentation is not a mild.

Under different assumptions, alternative methods for computing the LCS ranks of a group G can be found in [47, 43].

7.2. Mildness and graded formality. We now use Labute's work on the associated graded Lie algebra and our presentation of the holonomy Lie algebra to give two graded-formality criteria.

Corollary 7.4. *Let G be a group admitting a mild presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$. If $\omega(r) \leq 2$ for each $r \in \mathbf{r}$, then G is graded-formal.*

Proof. By Theorem 7.2, the associated graded Lie algebra $\text{gr}(H; \mathbb{Q})$ has a presentation of the form $\text{lie}(\mathbf{x})/\text{in}(\mathbf{r})$, with $\text{in}(\mathbf{r})$ a homogeneous ideal generated in degrees 1 and 2. Using the degree 1 relations to eliminate superfluous generators, we arrive at a presentation with only quadratic relations. The desired conclusion follows from Proposition 6.7. \square

An important sufficient condition for mildness of a presentation was given by Anick [1]. Recall that ι denotes the canonical injection from the free Lie algebra $\text{lie}(\mathbf{x})$ into $\mathbb{Q}\langle \mathbf{x} \rangle$. Fix an ordering on the set $\{\mathbf{x}\}$. The set of monomials in the homogeneous elements $\iota(\text{in}(r_1)), \dots, \iota(\text{in}(r_m))$ inherits

the lexicographic order. Let w_i be the highest term of $\iota(\text{in}(r_i))$ for $1 \leq i \leq m$. Suppose that (i) no w_i equals zero; (ii) no w_i is a submonomial of any w_j for $i \neq j$, i.e., $w_j = uw_i v$ cannot occur; and (iii) no w_i overlaps with any w_j , i.e., $w_i = uv$ and $w_j = vw$ cannot occur unless $v = 1$, or $u = w = 1$. Then, the set $\{r_1, \dots, r_n\}$ is mild (over \mathbb{Q}). We use this criterion to provide an example of a finitely-presented group G which is graded-formal, but not filtered-formal.

Example 7.5. Let G be the group with generators x_1, \dots, x_4 and relators $r_1 = [x_2, x_3]$, $r_2 = [x_1, x_4]$, and $r_3 = [x_1, x_3][x_2, x_4]$. Ordering the generators as $x_1 > x_2 > x_3 > x_4$, we find that the highest terms for $\{\iota(\text{in}(r_1)), \iota(\text{in}(r_2)), \iota(\text{in}(r_3))\}$ are $\{x_2 x_3, x_1 x_4, x_1 x_3\}$, and these words satisfy the above conditions of Anick. Thus, by Theorem 7.2, the Lie algebra $\text{gr}(G)$ is the quotient of $\text{lie}(x_1, \dots, x_4)$ by the ideal generated by $[x_2, x_3]$, $[x_1, x_4]$, and $[x_1, x_3] + [x_2, x_4]$. Hence, $\mathfrak{h}(G) \cong \text{gr}(G)$, that is, G is graded-formal. On the other hand, using the Tangent Cone Theorem from [7], one can show that the group G is not 1-formal. Therefore, G is not filtered-formal.

7.3. The rational Murasugi conjecture. Let $L = (L_1, \dots, L_n)$ be an n -component link in S^3 . The complement of the link, $X = S^3 \setminus \bigcup_{i=1}^n L_i$, is a connected, 3-dimensional manifold, which has the homotopy type of a finite, 2-dimensional CW-complex. The link group, $G = \pi_1(X)$, carries crucial information about the homotopy type of X . For instance, if $n = 1$ (i.e., the link is a knot), or if $n > 1$ and L is not a split link, then the complement X is a $K(G, 1)$.

Let $\ell_{i,j} = \text{lk}(L_i, L_j)$ be the linking numbers of L . The information coming from these numbers is conveniently encoded in a graph Γ with vertex set $\{1, \dots, n\}$, and edges (i, j) whenever $\ell_{i,j} \neq 0$. As noted in [24, 31], the holonomy Lie algebra $\mathfrak{h}(G) = \mathfrak{h}(X)$ is determined by these data:

$$(37) \quad \mathfrak{h}(G) = \text{lie}(y_1, \dots, y_n) / \left(\sum_{j=1}^n \ell_{i,j} [y_i, y_j] = 0, 1 \leq i < n \right).$$

Turning now to the associated graded Lie algebra of a link group G , Murasugi conjectured in [29] that $\text{gr}_k(G; \mathbb{Z}) = \text{gr}_k(F_{n-1}; \mathbb{Z})$ for all $k > 1$, provided that the link L has n components, and all the linking numbers are equal to ± 1 . This conjecture was proved by Massey–Traldi [26] and Labute [16], who also proved an analogous result for the Chen ranks of such links. In [46], Traldi computed the Chen groups $\text{gr}_k(G/G''; \mathbb{Z})$ for all links with connected linking graph. The next theorem, which can be viewed as a rational version of Murasugi’s conjecture, combines results of Anick [1], Berceanu–Papadima [3], and Papadima–Suciu [31].

Theorem 7.6. *Let L be an n -component link in S^3 , and let G be the fundamental group of its complement. Assuming the linking graph Γ is connected, the following hold.*

- (1) *The group G is graded-formal, and thus, the associated graded Lie algebra $\text{gr}(G)$ is isomorphic to the holonomy Lie algebra $\mathfrak{h}(G)$, with presentation given by (37).*
- (2) *There exists a graded Lie algebra isomorphism $\text{gr}(G/G'') \cong \mathfrak{h}(G)/\mathfrak{h}(G)''$.*
- (3) *If, furthermore, L is the closure of a pure braid, then G admits a mild presentation.*

Proof. The first assertion follows from Lemma 4.1 and Theorems 3.2 and 4.2 in [3], the second assertion is proved in [31, Thm. 10.1], while the last assertion follows from [1, Thm. 3.7]. \square

We conclude this section with several examples illustrating the concepts discussed above. In each example, L is a link in S^3 , and G is the corresponding link group. The first two examples were computed by Hain in [11] using a slightly different method.

Example 7.7. Let L be the Borromean rings. All the linking numbers are 0, and so $\mathfrak{h}(G) = \text{lie}(x, y, z)$. The link group G has a presentation with three generators x, y, z and two relators, $r_1 = [x, [y, z]]$ and $r_2 = [z, [y, x]]$. It is readily seen that the link L passes Anick's mildness test; hence G admits a mild presentation. Thus, $\text{gr}(G) = \text{lie}(x, y, z)/\text{ideal}([x, [y, z]], [z, [y, x]])$, and so G is not graded-formal.

Example 7.8. Let L be the Whitehead link. This is a 2-component link with linking number 0; its link group has presentation $G = \langle x, y \mid r \rangle$, where $r = x^{-1}y^{-1}xyx^{-1}yxy^{-1}xyx^{-1}y^{-1}xy^{-1}x^{-1}y$. Since G has only one relator, Theorem 7.2 insures that this presentation is mild. Direct computation shows that $\text{in}(r) = [x, [y, [x, y]]]$. Thus, $\text{gr}(G) = \text{lie}(x, y)/\text{ideal}([x, [y, [x, y]]])$, and G is not graded-formal.

Example 7.9. Let L be the link of great circles in S^3 corresponding to the arrangement of transverse planes through the origin of \mathbb{R}^4 denoted as $\mathcal{A}(31425)$ in Matei–Suciu [28]. By construction, L is the closure of a pure braid, and its linking graph is a complete graph. Thus, by Theorem 7.6, the link group G is graded-formal, and admits a mild presentation. On the other hand, as noted in [7, Example 8.2], the group G is not 1-formal.

8. ONE-RELATOR GROUPS

We turn now to some other specific classes of finitely presented groups where our approach applies. We start with a well-known and much-studied class of groups in group theory.

8.1. Holonomy and graded-formality. If the group G admits a finite presentation with a single relator, we saw in the previous section that G is mildly presented. In fact, more can be said.

Proposition 8.1. *Let $G = \langle \mathbf{x} \mid r \rangle$ be a 1-relator group.*

- (1) *If r is a commutator relator, then $\mathfrak{h}(G) = \text{lie}(\mathbf{x})/\text{ideal}(M_2(r))$.*
- (2) *If r is not a commutator relator, then $\mathfrak{h}(G) = \text{lie}(y_1, \dots, y_{n-1})$.*

Proof. Part (1) follows from Corollary 5.7. When r is not a commutator relator, the Jacobian matrix $J_G = (\varepsilon(\partial_i r))$ has rank 1. Part (2) then follows from Theorem 5.5. \square

Corollary 8.2. *Let $G = \langle x_1, \dots, x_n \mid r \rangle$ be a 1-relator group, and let $\mathfrak{h} = \mathfrak{h}(G)$. Then*

$$(38) \quad \text{Hilb}(U(\mathfrak{h}); t) = \begin{cases} 1/(1 - (n-1)t) & \text{if } \omega(r) = 1, \\ 1/(1 - nt + t^2) & \text{if } \omega(r) = 2, \\ 1/(1 - nt) & \text{if } \omega(r) \geq 3. \end{cases}$$

Proof. Let $\mathbf{x} = \{x_1, \dots, x_n\}$. By Proposition 8.1, the universal enveloping algebra $U(\mathfrak{h})$ is isomorphic to either $T(y_1, \dots, y_{n-1})$ if $\omega(r) = 1$, or to $T(\mathbf{x})/\text{ideal}(M_2(r))$ if $\omega(r) = 2$, or to $T(\mathbf{x})$ if $\omega(r) \geq 3$. The claim now follows from Theorem 7.2 and formula (35). \square

Theorem 8.3. *Let $G = \langle \mathbf{x} \mid r \rangle$ be a group defined by a single relation. Then G is graded-formal if and only if $\omega(r) \leq 2$.*

Proof. By Theorem 7.2, the given presentation of G is mild. The weight $\omega(r)$ can also be computed as $\omega(r) = \inf\{|I| \mid M(r)_I \neq 0\}$. If $\omega(r) \leq 2$, then, by Proposition 8.1, we have that $\mathfrak{h}(G) \cong \text{gr}(G) \cong \text{lie}(\mathbf{x})/\text{ideal}(\text{in}(r))$, and so G is graded-formal.

On the other hand, if $\omega(r) \geq 3$, then $\mathfrak{h}(G) = \text{lie}(\mathbf{x})$. However, $\text{gr}(G) = \text{lie}(\mathbf{x})/\text{ideal}(\text{in}(r))$. Hence, G is not graded-formal. \square

Example 8.4. Let $G = \langle x_1, x_2 \mid r \rangle$, where $r = [x_1, [x_1, x_2]]$. Clearly, $\omega(r) = 3$. Hence, G is not graded-formal.

The next example shows that there exists a graded-formal group which is not filtered-formal.

Example 8.5. Let $G = \langle x_1, \dots, x_5 \mid w \rangle$, where $w = [x_1, x_2][x_3, [x_4, x_5]]$. Since $\omega(w) = 2$, Theorem 8.3 implies that the group G is graded-formal. On the other hand, as shown in [41], G is not 1-formal, and so G is not filtered-formal.

8.2. Chen ranks. We now determine the ranks of the (rational) Chen Lie algebra associated to an arbitrary finitely presented, 1-relator, 1-formal group, thereby extending a result of Papadima and Suciu from [31]. By definition, the *Chen ranks* of a finitely generated group G are the LCS ranks of its maximal metabelian quotient,

$$(39) \quad \theta_k(G) := \dim_{\mathbb{Q}}(\mathrm{gr}_k(G/G'')).$$

The projection $\pi: G \rightarrow G/G''$ induces an epimorphism, $\mathrm{gr}(\pi): \mathrm{gr}(G) \rightarrow \mathrm{gr}(G/G'')$, which is an isomorphism in degrees $k \leq 3$. Consequently, $\phi_k(G) \geq \theta_k(G)$, with equality for $k \leq 3$. The Chen ranks were introduced and studied by K.-T. Chen [5], who showed that, for all $k \geq 2$,

$$(40) \quad \theta_k(F_n) = (k-1) \binom{n+k-2}{k}.$$

The *holonomy Chen ranks* of the group G are defined as $\bar{\theta}_k(G) := \dim(\mathfrak{h}/\mathfrak{h}'')_k$, where $\mathfrak{h} = \mathfrak{h}(G)$. It is readily seen that $\bar{\theta}_k(G) \geq \theta_k(G)$, with equality for $k \leq 2$. A basic result in the subject reads as follows: If G is 1-formal, then

$$(41) \quad \theta_k(G) = \bar{\theta}_k(G),$$

for all $k \geq 1$. This result was proved in [31, Cor. 9.4] for 1-formal groups admitting a finite, commutator-relators presentation, and in full generality in [43, Prop. 8.1 and Cor. 8.6].

Proposition 8.6. *Let $G = F/\langle r \rangle$ be a one-relator group, where $F = \langle x_1, \dots, x_n \rangle$, and suppose G is 1-formal. Then*

$$\mathrm{Hilb}(\mathrm{gr}(G/G''), t) = \begin{cases} 1 + nt - \frac{1 - nt + t^2}{(1-t)^n} & \text{if } r \in [F, F], \\ 1 + (n-1)t - \frac{1 - (n-1)t}{(1-t)^{n-1}} & \text{otherwise.} \end{cases}$$

Proof. First assume that $r \in [F, F]$. The claim is then proved in [31, Thm. 7.3].

Now assume that $r \notin [F, F]$. In that case, Theorem 5.5 implies that $\mathfrak{h}(G) \cong \mathrm{lie}(y_1, \dots, y_{n-1})$, which in turn is isomorphic to $\mathfrak{h}(F_{n-1})$. Since both G and F_{n-1} are 1-formal, formula (41) implies that

$$(42) \quad \theta_k(G) = \bar{\theta}_k(G) = \bar{\theta}_k(F_{n-1}) = \theta_k(F_{n-1}).$$

The claim then follows from Chen's formula (40). \square

8.3. Surface groups. The Riemann surface Σ_g is a compact Kähler manifold, and thus, a formal space. The formality of Σ_g also implies the 1-formality of Π_g . As a consequence, the associated

graded Lie algebra $\text{gr}(\Pi_g)$ is isomorphic to the holonomy Lie algebra $\mathfrak{h}(\Pi_g)$, which has a presentation $\mathfrak{h}(\Pi_g) = \text{lie}(2g) / \langle \sum_{i=1}^g [x_i, y_i] = 0 \rangle$, where $\text{lie}(2g) := \text{lie}(x_1, y_1, \dots, x_g, y_g)$. It follows from formula (36) that the LCS ranks of the 1-relator group Π_g are given by

$$(43) \quad \phi_k(\Pi_g) = \frac{1}{k} \sum_{d|k} \mu(k/d) \left[\sum_{i=0}^{\lfloor d/2 \rfloor} (-1)^i \frac{d}{d-i} \binom{d-i}{i} (2g)^{d-2i} \right].$$

Using now Theorem 5.11, we see that the Chen Lie algebra of Π_g has presentation

$$(44) \quad \text{gr}(\Pi_g/\Pi_g'') = \text{lie}(2g) / \langle \langle \sum_{i=1}^g [x_i, y_i] \rangle + \text{lie}''(2g) \rangle.$$

Furthermore, Proposition 8.6 shows that the Chen ranks of our surface group are given by $\theta_1(\Pi_g) = 2g$, $\theta_2(\Pi_g) = 2g^2 - g - 1$, and

$$(45) \quad \theta_k(\Pi_g) = (k-1) \binom{2g+k-2}{k} - \binom{2g+k-3}{k-2}, \quad \text{for } k \geq 3.$$

Let N_h be the nonorientable surface of genus $h \geq 1$. It is well known that N_h has the rational homotopy type of a wedge of $h-1$ circles, see [7, Example 6.18]. Hence, N_h is a formal space, and thus $\pi_1(N_h)$ is a 1-formal group. Furthermore, Proposition 8.1 shows that the holonomy Lie algebra of $\pi_1(N_h)$ is isomorphic to the free Lie algebra with $h-1$ generators, and Proposition 8.6 implies that the Chen ranks of $\pi_1(N_h)$ are given by $\theta_k(\pi_1(N_h)) = (k-1) \binom{h+k-3}{k}$ for $k \geq 2$.

9. SEIFERT FIBERED SPACES

We will consider here only orientable, closed Seifert manifolds with orientable base. Every such manifold M admits an effective circle action, with orbit space an orientable surface of genus g , and finitely many exceptional orbits, encoded in pairs of coprime integers $(\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)$ with $\alpha_j \geq 2$. The obstruction to trivializing the bundle $\eta: M \rightarrow \Sigma_g$ outside tubular neighborhoods of the exceptional orbits is given by an integer $b = b(\eta)$. A standard presentation for the fundamental group of M in terms of the Seifert invariants is given by

$$(46) \quad \begin{aligned} \pi_\eta := \pi_1(M) = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h \mid h \text{ central}, \\ [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_s = h^b, z_i^{\alpha_i} h^{\beta_i} = 1 \ (i = 1, \dots, s) \rangle. \end{aligned}$$

As shown by Scott in [39], the Euler number $e(\eta)$ of the Seifert bundle $\eta: M \rightarrow \Sigma_g$ satisfies $e(\eta) = -b(\eta) - \sum_{i=1}^s \beta_i / \alpha_i$.

9.1. Holonomy Lie algebra. We now give a presentation for the holonomy Lie algebra of a Seifert manifold group.

Theorem 9.1. *Let $\eta: M \rightarrow \Sigma_g$ be a Seifert fibration with orientable base. The rational holonomy Lie algebra of the group $\pi_\eta = \pi_1(M)$ is given by*

$$\mathfrak{h}(\pi_\eta; \mathbb{Q}) = \begin{cases} \text{lie}(x_1, y_1, \dots, x_g, y_g, h) / \langle \sum_{i=1}^s [x_i, y_i] = 0, h \text{ central} \rangle & \text{if } e(\eta) = 0; \\ \text{lie}(2g) & \text{if } e(\eta) \neq 0. \end{cases}$$

Proof. First assume $e(\eta) = 0$. In this case, the row-echelon approximation of π_η has presentation

$$(47) \quad \begin{aligned} \widetilde{\pi}_\eta = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h \mid z_i^{\alpha_i} h^{\beta_i} = 1 \ (i = 1, \dots, s), \\ ([x_1, y_1] \cdots [x_g, y_g])^{\alpha_1 \cdots \alpha_s} = 1, h \text{ central} \rangle \end{aligned}$$

It is readily seen that the rank of the Jacobian matrix associated to this presentation has rank s . Furthermore, the map $\pi: F_{\mathbb{Q}} \rightarrow H_{\mathbb{Q}}$ is given by $x_i \mapsto x_i, y_i \mapsto y_i, z_j \mapsto (-\beta_j/\alpha_j)h, h \mapsto h$. Let κ be the Magnus expansion from Definition 2.1. A Fox Calculus computation shows that κ takes the following values on the commutator-relators of $\widetilde{\pi}_\eta$: $\kappa([z_i, h]) = 1$,

$$\kappa([x_i, h]) = 1 + x_i h - h x_i + \text{terms of degree } \geq 3,$$

$$\kappa([y_i, h]) = 1 + y_i h - h y_i + \text{terms of degree } \geq 3,$$

$$\kappa(r) = 1 + (\alpha_1 \cdots \alpha_s)(x_1 y_1 - y_1 x_1 + \cdots + x_g y_g - y_g x_g) + \text{terms of degree } \geq 3,$$

where $r = ([x_1, y_1] \cdots [x_g, y_g])^{\alpha_1 \cdots \alpha_s}$. The first claim now follows from Theorem 5.5.

Next, assume $e(\eta) \neq 0$. Then the row-echelon approximation of π_η is given by

$$(48) \quad \begin{aligned} \widetilde{\pi}_\eta = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h \mid z_i^{\alpha_i} h^{\beta_i} = 1 \ (i = 1, \dots, s), \\ ([x_1, y_1] \cdots [x_g, y_g])^{\alpha_1 \cdots \alpha_s} h^{e(\eta)\alpha_1 \cdots \alpha_s} = 1, h \text{ central} \rangle, \end{aligned}$$

while the homomorphism $\pi: F_{\mathbb{Q}} \rightarrow H_{\mathbb{Q}}$ is given by $x_i \mapsto x_i, y_i \mapsto y_i, z_j \mapsto (-\beta_j/\alpha_j)h$, and $h \mapsto 0$. As before, the second claim follows from Theorem 5.5, and we are done. \square

The Malcev Lie algebra of π_η , given in [41, Thm.11.6], has an explicit presentation, which is the degree completion of the graded Lie algebra

$$(49) \quad L(\pi_\eta) = \begin{cases} \text{lie}(x_1, y_1, \dots, x_g, y_g, z) / \langle \sum_{i=1}^g [x_i, y_i] = 0, z \text{ central} \rangle & \text{if } e(\eta) = 0; \\ \text{lie}(x_1, y_1, \dots, x_g, y_g, w) / \langle \sum_{i=1}^g [x_i, y_i] = w, w \text{ central} \rangle & \text{if } e(\eta) \neq 0, \end{cases}$$

where $\deg(w) = 2$ and the other generators have degree 1. Moreover, $\text{gr}(\pi_\eta) \cong L(\pi_\eta)$. From the presentation of π_η and the definition of filtered formality, we immediately obtain that fundamental groups of orientable Seifert manifolds are filtered-formal.

9.2. LCS ranks. We end this section with a computation of the ranks of the various graded Lie algebras attached to the fundamental group of an orientable Seifert manifold. Comparing these ranks, we derive some consequences regarding the non-formality properties of such groups.

We start with the LCS ranks $\phi_k(\pi_\eta) = \dim \text{gr}_k(\pi_\eta)$ and the holonomy ranks $\bar{\phi}_k(\pi_\eta) = \dim \mathfrak{h}(\pi_\eta)_k$.

Proposition 9.2. *The LCS ranks and the holonomy ranks of a Seifert manifold group π_η are computed as follows.*

(1) *If $e(\eta) = 0$, then $\phi_1(\pi_\eta) = \bar{\phi}_1(\pi_\eta) = 2g + 1$, and $\phi_k(\pi_\eta) = \bar{\phi}_k(\pi_\eta) = \phi_k(\Pi_g)$ for $k \geq 2$.*

(2) *If $e(\eta) \neq 0$, then $\bar{\phi}_k(\pi_\eta) = \phi_k(F_{2g})$ for $k \geq 1$.*

(3) *If $e(\eta) \neq 0$, then $\phi_1(\pi_\eta) = 2g$, $\phi_2(\pi_\eta) = g(2g - 1)$, and $\phi_k(\pi_\eta) = \phi_k(\Pi_g)$ for $k \geq 3$.*

Here the LCS ranks $\phi_k(\Pi_g)$ are given by formula (43).

Proof. If $e(\eta) = 0$, then $\pi_\eta \cong \Pi_g \times \mathbb{Z}$, and claim (1) readily follows. So suppose that $e(\eta) \neq 0$. In this case, we know from Theorem 9.1 that $\mathfrak{h}(\pi_\eta) = \mathfrak{h}(F_{2g})$, and thus claim (2) follows.

By (49), the associated graded Lie algebra $\text{gr}(\pi_\eta)$ is isomorphic to the quotient of the free Lie algebra $\text{lie}(x_1, y_1, \dots, x_g, y_g, w)$ by the ideal generated by the elements $\sum_{i=1}^g [x_i, y_i] - w$, $[w, x_i]$, and $[w, y_i]$. Define a morphism $\chi: \text{gr}(\pi_\eta) \rightarrow \text{gr}(\Pi_g)$ by sending $x_i \mapsto x_i, y_i \mapsto y_i$, and $w \mapsto 0$. It is readily

seen that the kernel of χ is the Lie ideal of $\text{gr}(\pi_\eta)$ generated by w , and this ideal is isomorphic to the free Lie algebra on w . Thus, we get a short exact sequence of graded Lie algebras,

$$(50) \quad 0 \longrightarrow \text{lie}(w) \longrightarrow \text{gr}(\pi_\eta) \xrightarrow{\chi} \text{gr}(\Pi_g) \longrightarrow 0 .$$

Comparing Hilbert series in this sequence establishes claim (3) and completes the proof. \square

Corollary 9.3. *If $g = 0$, the group π_η is always 1-formal, while if $g > 0$, the group π_η is graded-formal if and only if $e(\eta) = 0$.*

Proof. First suppose $e(\eta) = 0$. In this case, we know from (49) that $\text{gr}(\pi_\eta) \cong \text{gr}(\Pi_g) \times \text{gr}(\mathbb{Z})$. It easily follows that $\text{gr}(\pi_\eta) \cong \mathfrak{h}(\pi_\eta)$ by comparing the presentations of these two Lie algebras. Hence, π_η is graded-formal, and thus 1-formal, by the fact that π_η is filtered formal.

It is enough to assume that $g > 0$ and $e(\eta) \neq 0$, since the other claims are clear. By Proposition 9.2, we have that $\bar{\phi}_3(\pi_\eta) = (8g^3 - 2g)/3$, whereas $\phi_3(\pi_\eta) = (8g^3 - 8g)/3$. Hence, $\mathfrak{h}(\pi_\eta)$ is not isomorphic to $\text{gr}(\pi_\eta)$, proving that π_η is not graded-formal. \square

9.3. Chen ranks. Recall that the Chen ranks are defined as $\theta_k(\pi) = \dim \text{gr}_k(\pi/\pi'')$, while the holonomy Chen ranks are defined as $\bar{\theta}_k(\pi) = \dim(\mathfrak{h}/\mathfrak{h}'')_k$, where $\mathfrak{h} = \mathfrak{h}(\pi)$.

Proposition 9.4. *The Chen ranks and the holonomy Chen ranks of a Seifert manifold group π_η are computed as follows.*

- (1) *If $e(\eta) = 0$, then $\theta_1(\pi_\eta) = \bar{\theta}_1(\pi_\eta) = 2g + 1$, and $\theta_k(\pi_\eta) = \bar{\theta}_k(\pi_\eta) = \theta_k(\Pi_g)$ for $k \geq 2$.*
- (2) *If $e(\eta) \neq 0$, then $\bar{\theta}_k(\pi_\eta) = \theta_k(F_{2g})$ for $k \geq 1$.*
- (3) *If $e(\eta) \neq 0$, then $\theta_1(\pi_\eta) = 2g$, $\theta_2(\pi_\eta) = g(2g - 1)$, and $\theta_k(\pi_\eta) = \theta_k(\Pi_g)$ for $k \geq 3$.*

Here the Chen ranks $\theta_k(F_{2g})$ and $\theta_k(\Pi_g)$ are given by formulas (40) and (45), respectively.

Proof. Claims (1) and (2) are easily proved, as in Proposition 9.2. To prove claim (3), start by recalling that the group π_η is filtered-formal. Hence, from [41, Theorem 9.3], the Chen Lie algebra $\text{gr}(\pi_\eta/\pi_\eta'')$ is isomorphic to $\text{gr}(\pi_\eta)/\text{gr}(\pi_\eta)''$. As before, we obtain a short exact sequence of graded Lie algebras,

$$(51) \quad 0 \longrightarrow \text{lie}(w) \longrightarrow \text{gr}(\pi_\eta/\pi_\eta'') \longrightarrow \text{gr}(\Pi_g/\Pi_g'') \longrightarrow 0 .$$

Comparing Hilbert series in this sequence completes the proof. \square

Remark 9.5. The above result can be used to give another proof of Corollary 9.3. Indeed, suppose $e(\eta) \neq 0$. Then, by Proposition 9.4, we have that $\bar{\theta}_3(\pi_\eta) - \theta_3(\pi_\eta) = 2g$. Consequently, the group π_η is not 1-formal. The group π_η is not graded-formal, since it is filtered-formal.

REFERENCES

1. David J. Anick, *Inert sets and the Lie algebra associated to a group*, J. Algebra **111** (1987), no. 1, 154–165. [MR913201](#) [1.3](#), [7.1](#), [7.2](#), [7.3](#), [7.3](#)
2. Dror Bar-Natan, *Expansions and quadraticity for groups*, in preparation (2016). [1.1](#), [2.1](#)
3. Barbu Berceanu and Ștefan Papadima, *Cohomologically generic 2-complexes and 3-dimensional Poincaré complexes*, Math. Ann. **298** (1994), no. 3, 457–480. [MR1262770](#) [7.3](#), [7.3](#)
4. Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. [MR1324339](#) [3.1](#), [4.1](#)
5. Kuo-Tsai Chen, *Integration in free groups*, Ann. of Math. (2) **54** (1951), no. 1, 147–162. [MR0042414](#) [8.2](#)
6. Kuo-Tsai Chen, *Iterated integrals of differential forms and loop space homology*, Ann. of Math. (2) **97** (1973), 217–246. [MR0380859](#) [1.2](#), [5.1](#), [6.4](#)

7. Alexandru Dimca, Ștefan Papadima, and Alexander I. Suci, *Topology and geometry of cohomology jump loci*, Duke Math. J. **148** (2009), no. 3, 405–457. [MR2527322](#) [7.5](#), [7.9](#), [8.3](#)
8. William G. Dwyer, *Homology, Massey products and maps between groups*, J. Pure Appl. Algebra **6** (1975), no. 2, 177–190. [MR0385851](#) [1.3](#)
9. Roger Fenn, *Techniques of geometric topology*, London Mathematical Society Lecture Notes Series, vol. 57, Cambridge University Press, Cambridge, 1983. [MR0787801](#) [2.2](#), [3.1](#)
10. Roger Fenn and Denis Sjerve, *Massey products and lower central series of free groups*, Canad. J. Math. **39** (1987), no. 2, 322–337. [MR899840](#) [1.1](#), [1.4](#), [2.2](#), [3.1](#), [4.1](#), [4.1](#), [4.1](#), [4.3](#), [4.5](#)
11. Richard M. Hain, *Iterated integrals, intersection theory and link groups*, Topology **24** (1985), no. 1, 45–66. [MR790675](#) [6.4](#), [7.3](#)
12. Marshall Hall, Jr., *A basis for free Lie rings and higher commutators in free groups*, Proc. Amer. Math. Soc. **1** (1950), 575–581. [MR0038336](#) [1.3](#)
13. Toshitake Kohno, *On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces*, Nagoya Math. J. **92** (1983), 21–37. [MR726138](#) [1.2](#), [5.1](#), [6.4](#)
14. John P. Labute, *On the descending central series of groups with a single defining relation*, J. Algebra **14** (1970), 16–23. [MR0251111](#) [1.3](#), [7.2](#)
15. John P. Labute, *The determination of the Lie algebra associated to the lower central series of a group*, Trans. Amer. Math. Soc. **288** (1985), no. 1, 51–57. [MR773046](#) [1.3](#), [6.4](#), [7.1](#), [7.1](#), [7.2](#), [7.1](#)
16. John P. Labute, *The Lie algebra associated to the lower central series of a link group and Murasugi’s conjecture*, Proc. Amer. Math. Soc. **109** (1990), no. 4, 951–956. [MR1013973](#) [7.3](#)
17. Larry A. Lambe, *Two exact sequences in rational homotopy theory relating cup products and commutators*, Proc. Amer. Math. Soc. **96** (1986), no. 2, 360–364. [MR818472](#) [6.2](#)
18. Michel Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. École Norm. Sup. (3) **71** (1954), 101–190. [MR0088496](#) [1.3](#), [6.1](#)
19. Peter Lee, *The pure virtual braid group is quadratic*, Selecta Math. (N.S.) **19** (2013), no. 2, 461–508. [MR3090235](#) [6.4](#)
20. Xiao-Song Lin, *Power series expansions and invariants of links*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 184–202. [MR1470727](#) [1.1](#), [2.1](#)
21. Wilhelm Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. **111** (1935), no. 1, 259–280. [MR1512992](#) [1.1](#)
22. Wilhelm Magnus, *Über Beziehungen zwischen höheren Kommutatoren*, J. Reine Angew. Math. **177** (1937), 105–115. [MR1581549](#) [1.3](#)
23. Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers, New York-London-Sydney, 1966. [MR0207802](#) [1.3](#), [2.2](#), [6.1](#)
24. Martin Markl and Ștefan Papadima, *Homotopy Lie algebras and fundamental groups via deformation theory*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 905–935. [MR1196099](#) [1.2](#), [5.1](#), [5.2](#), [6.1](#), [7.3](#)
25. Gwénaél Massuyeau, *Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant*, Bull. Soc. Math. France **140** (2012), no. 1, 101–161. [MR2903772](#) [1.1](#)
26. William S. Massey and Lorenzo Traldi, *On a conjecture of K. Murasugi*, Pacific J. Math. **124** (1986), no. 1, 193–213. [MR850676](#) [7.3](#)
27. Daniel Matei and Alexander I. Suci, *Cohomology rings and nilpotent quotients of real and complex arrangements, Arrangements—Tokyo 1998*, Adv. Stud. Pure Math., vol. 27, Kinokuniya, Tokyo, 2000, pp. 185–215. [MR1796900](#) [1.1](#), [1.2](#), [2.2](#), [2.3](#), [3.1](#)
28. Daniel Matei and Alexander I. Suci, *Homotopy types of complements of 2-arrangements in \mathbf{R}^4* , Topology **39** (2000), no. 1, 61–88. [MR1710992](#) [7.9](#)
29. Kunio Murasugi, *On Milnor’s invariant for links. II. The Chen group*, Trans. Amer. Math. Soc. **148** (1970), 41–61. [MR0259890](#) [7.3](#)
30. Ștefan Papadima, *Finite determinacy phenomena for finitely presented groups*, Proceedings of the 2nd Gauss Symposium. Conference A: Mathematics and Theoretical Physics (Munich, 1993), Sympos. Gaussiana, de Gruyter, Berlin, 1995, pp. 507–528. [MR1352516](#) [1.1](#)
31. Ștefan Papadima and Alexander I. Suci, *Chen Lie algebras*, Int. Math. Res. Not. (2004), no. 21, 1057–1086. [MR2037049](#) [1.2](#), [1.2](#), [1.3](#), [1.4](#), [5.1](#), [5.1](#), [5.2](#), [5.7](#), [5.3](#), [6.1](#), [6.3](#), [7.3](#), [7.3](#), [7.3](#), [7.3](#), [8.2](#), [8.2](#), [8.2](#)

32. Stefan Papadima and Alexandru Suciu, *Geometric and algebraic aspects of 1-formality*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **52(100)** (2009), no. 3, 355–375. [MR2554494](#) [1.3](#), [2.1](#)
33. Stefan Papadima and Alexander I. Suciu, *The spectral sequence of an equivariant chain complex and homology with local coefficients*, Trans. Amer. Math. Soc. **362** (2010), no. 5, 2685–2721. [MR2584616](#) [3.1](#)
34. Stefan Papadima and Alexander I. Suciu, *Jump loci in the equivariant spectral sequence*, Math. Res. Lett. **21** (2014), no. 4, 863–883. [MR3275650](#) [1.2](#), [5.1](#)
35. Stefan Papadima and Alexander I. Suciu, *Vanishing resonance and representations of Lie algebras*, J. Reine Angew. Math. **706** (2015), 83–101. [MR3393364](#) [1.1](#), [1.2](#), [5.2](#)
36. Inder Bir Singh Passi, *The associated graded ring of a group ring*, Bull. London Math. Soc. **10** (1978), no. 3, 241–255. [MR0519902](#) [2.1](#)
37. Daniel G. Quillen, *On the associated graded ring of a group ring*, J. Algebra **10** (1968), 411–418. [MR0231919](#) [2.1](#), [2.1](#)
38. Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. [MR0258031](#) [2.1](#)
39. Peter Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487. [MR705527](#) [9](#)
40. John Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181. [MR0175956](#) [1.3](#)
41. Alexander I. Suciu and He Wang, *Formality properties of finitely generated groups and Lie algebras*, preprint (2018), [arXiv:1504.08294v4](#). [1.1](#), [1.2](#), [1.3](#), [1.4](#), [2.1](#), [5.1](#), [5.1](#), [6.3](#), [6.5](#), [6.4](#), [8.5](#), [9.1](#), [9.3](#)
42. Alexander I. Suciu and He Wang, *The pure braid groups and their relatives*, *Perspectives in Lie theory*, 403–426, Springer INdAM series, vol. 19, Springer, Cham, 2017. [MR3751136](#) [1.4](#)
43. Alexander I. Suciu and He Wang, *Pure virtual braids, resonance, and formality*, Math. Zeit. **286** (2017), no. 3-4, 1495–1524. [MR3671586](#) [1.2](#), [1.4](#), [7.1](#), [8.2](#)
44. Alexander I. Suciu and He Wang, *Chen ranks and resonance varieties of the upper McCool groups*, preprint (2018), [arXiv:1804.06006v1](#). [1.2](#), [1.4](#)
45. Dennis Sullivan, *On the intersection ring of compact three manifolds*, Topology **14** (1975), no. 3, 275–277. [MR0383415](#) [6.2](#)
46. Lorenzo Traldi, *Linking numbers and Chen groups*, Topology Appl. **31** (1989), no. 1, 55–71. [MR984104](#) [7.3](#)
47. Thomas Weigel, *Graded Lie algebras of type FP*, Israel J. Math. **205** (2015), no. 1, 185–209. [MR3314587](#) [7.1](#)
48. Ernst Witt, *Treue Darstellungen Liescher Ringe*, J. Reine Angew. Math. **177** (1937), 152–160. [MR1581553](#) [1.3](#)

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, USA

E-mail address: a.suciu@northeastern.edu

URL: <http://web.northeastern.edu/suciu/>

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEVADA, RENO, NV 89557, USA

E-mail address: hew@unr.edu

URL: <http://wol.fweb.unr.edu/homepage/hew/>