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COMPLEX HYPERPLANE ARRANGEMENTS

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In the Fall of 2004, we were fortunate to spend the semester in residence at MSRI for the program "Hyperplane Arrangements and Applications." It was an intense, stimulating, productive, enlightening, eventful and most enjoyable experience. It was especially so for us "long-timers" in the field because the program truly marked a coming-of-age in the evolution of the subject from relative obscurity thirty years ago. We had an opportunity to introduce the wonders of arrangements to a group of graduate students during the two-week MSRI graduate school in Eugene in early August, and to an impressive group of post-docs and many other unsuspecting mathematicians during the program. We are glad to have this chance to bring some of the ideas to a wider audience. For general reference, we suggest the reader consult the books and survey articles listed on the summer school web page, `www.math.neu.edu/~suciu/eugene04.html`.

In its simplest manifestation, an arrangement is merely a finite collection of lines in the real plane. The complement of the lines consists of a finite number of polygonal regions. Determining the number of regions turns out to be a purely combinatorial problem: one can easily find a recursion for the number of regions, whose solution is given by a formula involving only the number of lines and the numbers of lines through each intersection point. This formula generalizes to collections of hyperplanes in $\mathbb{R}^\ell$, where the recursive formula is satisfied by an evaluation of the characteristic polynomial of the (reverse-ordered) poset of intersections. The study of characteristic polynomials forms the backbone of the combinatorial, and much of the algebraic theory of arrangements, which were featured in the MSRI workshop "Combinatorial Aspects of Hyperplane Arrangements" last November.

From the topological standpoint, a richer situation is presented by arrangements of complex hyperplanes, that is, finite collections of hyperplanes in $\mathbb{C}^\ell$ (or in projective space $\mathbb{P}^\ell$). In this case, the complement is connected, and its topology, as reflected in the fundamental group or the cohomology ring for instance, is much more interesting.

The motivation and many of the applications of the topological theory arose initially from the connection with braids. Let $\mathcal{A}_\ell = \{z_i = z_j\}_{1\le i<j\le \ell}$ be the arrangement of diagonal hyperplanes in $\mathbb{C}^\ell$, with complement the configuration space X_ℓ. In 1962, Fox and Neuwirth showed that $\pi_1(X_\ell) = P_\ell$, the pure braid group on ℓ strings, while Neuwirth and Fadell showed that X_ℓ is aspherical. A few years later, as part of his approach to Hilbert's thirteenth problem, Arnol'd computed the cohomology ring $H^*(X_\ell, \mathbb{C})$.

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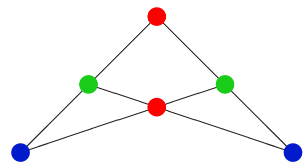

FIGURE 1. The $\mathcal{A}_3$ matroid, and a neighborly partition

For an arbitrary hyperplane arrangement in $\mathbb{C}^\ell$, the fundamental group of the complement, $G = \pi_1(X)$, can be computed algorithmically, using the braid monodromy associated to a generic projection of a generic slice in $\mathbb{C}^2$, see [3] and references there. The end result is a finite presentation with generators x_i corresponding to meridians around the n hyperplanes, and commutator relators of the form $x_i\alpha_j(x_i)^{-1}$, where $\alpha_j \in P_n$ are the (pure) braid monodromy generators, acting on the meridians via the Artin representation $P_n \hookrightarrow \mathrm{Aut}(F_n)$.

The cohomology ring $H^*(X, \mathbb{Q})$ was computed by Brieskorn in the early 1970's. His proof shows that X is a formal space, in the sense of Sullivan: the rational homotopy type of X is determined by $H^*(X, \mathbb{Q})$. In particular, all rational Massey products vanish. In 1980, Orlik and Solomon gave a simple combinatorial description of the $\Bbbk$-algebra $H^*(X, \Bbbk)$, for any field $\Bbbk$: it is the quotient $A = E/I$ of the exterior algebra E on classes dual to the meridians, modulo a certain ideal I determined by the intersection poset, see [17, 18].

For each $a \in A^1 \cong \Bbbk^n$, the Orlik-Solomon algebra can be turned into a cochain complex (A, a), with i-th term the degree i graded piece of A, and with differential given by multiplication by a, cf. [29]. The *resonance varieties* of A were defined in [7] to be the jumping loci for the cohomology of this cochain complex:

$$R^i_d(A) = \{a \in A^1 \mid \dim_\Bbbk H^i(A, a) \geq d\}. \tag{1}$$

The case of a line arrangement in $\mathbb{P}^2$ is already quite fascinating. The subarrangements that contribute components to $R^1_d(A)$ have very special combinatorial and geometric properties. To be eligible, the incidence matrix for the lines and intersection points must have null-space of dimension at least two, with full support. In addition, the subarrangement must have a partition into at least three classes such that no point p is incident with one line of one class, while all other lines incident with p belong to a second class. Such partitions are termed "neighborly." The simplest non-trivial example is provided by the braid arrangement $\mathcal{A}_3$, see Figure 1. In this figure the points represent hyperplanes and the lines correspond to the points of multiplicity greater than two. This is a diagram of the matroid associated with the arrangement.

When $\Bbbk$ has characteristic zero, the Vinberg classification of generalized Cartan matrices implies an even more exceptional situation, see [12]. One can assign multiplicities to the lines so that the partition is into classes of equal size d, with the same number of lines from each class containing each "inter-class" intersection point. This partition gives rise to a pencil of degree d curves which interpolates the completely reducible (not necessarily reduced) curves formed by the classes in the partition. The pencil that corresponds to Figure 1 consists of the curves $ax^2 + by^2 + cz^2 = 0$, with $a + b + c = 0$, see Figure 2. The

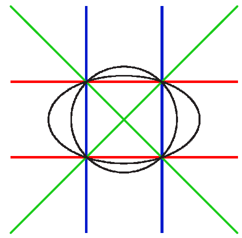

FIGURE 2. A pencil of conics including the braid arrangement

singular fibers are given by $a = 0$, $b = 0$, and $c = 0$. A non-reduced example is provided by the arrangement of symmetry planes of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, with the coordinate hyperplanes having multiplicity two. This multi-arrangement is interpolated by a pencil of quartics. Such pencils often yield (non-linear) fiberings of the complement by punctured surfaces, showing in particular that the complement is aspherical.

There is also an apparent connection between the cohomology of (A, a) and critical points of certain multi-variate rational functions. A resonant degree-one element a is represented (up to a scalar) by a logarithmic deRham one-form $d \log \Phi$, where Φ is a product of the defining linear forms of the hyperplanes, raised to integral powers. The dimension of $H^i(A, a)$ is related to the number of components in the critical locus of Φ of codimension i. In particular we expect Φ to have nonisolated critical points when a is "generically resonant." This is known to be the case for certain high-dimensional arrangements with certain weights [6], and was established for arrangements of rank three during the Fall program. A precise description of this relationship in general is a topic of current study.

In our example of the $\mathcal{A}_3$ arrangement, $d \log \Phi$ is resonant precisely when $\Phi(x, y, z) = (x^2 - y^2)^\alpha (y^2 - z^2)^\beta (z^2 - x^2)^\gamma$, with $\alpha + \beta + \gamma = 0$. The critical set $d\Phi = 0$ is given (projectively) by $[x^2 - y^2 : y^2 - z^2 : z^2 - x^2] = [\alpha : \beta : \gamma]$. It is not a coincidence that these critical loci are curves in the pencil of Figure 2.

Through the connection with generalized hypergeometric functions, the critical locus of Φ is of interest in relation to the Bethe Ansatz in mathematical physics, see [28]. This was a major topic of discussion in the MSRI workshop "Topology of Arrangements and Applications" last October. Somewhat serendipitously, the same problem for $\Bbbk = \mathbb{R}$ is of interest to the combinatorialists studying algebraic statistics, who were well-represented in Berkeley last fall.

The *characteristic varieties* of a space X are the jumping loci for the cohomology of X with coefficients in rank 1 local systems:

$$V_d^i(X) = \{\mathbf{t} \in \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \mid \dim_{\mathbb{C}} H^i(X, \mathbb{C}_{\mathbf{t}}) \geq d\}, \tag{2}$$

where $\mathbb{C}_{\mathbf{t}}$ denotes the abelian group $\mathbb{C}$, with $\pi_1(X)$-module structure given by the representation $\mathbf{t}\colon \pi_1(X) \to \mathbb{C}^*$.

Now suppose X is the complement of an arrangement of n hyperplanes. By work of Arapura [1], the irreducible components of the characteristic varieties of X are algebraic subtori of the character torus $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^n$, possibly translated by unitary characters. It turns out that the tangent cone at the origin to $V_d^i(X)$ coincides with the resonance variety $R^i_d(A)$, see [4, 11, 2]. Consequently, the resonance varieties are unions of linear subspaces; moreover, the algebraic subtori in the characteristic varieties are determined by the intersection lattice. Nevertheless, there exist arrangements for which the characteristic varieties have components that do not pass through the origin, [26]; it is an open question whether such components are combinatorially determined.

Counting certain torsion points on the character torus, according to their depth with respect to the stratification by the characteristic varieties, yields information about the homology of finite abelian covers of the complement, see [16]. This approach gives a practical algorithm for computing the Betti numbers of the Milnor fiber F of a central arrangement in $\mathbb{C}^3$. It has also led to examples of multi-arrangements with torsion in $H_1(F)$, see [5]. There are no known examples of ordinary arrangements with this property.

The tangent-cone theorem, and the linearity of resonance components, both fail over fields of characteristic $p > 0$. There is evidence that this failure is related to the existence of non-vanishing Massey products over $\mathbb{Z}_p$, cf. [14]. In addition, there is an empirical connection between translated components of characteristic varieties over $\mathbb{C}$ and resonance varieties over fields or rings of positive characteristic. The study of resonance varieties in prime characteristic, started in [15], leads naturally to the theory of line complexes and ruled varieties, developed in [8]. The counter-example to the linearity question, raised in [26], is a singular, irreducible cubic threefold in $\mathbb{P}^4$ ruled by lines, in characteristic three. The underlying arrangement is the Hessian arrangement of 12 lines determined by the inflection points on a general cubic; see [8].

As noted by Rybnikov [22], the fundamental group of the complement, $G = \pi_1(X)$, is not necessarily determined by the intersection poset. Even so, the ranks $\phi_k(G)$ of the successive quotients of the lower central series $\{G_k\}_{k\geq 1}$, where $G_1 = G$ and $G_{k+1} = [G, G_k]$, are combinatorially determined. Indeed, according to Sullivan, the formality of X implies that the graded Lie algebra $\operatorname{gr}(G) = \bigoplus_{k\geq 1} G_k/G_{k+1}$ is rationally isomorphic to the holonomy Lie algebra $\mathfrak{h}_A$ associated to $A = H^*(X;\mathbb{Q})$. Furthermore, the Chen Lie algebra $\operatorname{gr}(G/G'')$, associated to the lower central series of G/G'', is rationally isomorphic to $\mathfrak{h}_A/\mathfrak{h}''_A$, and so the Chen ranks $\theta_k(G)$ are also combinatorially determined, see [19].

Much effort has been put in computing explicitly the LCS and Chen ranks of an arrangement group G. It turns out that both can be expressed in terms of the Betti numbers of the linear strands in certain free resolutions (over A or E):

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(G)} = \sum_{i=0}^{\infty} \dim \operatorname{Tor}^A_i(\mathbb{Q}, \mathbb{Q})_i \, t^i, \tag{3}$$

$$\theta_k(G) = \dim \operatorname{Tor}^E_{k-1}(A, \mathbb{Q})_k, \quad \text{for } k \geq 2. \tag{4}$$

When the arrangement is of fiber-type (equivalently, the intersection lattice is supersolvable), A is a Koszul algebra. As noted in [21, 25], formula (3) and Koszul duality yield the classical LCS formula of Kohno [10] and Falk-Randell [9]:

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k(G)} = \operatorname{Hilb}(A, -t). \tag{5}$$

In [26], two conjectures were made, expressing (under some conditions) the LCS and Chen ranks of an arrangement group in terms of the dimensions of the components of the first resonance variety. Write $R^1_1(A) = L_1 \cup \cdots \cup L_q$, with $\dim L_i = d_i$. Then:

$$\prod_{k=2}^{\infty}(1-t^k)^{\phi_k(G)} = \prod_{i=1}^{q}\frac{1-d_it}{(1-t)^{d_i}}, \qquad \text{provided } \phi_4(G)=\theta_4(G), \tag{6}$$

$$\theta_k(G) = (k-1)\sum_{i=1}^{q}\binom{k+d_i-2}{k}, \qquad \text{for } k \text{ sufficiently large.} \tag{7}$$

The inequality $\geq$ from (7) has been proven in [24]. The reverse inequality has an algebro-geometric interpretation in terms of the sheaf on $\mathbb{C}^n$ determined by the linearized Alexander invariant. Equality in both (6) and (7) has been verified for two important classes of arrangements: decomposable arrangements (essentially, those for which all components of $R^1_1(A)$ arise from sub-arrangements of rank two), and graphic arrangements (i.e., sub-arrangements of the braid arrangement); see [23, 20, 24, 13].

Many of the results and observations reported on here represent joint work (or work in progress) with our friends and collaborators: Dan Cohen, Graham Denham, Dani Matei, Stefan Papadima, Hal Schenck, Sasha Varchenko, and Sergey Yuzvinsky. Our thanks go to them. In addition, we are grateful to many other unnamed participants in the MSRI program last Fall, for the countless hours spent in helpful and stimulating conversations about arrangements.

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