

TAYLOR EXPANSIONS OF GROUPS AND FILTERED-FORMALITY

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To the memory of Ștefan Papadima, 1953–2018

ABSTRACT. Let G be a finitely generated group, and let $\mathbb{k}G$ be its group algebra over a field of characteristic 0. A Taylor expansion is a certain type of map from G to the degree completion of the associated graded algebra of $\mathbb{k}G$ which generalizes the Magnus expansion of a free group. The group G is said to be filtered-formal if its Malcev Lie algebra is isomorphic to the degree completion of its associated graded Lie algebra. We show that G is filtered-formal if and only if it admits a Taylor expansion, and derive some consequences.

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1. INTRODUCTION

1.1. Expansions of groups. Group expansions were first introduced by Magnus in [30], in order to show that finitely generated free groups are residually nilpotent. This technique has been generalized and used in many ways. For instance, the exponential expansion of a free group was used to give a presentation for the Malcev Lie algebra of a finitely presented group by Papadima [39] and Massuyeau [34]. Expansions of pure braid groups and their applications in knot theory have been studied since the 1980s by several authors, see for instance Kohno's papers [24, 25, 26]. X.-S. Lin studied in [29] expansions of fundamental groups of smooth manifolds, using K.T. Chen's theory [12] of formal power series connections and their induced monodromy representations. More generally, Bar-Natan has explored in [4] the Taylor expansions of an arbitrary ring.

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Let G be a finitely generated group, and fix a coefficient field \mathbb{k} of characteristic zero. We let $\mathrm{gr}(\mathbb{k}G)$ be the associated graded algebra of $\mathbb{k}G$ with respect to the filtration by powers of the augmentation ideal, and we let $\widehat{\mathrm{gr}}(\mathbb{k}G)$ be the degree completion of this algebra. Developing an idea from [4], we say that a map $E: G \rightarrow \widehat{\mathrm{gr}}(\mathbb{k}G)$ is a multiplicative expansion of G if the induced algebra morphism, $\bar{E}: \mathbb{k}G \rightarrow \widehat{\mathrm{gr}}(\mathbb{k}G)$, is filtration-preserving and induces the identity at the associated graded level. Such a map E is called a *Taylor expansion* if it sends each element of G to a group-like element of the Hopf algebra $\widehat{\mathrm{gr}}(\mathbb{k}G)$.

1.2. Expansions and filtered-formality. Once again, let G be a finitely generated group. The concept of filtered-formality relates an object from rational homotopy theory to a group-theoretic object. The first object is the Malcev Lie algebra $\mathfrak{m}(G, \mathbb{k})$, defined by Quillen [45] as the set of primitive elements of the I -adic completion of the group algebra of G , where I is the augmentation ideal of $\mathbb{k}G$. This Lie algebra comes endowed with a (complete) filtration induced from the natural filtration on $\widehat{\mathbb{k}G}$, and is isomorphic to the dual of Sullivan's 1-minimal model of a $K(G, 1)$ space. The second object is the graded Lie algebra $\mathrm{gr}(G, \mathbb{k})$, defined by taking the direct sum of the successive quotients of the lower central series of G , tensored with \mathbb{k} . As shown by Quillen in [44], the associated graded algebra $\mathrm{gr}(\mathbb{k}G)$ is isomorphic to the universal enveloping algebra of $\mathrm{gr}(G, \mathbb{k})$.

The group G is called *filtered-formal* if its Malcev Lie algebra, $\mathfrak{m}(G, \mathbb{k})$, is isomorphic to $\widehat{\mathrm{gr}}(G; \mathbb{k})$, the degree completion of its associated graded Lie algebra, as filtered Lie algebras. If, in addition, the graded Lie algebra $\mathrm{gr}(G; \mathbb{k})$ is quadratic, the group G is said to be *1-formal*. For more details on these notions we refer to [41, 42, 49] and references therein.

The following result, which elucidates the relationship between Taylor expansions and formality properties, is a combination of Theorem 6.1 and Corollary 6.4.

Theorem 1.1. *Let G be a finitely generated group. Then:*

- (1) G is filtered-formal if and only if G has a Taylor expansion $G \rightarrow \widehat{\mathrm{gr}}(\mathbb{k}G)$.
- (2) G is 1-formal if and only if G has a Taylor expansion and $\mathrm{gr}(\mathbb{k}G)$ is a quadratic algebra.

Combining this theorem with our results on filtered-formality from [49], we conclude that the following propagation property of Taylor expansions holds. This is a combination of Theorems 6.3 and 6.6.

Proposition 1.2. *The existence of a Taylor expansion is preserved under field extensions, and taking finite products and coproducts, split injections, nilpotent quotients or solvable quotients of groups.*

In particular, if a finitely generated group G has a Taylor expansion over \mathbb{C} , then it also has a Taylor expansion over \mathbb{Q} .

1.3. Residual properties and Taylor expansions. A group G is said to be *residually torsion-free nilpotent* if any non-trivial element of G can be detected in a torsion-free nilpotent quotient. If G is finitely generated, this condition is equivalent to the injectivity of the canonical map to the Malcev group completion, $\kappa: G \rightarrow \mathfrak{M}(G, \mathbb{k})$. An expansion $\bar{E}: \mathbb{k}G \rightarrow \widehat{\mathrm{gr}}(\mathbb{k}G)$ is said to be *faithful* if the map $E: G \rightarrow \widehat{\mathrm{gr}}(\mathbb{k}G)$ is injective.

The next proposition relates the property of being residually torsion-free nilpotent to the existence of a faithful Taylor expansion.

Proposition 1.3. *A finitely generated group G has a faithful Taylor expansion if and only if G is residually torsion-free nilpotent and filtered-formal.*

The work of Magnus [30, 32] shows that all the free groups F_n are residually torsion-free nilpotent (RTFN). Furthermore, as shown by Hain [22] and Berceanu–Papadima [7], the Torelli groups $\text{IA}_n = \ker(\text{Aut}(F_n) \rightarrow \text{Aut}((F_n)_{\text{ab}}))$ are also RTFN. Consequently, all the subgroups of IA_n , for instance, the pure braid group P_n , the McCool group wP_n , and the upper McCool group wP_n^+ , inherit this property.

Let Π_n be the direct product of the free groups F_1, \dots, F_{n-1} . The graded Lie algebras $\text{gr}(P_n, \mathbb{k})$, $\text{gr}(\Pi_n, \mathbb{k})$ and $\text{gr}(wP_n^+, \mathbb{k})$ are isomorphic as vector spaces. Hence, their universal enveloping algebras, which are domains for the Taylor expansions of P_n , Π_n , and wP_n^+ , are also isomorphic as vector spaces. The next proposition shows that they are not isomorphic as algebras.

Proposition 1.4. *For each $n \geq 4$, the graded Lie algebras $\text{gr}(P_n, \mathbb{k})$, $\text{gr}(\Pi_n, \mathbb{k})$, and $\text{gr}(wP_n^+, \mathbb{k})$ are pairwise non-isomorphic.*

1.4. Braid-like groups and further directions. Explicit Taylor expansions have been constructed for several classes of filtered-formal groups, including finitely generated free groups, free abelian groups, surface groups, the pure braid groups, and the McCool groups.

When G is the fundamental group of a smooth manifold M , an important construction for a Taylor expansion arises from Chen’s theory of formal power series connections and their induced monodromy representations. Using this technique, Kohno [25, 26] gave explicit Taylor expansions for the pure braid groups P_n . Using a completely different approach, Papadima constructed in [40] integral Taylor expansions for the braid groups B_n . In another direction, Hain studied expansions for link groups [21], fundamental groups of algebraic varieties [23], and the Torelli groups [22], while Lin [29] further investigated the relationship between expansions and link invariants, including Vassiliev invariants, Milnor’s link variants and the Kontsevich integral.

There is also a strong interplay between Taylor expansions of the pure braid groups and the finite-type (or Vassiliev) invariants in knot theory. In this context, the relevant formal power series connection is a version of the Knizhnik–Zamolodchikov connection. The Taylor expansions of the groups constructed from Chen’s theory of formal power series connections yield finite-type invariants for pure braids, and provide a prototype for the Kontsevich integral for knots. For more on all of this, we refer the reader to [20, 38, 40, 29, 3, 4].

2. HOPF ALGEBRAS AND EXPANSIONS OF GROUPS

2.1. Group algebras, completions, and associated graded algebras. Let G be a finitely generated group, and let $\mathbb{k}G$ be its group algebra over a field \mathbb{k} . Let $\varepsilon: \mathbb{k}G \rightarrow \mathbb{k}$ be the augmentation homomorphism, defined by $\varepsilon(g) = 1$ for all $g \in G$. The powers of the augmentation ideal, $I = \ker(\varepsilon)$, define the I -adic filtration on the group algebra, $\{I^k\}_{k \geq 0}$. This filtration is multiplicative, in the sense that $I^k \cdot I^\ell \subset I^{k+\ell}$. The corresponding completion,

$$(1) \quad \widehat{\mathbb{k}G} = \varprojlim_k \mathbb{k}G/I^k,$$

comes equipped with the inverse limit filtration, $\{\widehat{I}^k\}_{k \geq 0}$. The multiplication in $\mathbb{k}G$ extends to a multiplication in $\widehat{\mathbb{k}G}$, compatible with this filtration.

On the other hand, the associated graded group,

$$(2) \quad \text{gr}(\mathbb{k}G) = \bigoplus_{k \geq 0} I^k/I^{k+1},$$

is a graded algebra, with multiplication inherited from the product in $\mathbb{k}G$. This algebra comes endowed with the degree filtration, $\mathcal{F}_k(\mathrm{gr}(\mathbb{k}G)) = \bigoplus_{j \geq k} I^j/I^{j+1}$. The completion of $\mathrm{gr}(\mathbb{k}G)$ with respect to this filtration,

$$(3) \quad \widehat{\mathrm{gr}}(\mathbb{k}G) = \prod_{k \geq 0} I^k/I^{k+1},$$

comes endowed with the inverse limit filtration,

$$(4) \quad \widehat{\mathcal{F}}_k(\widehat{\mathrm{gr}}(\mathbb{k}G)) = \prod_{j \geq k} I^j/I^{j+1}.$$

The associated graded algebra of $\widehat{\mathrm{gr}}(\mathbb{k}G)$ is canonically identified with $\mathrm{gr}(\mathbb{k}G)$.

For example, if $G = F_n$ is a free group of rank $n \geq 1$, then $\mathrm{gr}(\mathbb{k}G)$ is the tensor \mathbb{k} -algebra on n generators t_i while the completion $\widehat{\mathrm{gr}}(\mathbb{k}G)$ is the power series ring in n non-commuting variables $x_i = t_i - 1$.

2.2. Hopf algebras. A *Hopf algebra* is an associative and coassociative bialgebra over a field \mathbb{k} , with multiplication $\nabla: A \otimes A \rightarrow A$, comultiplication $\Delta: A \rightarrow A \otimes A$, unit $\eta: \mathbb{k} \rightarrow A$, and counit $\varepsilon: A \rightarrow \mathbb{k}$, endowed with a \mathbb{k} -linear map $T: A \rightarrow A$ (called the antipode), such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & A \otimes A & \xrightarrow{T \otimes \mathrm{id}} & A \otimes A & & \\
 & \Delta \nearrow & & & & \searrow \nabla & \\
 A & & & \xrightarrow{\varepsilon} & \mathbb{k} & \xrightarrow{\eta} & A \\
 & \Delta \searrow & & & & \nearrow \nabla & \\
 & & A \otimes A & \xrightarrow{\mathrm{id} \otimes T} & A \otimes A & &
 \end{array}$$

An element $x \in A$ is called *group-like* if $\Delta(x) = x \otimes x$, and it is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. The set of group-like elements of A form a group, with multiplication inherited from A and inverse given by the antipode, while the set of primitive elements of A form a Lie algebra, with Lie bracket $[x, y] = \nabla(x, y) - \nabla(y, x)$.

For instance, if \mathfrak{g} is a Lie algebra, then its universal enveloping algebra, $U(\mathfrak{g})$, is a Hopf algebra, with $\Delta x = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $T(x) = -x$ for all $x \in \mathfrak{g}$. By construction, the set of primitive elements in $U(\mathfrak{g})$ coincides with \mathfrak{g} . Suppose now that $\mathfrak{g} \cong \mathbb{k}^n$, with Lie bracket equal to 0. We may then identify $U(\mathfrak{g})$ with the polynomial ring $\mathbb{k}[x_1, \dots, x_n]$. Likewise, if $\widehat{U}(\mathfrak{g})$ denotes the completion of $U(\mathfrak{g})$ with respect to the filtration by powers of the augmentation ideal $J = \ker(\varepsilon)$, we may then identify $\widehat{U}(\mathfrak{g})$ with the power series ring $\mathbb{k}[[x_1, \dots, x_n]]$.

From now on, we will assume that \mathbb{k} is a field of characteristic 0. As is well-known, the group algebra $\mathbb{k}G$ of a group G is a Hopf algebra, with comultiplication $\Delta: \mathbb{k}G \rightarrow \mathbb{k}G \otimes \mathbb{k}G$ given by $\Delta(g) = g \otimes g$ for $g \in G$, counit $\varepsilon: \mathbb{k}G \rightarrow \mathbb{k}$ the augmentation map, and antipode $T: \mathbb{k}G \rightarrow \mathbb{k}G$ given by $T(g) = g^{-1}$. In [45], Quillen showed that the I -adic completion of the group algebra, $\widehat{\mathbb{k}G}$, is a complete Hopf algebra, with comultiplication map

$$(5) \quad \widehat{\Delta}: \widehat{\mathbb{k}G} \longrightarrow \widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G}.$$

where $\widehat{\otimes}$ denotes the completed tensor product, defined in this case as $\widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G} = \varprojlim_k \mathbb{k}G/I^k \otimes \mathbb{k}G/I^k$. Identifying the associated graded algebra $\mathrm{gr}(\mathbb{k}G \otimes \mathbb{k}G)$ with $\mathrm{gr}(\mathbb{k}G) \otimes \mathrm{gr}(\mathbb{k}G)$, we see that the degree

completion $\widehat{\text{gr}}(\mathbb{k}G)$ is also a complete Hopf algebra, with comultiplication map

$$(6) \quad \bar{\Delta} := \widehat{\text{gr}}(\Delta): \widehat{\text{gr}}(\mathbb{k}G) \longrightarrow \widehat{\text{gr}}(\mathbb{k}G) \hat{\otimes} \widehat{\text{gr}}(\mathbb{k}G) .$$

2.3. Multiplicative expansions and Taylor expansions. Given a map $f: G \rightarrow R$, where R is a ring, we will denote by $\bar{f}: \mathbb{k}G \rightarrow R$ its linear extension to the group algebra.

Definition 2.1. A (multiplicative) expansion of a group G is a map

$$(7) \quad E: G \longrightarrow \widehat{\text{gr}}(\mathbb{k}G)$$

such that the linear extension $\bar{E}: \mathbb{k}G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ is a filtration-preserving algebra morphism with the property that $\text{gr}(\bar{E}) = \text{id}$. Furthermore, we say that the expansion E is *faithful* if E is injective.

Alternatively, an expansion of G is a (multiplicative) monoid map $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ such that the following property holds: If $f \in I^k \setminus I^{k+1}$, then $\bar{E}(f)$ starts with $[f] \in I^k/I^{k+1}$, that is, $\bar{E}(f) = (0, \dots, 0, [f], *, *, \dots)$.

Following Bar-Natan [4], we make the following definition.

Definition 2.2. An expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ is called a *Taylor expansion* (or, a *group-like expansion*) if it sends all elements of G to group-like elements of $\widehat{\text{gr}}(\mathbb{k}G)$, that is,

$$(8) \quad \bar{\Delta}(E(g)) = E(g) \hat{\otimes} E(g)$$

for all $g \in G$.

Equivalently, an expansion E is a Taylor expansion if it is *co-multiplicative*, i.e., the following diagram commutes:

$$(9) \quad \begin{array}{ccc} \mathbb{k}G & \xrightarrow{\Delta} & \mathbb{k}G \otimes \mathbb{k}G \\ \downarrow \bar{E} & & \downarrow \bar{E} \hat{\otimes} \bar{E} \\ \widehat{\text{gr}}(\mathbb{k}G) & \xrightarrow{\bar{\Delta}} & \widehat{\text{gr}}(\mathbb{k}G) \hat{\otimes} \widehat{\text{gr}}(\mathbb{k}G). \end{array}$$

Proposition 2.3. A Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ induces a filtration-preserving isomorphism of complete Hopf algebras, $\widehat{E}: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, such that $\text{gr}(\widehat{E})$ is the identity on $\text{gr}(\mathbb{k}G)$.

Proof. As in the above definition, the expansion E induces a filtration-preserving algebra morphism, $\bar{E}: \mathbb{k}G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$. Applying the I -adic completion functor, we obtain an algebra morphism, $\widehat{E}: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$. By the above discussion, the expansion E is group-like if and only if \bar{E} is co-multiplicative. Applying the completion functor to diagram (9) yields another commuting diagram,

$$(10) \quad \begin{array}{ccc} \widehat{\mathbb{k}G} & \xrightarrow{\widehat{\Delta}} & \widehat{\mathbb{k}G} \hat{\otimes} \widehat{\mathbb{k}G} \\ \downarrow \widehat{E} & & \downarrow \widehat{E} \hat{\otimes} \widehat{E} \\ \widehat{\text{gr}}(\mathbb{k}G) & \xrightarrow{\bar{\Delta}} & \widehat{\text{gr}}(\mathbb{k}G) \hat{\otimes} \widehat{\text{gr}}(\mathbb{k}G). \end{array}$$

Since \bar{E} is filtration-preserving and $\text{gr}(\bar{E}) = \text{id}$, this implies that the Hopf algebra morphism \widehat{E} preserves filtrations and that $\text{gr}(\widehat{E}) = \text{id}$. By induction on k , all induced maps $\widehat{\mathbb{k}G}/\widehat{I}^k \rightarrow \widehat{\text{gr}}(\mathbb{k}G)/\widehat{\mathcal{F}}_k$ are isomorphisms, where $\widehat{\mathcal{F}}_k$ is the filtration from display (4). It follows from the next lemma that \widehat{E} is an isomorphism. \square

Lemma 2.4. *Let $f: A \rightarrow B$ be a morphism of filtered, complete, and separated algebras. If $\text{gr}(f): \text{gr}^{\mathcal{F}}(A) \rightarrow \text{gr}^{\mathcal{G}}(B)$ is an isomorphism, then f is also an isomorphism.*

Proof. By assumption, the homomorphisms $\text{gr}_k(f): \mathcal{F}_k A / \mathcal{F}_{k+1} A \rightarrow \mathcal{G}_k B / \mathcal{G}_{k+1} B$ are isomorphisms, for all $k \geq 1$. An easy induction on k shows that all maps $f_k: A / \mathcal{F}_{k+1} A \rightarrow B / \mathcal{G}_{k+1} B$ are isomorphisms. Therefore, the map $\hat{f}: \widehat{A} \rightarrow \widehat{B}$ is an isomorphism. On the other hand, both A and B are complete and separated, and so $A = \widehat{A}$ and $B = \widehat{B}$. Hence $f = \hat{f}$, and we are done. \square

2.4. On the existence of Taylor expansions. As we shall see, not all finitely generated groups admit a Taylor expansion. We conclude this section with an if-and-only-if criterion for the existence of a such expansion.

Proposition 2.5. *A filtration-preserving isomorphism of complete Hopf algebras, $\phi: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, induces a Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$.*

Proof. The isomorphism ϕ induces a filtration-preserving isomorphism of complete Hopf algebras, $\tilde{\phi} := (\widehat{\text{gr}}(\phi))^{-1} \circ \phi$, from $\widehat{\mathbb{k}G}$ to $\widehat{\text{gr}}(\mathbb{k}G)$, such that $\text{gr}(\tilde{\phi}) = \text{id}$. Let $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ be the composite

$$(11) \quad G \hookrightarrow \mathbb{k}G \xrightarrow{j} \widehat{\mathbb{k}G} \xrightarrow{\tilde{\phi}} \widehat{\text{gr}}(\mathbb{k}G).$$

Since both $\tilde{\phi}$ and j are morphisms of Hopf algebras, and since the inclusion $G \hookrightarrow \mathbb{k}G$ is a monoid map sending G to the group-like elements of $\mathbb{k}G$, the composite E is also a monoid map. It is clear that $\widehat{E} = \tilde{\phi}$ and $\bar{E} = \tilde{\phi} \circ j$. Since both $\tilde{\phi}$ and j are filtration-preserving, and $\text{gr}(j) = \text{gr}(\bar{E}) = \text{id}$, we infer that \bar{E} is filtration-preserving and $\text{gr}(\bar{E}) = \text{id}$. Finally, by construction, E is a group-like expansion. \square

Propositions 2.3 and 2.5 can be summarized as follows.

Theorem 2.6. *The assignment $E \rightsquigarrow \widehat{E}$ establishes a one-to-one correspondence between Taylor expansions $G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ and filtration-preserving isomorphisms of complete Hopf algebras $\widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ for which the associated graded morphism is the identity on $\text{gr}(\mathbb{k}G)$.*

This theorem generalizes a result of Massuyeau ([34, Proposition 2.10]), from finitely generated free groups to arbitrary finitely generated groups. Proposition 2.5 and Theorem 2.6 have as an immediate corollary the aforementioned criterion for the existence of a Taylor expansion.

Corollary 2.7. *A finitely generated group G has a Taylor expansion if and only if there is an isomorphism of filtered Hopf algebras, $\widehat{\mathbb{k}G} \cong \widehat{\text{gr}}(\mathbb{k}G)$.*

3. CHEN ITERATED INTEGRALS AND TAYLOR EXPANSIONS

3.1. Chen iterated integrals. In [11, 12], Chen developed a theory of formal power series connections and iterated integrals on smooth manifolds. His original motivation was to describe the homology of the loop space of a smooth manifold M in terms of the differential graded algebra formed by tensoring the de Rham algebra $\Omega_{\text{DR}}(M)$ with the tensor algebra on the vector space $H_{>0}(M, \mathbb{R})$, completed with respect to the powers of the augmentation ideal. As summarized below, Chen's theory leads to monodromy representations of the fundamental group of M (see also Lin [29] and Kohno [26] for further details).

For simplicity, we will assume the manifold M has the homotopy type of a connected, finite-type CW-complex. Upon choosing a basis $\mathbf{X} = \{X_i\}_i$ for $\widetilde{H}_*(M, \mathbb{k})$, we may identify the algebra

$\Omega_{\text{DR}}(M) \otimes_{\mathbb{k}} \widehat{T}(\widetilde{H}_*(M, \mathbb{k}))$ with $\Omega_{\text{DR}}(M)\langle\langle \mathbf{X} \rangle\rangle$. (Here, $\mathbb{k} = \mathbb{R}$ or \mathbb{C} .) A *formal power series connection* on M is an element $\omega \in \Omega_{\text{DR}}(M)\langle\langle \mathbf{X} \rangle\rangle$. We may write such an element (which may also be viewed as a usual connection on the trivial bundle $M \times \mathbb{k}\langle\langle \mathbf{X} \rangle\rangle$) as

$$(12) \quad \omega = \sum w_i X_i + \cdots + \sum w_{i_1 \dots i_r} X_{i_1} \cdots X_{i_r} + \cdots,$$

where the coefficients are smooth forms of positive degree on M . A connection ω as above is said to be *flat* if it satisfies the Maurer–Cartan equation, $d\omega - \omega \wedge \omega = 0$.

For a homology class $X \in \widetilde{H}_p(M, \mathbb{k})$ we set $\deg X = p-1$; more generally, we set $\deg(X_{i_1} \cdots X_{i_r}) := \deg X_{i_1} + \cdots + \deg X_{i_r}$. We denote by $\omega_0 \in \Omega_{\text{DR}}^1(M) \otimes_{\mathbb{k}} \widehat{T}(H_1(M, \mathbb{k}))$ the degree 0 part of ω .

Now let $G = \pi_1(M, x_0)$ and suppose $\widehat{\text{gr}}(\mathbb{k}G)$ admits a presentation of the form $\widehat{T}(H_1(M, \mathbb{k}))/I$, for some closed Hopf ideal I in the completed tensor algebra on $H_1(M, \mathbb{k})$. If the connection ω_0 is flat modulo the relations in I , the corresponding holonomy homomorphism, $J: G \rightarrow \widehat{T}(H_1(M, \mathbb{k}))/I$, may be defined by means of iterated integrals, as follows:

$$(13) \quad \begin{aligned} J(g) &= 1 + \sum_{k=1}^{\infty} \int_{0 \leq t_1 \leq \dots \leq t_k \leq 1} \omega_0(\dot{\gamma}(t_k)) \wedge \cdots \wedge \omega_0(\dot{\gamma}(t_1)) \\ &= 1 + \sum_{k=1}^{\infty} \int_0^1 \omega_0(\dot{\gamma}(t_k)) \cdots \int_0^{t_3} \omega_0(\dot{\gamma}(t_2)) \int_0^{t_2} \omega_0(\dot{\gamma}(t_1)), \end{aligned}$$

where $g \in G$ is represented by a piecewise smooth loop $\gamma: [0, 1] \rightarrow M$ at x_0 . As shown in [10] (see also [29, 26]), the holonomy homomorphism $J: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ is multiplicative and maps G to group-like elements in $\widehat{\text{gr}}(\mathbb{k}G)$; thus, J is a Taylor expansion for G .

3.2. Expansions of free groups. Let F_n be a finitely-generated free group on generators x_1, \dots, x_n . The complete Hopf algebra $\widehat{\text{gr}}(\mathbb{k}F_n)$ can be identified with $\mathbb{k}\langle\langle \mathbf{X} \rangle\rangle = \mathbb{k}\langle\langle X_1, \dots, X_n \rangle\rangle$, the power series ring over \mathbb{k} in n non-commuting variables. There are three well-known expansions of this group.

- (1) The first one is the Magnus expansion, $M: F_n \rightarrow \mathbb{k}\langle\langle \mathbf{X} \rangle\rangle$, given by $M(x_i) = 1 + X_i$, see [32]. This expansion is multiplicative but not co-multiplicative if $n > 1$; thus, it is not a Taylor expansion.
- (2) The second one is the power series expansion, $L: F_n \rightarrow \mathbb{k}\langle\langle \mathbf{X} \rangle\rangle$, given by $L(x_i) = \exp(X_i)$. As shown by Lin in [29], this is a Taylor expansion.
- (3) The third type of expansion arises from the construction outlined in §3.1, with $\mathbb{k} = \mathbb{C}$. Let $C_n = \mathbb{C} \setminus \{1, \dots, n\}$ be the complex plane \mathbb{C} with n punctures, so that $F_n = \pi_1(C_n, 0)$. Let $w_i = \frac{1}{2\pi\sqrt{-1}} \cdot \frac{dz}{z-i}$ be closed 1-forms on C_n dual to the cycles x_i . Then $\omega = \sum_{i=1}^n w_i X_i$ is a degree 0 flat connection on the trivial bundle $C_n \times \mathbb{C}\langle\langle \mathbf{X} \rangle\rangle \rightarrow C_n$. The corresponding monodromy representation, $J: F_n \rightarrow \mathbb{C}\langle\langle \mathbf{X} \rangle\rangle$, is given by

$$(14) \quad J(f) = 1 + \sum_{k=1}^{\infty} \sum_{0 \leq i_1, \dots, i_k \leq n} \left(\frac{1}{2\pi\sqrt{-1}} \right)^k \int_{0 \leq t_1 \leq \dots \leq t_k \leq 1} \bigwedge_{r=1}^k \frac{d\gamma(t_r)}{\gamma(t_r) - i_{i_r}} X_{i_1} \cdots X_{i_k},$$

where $f \in F_n$ is represented by a piecewise smooth loop $\gamma: [0, 1] \rightarrow C_n$ at 0. This gives another Taylor expansion over \mathbb{C} for the free group F_n .

3.3. Expansions of free abelian groups. Let \mathbb{Z}^n be the free abelian group of rank $n > 0$. This group admits a presentation of the form $\mathbb{Z}^n = F_n/N$, where N is the normal subgroup of F_n generated by the commutators $[x_i, x_j] := x_i x_j x_i^{-1} x_j^{-1}$ for $1 \leq i < j \leq n$.

The complete Hopf algebra $\widehat{\text{gr}}(\mathbb{k}\mathbb{Z}^n)$ may be identified with $\mathbb{k}[\mathbf{X}] = \mathbb{k}[X_1, \dots, X_n]$, the power series ring over \mathbb{k} in n commuting variables. The power series expansion of the free group F_n induces a Taylor expansion of the free abelian group \mathbb{Z}^n ; this expansion, $L: \mathbb{Z}^n \rightarrow \mathbb{k}[\mathbf{X}]$, is given by $L(x_i) = \exp(X_i)$.

3.4. Taylor expansions for surface groups. Let $G = \pi_1(S_g) = F_{2g}/\langle r \rangle$ be the fundamental group of a compact, connected, orientable surface of genus $g \geq 1$. Such a group has a presentation with generators x_i, y_i for $i = 1, \dots, g$ and a single relator $r = \sum_{i=1}^g [x_i, y_i]$. It is well-known that G is 1-formal. In particular, there is a Taylor expansion $G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, for any field \mathbb{k} of characteristic 0. Here, the complete Hopf algebra $\widehat{\text{gr}}(\mathbb{k}G)$ is generated by X_i, Y_i for $i = 1, \dots, g$, and subjects to a relation $\sum_{i=1}^g [X_i, Y_i] = 0$. However, actually constructing such an expansion is not an easy task.

Using Chen's theory of iterated integrals, Lin constructed in [29] an explicit Taylor expansion over $\mathbb{k} = \mathbb{C}$ for the group $G = \pi_1(S_g)$. Let α_i, β_i be closed 1-forms dual to x_i, y_i , respectively. Set $\omega = \sum_{r=1}^{\infty} \omega^{(r)}$, where $\omega^{(1)} = \sum_{i=1}^g \alpha_i X_i + \sum_{i=1}^g \beta_i Y_i$, and $\omega^{(r)}$ is the homogeneous polynomial of degree r defined inductively by solving the equation $d\omega - \omega \wedge \omega = 0$. Then ω is a flat formal power series connection on S_g . The corresponding expansion, $J: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, is defined by means of the iterated integral (13). By Theorem 6.3(2), there exists a rational Taylor expansion for G .

Recently, Massuyeau [34] constructed rational Taylor expansions for the surface groups $G = \pi_1(S_g)$ by suitably deforming the power series expansion of the free groups F_{2g} .

4. LOWER CENTRAL SERIES AND HOLONOMY LIE ALGEBRAS

4.1. Associated graded Lie algebras. Let G be a group. The *lower central series* of G is the sequence of subgroups $\{\Gamma_k G\}_{k \geq 1}$ defined inductively by $\Gamma_1 G = G$ and

$$(15) \quad \Gamma_{k+1} G = [\Gamma_k G, G]$$

for $k \geq 1$. Here, for any subgroups H and K of G , we denote $[H, K]$ the subgroup of G generated by all group commutators $[h, g] := hgh^{-1}g^{-1}$ with $h \in H$ and $g \in K$. In particular, $\Gamma_2 G$ equals G' , the commutator subgroup of G . Clearly, each term in the LCS series is a normal subgroup (in fact, a characteristic subgroup) of G . Moreover, $\Gamma_{k+1} G$ contains the commutator subgroup of $\Gamma_k G$, and so the quotient group, $\text{gr}_k(G) := \Gamma_k G / \Gamma_{k+1} G$, is abelian.

Let us fix a coefficient field \mathbb{k} of characteristic 0. The associated graded Lie algebra of G over \mathbb{k} is defined by

$$(16) \quad \text{gr}(G, \mathbb{k}) := \bigoplus_{k \geq 1} \text{gr}_k(G) \otimes \mathbb{k},$$

with the Lie bracket induced by the group commutator. This construction is functorial: if $\varphi: G \rightarrow H$ is a group homomorphism, then φ preserves the respective lower central series, and so it induces a morphism of graded Lie algebras, $\text{gr}(\varphi, \mathbb{k}): \text{gr}(G, \mathbb{k}) \rightarrow \text{gr}(H, \mathbb{k})$.

Assume now that G is a finitely generated group. Then each LCS quotient $\text{gr}_k(G)$ is a finitely generated abelian group. Furthermore, $\text{gr}(G, \mathbb{k})$ is a finitely generated graded Lie algebra, that can be presented as $\text{gr}(G, \mathbb{k}) = \text{lie}(V)/\mathfrak{r}$, where $\text{lie}(V)$ is the free Lie algebra on a finite-dimensional \mathbb{k} -vector space V (with non-zero elements in degree 1), and \mathfrak{r} is a homogeneous Lie ideal. We let $\phi_k(G) := \dim \text{gr}_k(G, \mathbb{k})$ be the LCS ranks of G .

4.2. Chen Lie algebras. Another descending series associated to a group G is the *derived series*, starting at $G^{(0)} = G$, $G^{(1)} = G'$, and $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Note that any homomorphism $G \rightarrow H$ takes $G^{(i)}$ to $H^{(i)}$. The quotient groups, $G/G^{(i)}$, are solvable; in particular, $G/G' = G_{\text{ab}}$, while G/G'' is the maximal metabelian quotient of G .

Assume now that G is finitely generated. For each $i \geq 2$, the *i -th Chen Lie algebra* of G is defined to be the associated graded Lie algebra of the corresponding solvable quotient,

$$(17) \quad \text{gr}(G/G^{(i)}, \mathbb{k}).$$

Clearly, this construction is functorial. The quotient map, $p_i: G \twoheadrightarrow G/G^{(i)}$, induces a surjective morphism $\text{gr}(p_i)$ between associated graded Lie algebras $\text{gr}_k(G, \mathbb{k})$ and $\text{gr}_k(G/G^{(i)}, \mathbb{k})$. Plainly, this morphism is the canonical identification in degree 1. In fact, the map $\text{gr}(p_i)$ is an isomorphism for each $k \leq 2^i - 1$, see [48].

We now specialize to the case when $i = 2$, originally studied by K.-T. Chen in [9]. The *Chen ranks* of G are defined as $\theta_k(G) := \dim_{\mathbb{k}}(\text{gr}_k(G/G'', \mathbb{k}))$. By the above remarks, $\phi_k(G) \geq \theta_k(G)$, with equality for $k \leq 3$.

4.3. Holonomy Lie algebras. Once again, let G be a finitely generated group. Write $V = H_1(G, \mathbb{k})$ and let $\mu_G^\vee: H_2(G, \mathbb{k}) \rightarrow V \wedge V$ be the dual of the cup product map $\mu_G: H^1(G, \mathbb{k}) \wedge H^1(G, \mathbb{k}) \rightarrow H^2(G, \mathbb{k})$. The *holonomy Lie algebra* of G is the quadratic Lie algebra defined as

$$(18) \quad \mathfrak{h}(G, \mathbb{k}) = \text{lie}(V) / \langle \text{im } \mu_G^\vee \rangle.$$

Clearly, this construction is functorial. Furthermore, there is a natural surjective morphism of graded Lie algebras,

$$(19) \quad \psi_G: \mathfrak{h}(G, \mathbb{k}) \twoheadrightarrow \text{gr}(G, \mathbb{k}),$$

inducing isomorphisms in degree 1 and 2. (See [48, Lemma 6.1] and references therein.) If the map ψ_G is an isomorphism, then we say that the group G is *graded-formal* (over \mathbb{k}).

4.4. Free groups and surface groups. We conclude this section with some simple examples illustrating the above concepts.

Example 4.1. Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n . Then $\text{gr}(F_n, \mathbb{k}) = \text{lie}(x_1, \dots, x_n)$, the free Lie algebra on n generators, and the map $\psi: \mathfrak{h}(F_n, \mathbb{k}) \rightarrow \text{gr}(F_n, \mathbb{k})$ is an isomorphism. Moreover, as shown by Witt [52] and Magnus [31], the LCS ranks are given by

$$(20) \quad \prod_{k \geq 1} (1 - t^k)^{\phi_k(F_n)} = 1 - nt,$$

or, equivalently, $\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d}$, where μ denotes the Möbius function. Finally, as shown in [9], the Chen ranks of the free groups are given by $\theta_1(F_n) = n$ and

$$(21) \quad \theta_k(F_n) = (k-1) \binom{n+k-2}{k} \quad \text{for } k \geq 2.$$

Example 4.2. Let S_g be a closed, orientable surface of genus $g \geq 1$. Its fundamental group, $\Pi_g = \pi_1(S_g)$, has a presentation with generators $x_1, y_1, \dots, x_g, y_g$ and a single relator, $[x_1, y_1] \cdots [x_g, y_g]$. As shown by Labute [27], $\text{gr}(\Pi_g, \mathbb{k}) = \text{lie}(x_1, y_1, \dots, x_g, y_g) / \langle \sum_{i=1}^g [x_i, y_i] \rangle$. Again, it is readily seen that $\mathfrak{h}(\Pi_g, \mathbb{k}) \cong \text{gr}(\Pi_g, \mathbb{k})$. Furthermore, the LCS ranks of Π_g are given by

$$(22) \quad \prod_{k \geq 1} (1 - t^k)^{\phi_k(\Pi_g)} = 1 - 2gt + t^2,$$

while the Chen ranks are given by $\theta_1(\Pi_g) = 2g$, $\theta_2(\Pi_g) = 2g^2 - g - 1$, and

$$(23) \quad \theta_k(\Pi_g) = (k-1) \binom{2g+k-2}{k} - \binom{2g+k-3}{k-2} \quad \text{for } k \geq 3.$$

5. MALCEV LIE ALGEBRAS AND FORMALITY PROPERTIES

5.1. Malcev Lie algebras. As before, let G be a finitely generated group, let \mathbb{k} be a field of characteristic 0, and let $\widehat{\mathbb{k}G}$ be the I -adic completion of the group algebra of G , where I is the augmentation ideal of $\mathbb{k}G$. Following Quillen [45], we define the *Malcev Lie algebra* of G as the set $\mathfrak{m}(G, \mathbb{k})$ of all primitive elements in $\widehat{\mathbb{k}G}$, with bracket $[x, y] = xy - yx$. By construction, $\mathfrak{m}(G, \mathbb{k})$ is a complete, filtered Lie algebra. Moreover, if we complete the universal enveloping algebra $U(\mathfrak{m}(G, \mathbb{k}))$ with respect to the powers of its augmentation ideal, then $\widehat{U}(\mathfrak{m}(G, \mathbb{k})) \cong \widehat{\mathbb{k}G}$, as complete Hopf algebras.

The set of all primitive elements in $\text{gr}(\widehat{\mathbb{k}G})$ forms a graded Lie algebra, which is isomorphic to $\text{gr}(G, \mathbb{k})$. An important connection between the Malcev Lie algebra $\mathfrak{m}(G, \mathbb{k})$ and the associated graded Lie algebra $\text{gr}(G; \mathbb{k})$ was discovered by Quillen, who showed in [44] that there is an isomorphism of graded Lie algebras,

$$(24) \quad \text{gr}(\mathfrak{m}(G, \mathbb{k})) \cong \text{gr}(G, \mathbb{k}).$$

The set of all group-like elements in $\widehat{\mathbb{k}G}$ forms a group, denoted $\mathfrak{M}(G; \mathbb{k})$. This group comes endowed with a complete, separated filtration, whose k -th term is $\mathfrak{M}(G; \mathbb{k}) \cap (1 + \widehat{I}^k)$. As explained for instance in [34], there is a one-to-one, filtration-preserving correspondence between primitive elements and group-like elements of $\widehat{\mathbb{k}G}$ via the exponential and logarithmic maps

$$(25) \quad \mathfrak{M}(G; \mathbb{k}) \subset 1 + \widehat{I} \begin{array}{c} \xleftarrow{\text{exp}} \\ \xrightarrow{\text{log}} \end{array} \widehat{I} \supset \mathfrak{m}(G; \mathbb{k}).$$

Let G be a group which admits a finite presentation of the form $G = F/R$. Using a Taylor expansion for the finitely generated free group F , we may find a presentation for the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$, using the approach of Papadima [39] and Massuyeau [34], which may be summarized in the following theorem.

Theorem 5.1 ([34, 39]). *Let G be a group with generators x_1, \dots, x_n and relators r_1, \dots, r_m . Let E be a Taylor expansion of the free group $F = \langle x_1, \dots, x_n \rangle$. There exists then a unique filtered Lie algebra isomorphism*

$$\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{lie}}(\mathbb{k}^n) / \langle\langle W \rangle\rangle,$$

where $\langle\langle W \rangle\rangle$ denotes the closed ideal of the completed free Lie algebra $\widehat{\text{lie}}(\mathbb{k}^n)$ generated by the subset $\{\log(E(r_1)), \dots, \log(E(r_m))\}$.

5.2. Formality and filtered-formality. The notion of formality first appeared in the study of rational homotopy types of topological spaces initiated by Sullivan [51, 17]. Since then, it has been broadly used in investigating a variety of differential graded objects. We recall now a formality notion introduced in [49].

Definition 5.2. A finitely generated group G is called *filtered-formal* (over \mathbb{k}), if there is a filtered Lie algebra isomorphism from the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ to the degree completion $\widehat{\text{gr}}(G; \mathbb{k})$ inducing the identity on associated graded Lie algebras.

As shown in [49, Lemma 2.4], the following holds: if $\mathfrak{m}(G; \mathbb{k})$ is isomorphic (as a filtered Lie algebras) to the degree completion of a graded Lie algebra \mathfrak{g} , then the group G is filtered-formal (over \mathbb{k}). The notion of filtered-formality satisfies the following propagation properties.

Theorem 5.3 ([49]). *Let G be a finitely generated group.*

- (1) *Suppose there is a split monomorphism $\iota: K \rightarrow G$. If G is filtered-formal, then K is also filtered-formal.*
- (2) *The group G is filtered-formal over a field \mathbb{k} of characteristic 0 if and only if G is filtered-formal over \mathbb{Q} .*
- (3) *G_1 and G_2 are filtered-formal if and only if $G_1 * G_2$ is filtered-formal if and only if $G_1 \times G_2$ is filtered-formal.*

Proof. This theorem is a combination of the following results from [49]: Theorem 5.11 for (1); Theorem 6.6 for (2); Theorem 7.17 for (3). \square

In particular, if a finitely generated group G is filtered-formal over \mathbb{C} , then it is also filtered-formal over \mathbb{Q} .

A finitely generated group G is said to be *1-formal* (over \mathbb{k}) if $\mathfrak{m}(G, \mathbb{k}) \cong \widehat{\mathfrak{h}}(G, \mathbb{k})$ as filtered Lie algebras. It is readily seen that G is 1-formal if and only if it is graded-formal and filtered-formal.

5.3. Chen Lie algebras and formality. The next theorem is the Lie algebra version of Theorem 3.5 from [41], which describes the relationship between the Malcev Lie algebras of the derived quotients of a group G and the corresponding quotients of the Malcev Lie algebra of G .

Theorem 5.4 ([41]). *Let G be a finitely generated group. There is an isomorphism of complete, separated filtered Lie algebras,*

$$\mathfrak{m}(G/G^{(i)}; \mathbb{k}) \cong \mathfrak{m}(G; \mathbb{k}) / \overline{\mathfrak{m}(G; \mathbb{k})^{(i)}},$$

for each $i \geq 2$, where $\overline{\mathfrak{m}(G; \mathbb{k})^{(i)}}$ is the closure of $\mathfrak{m}(G; \mathbb{k})^{(i)}$ with respect to the filtration topology on $\mathfrak{m}(G; \mathbb{k})$.

One important application of Theorem 5.4 is the next theorem, which delineates the relationship between associated graded Lie algebras of derived quotients and derived quotients of associated graded Lie algebras. This theorem also shows that filtered-formality is preserved under the operation of taking derived quotients.

Theorem 5.5 ([49]). *The quotient map $p_i: G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded \mathbb{k} -Lie algebras,*

$$\Psi_G^{(i)}: \text{gr}(G; \mathbb{k}) / \text{gr}(G; \mathbb{k})^{(i)} \twoheadrightarrow \text{gr}(G/G^{(i)}; \mathbb{k}),$$

for each $i \geq 2$. Moreover, if the group G is filtered-formal, then $\Psi_G^{(i)}$ is an isomorphism and the derived quotient $G/G^{(i)}$ is filtered-formal.

5.4. filtered-formality and Chen Lie algebras. As mentioned previously, any homomorphism $G_1 \rightarrow G_2$ induces morphisms of graded Lie algebras, $\text{gr}(G_1; \mathbb{k}) \rightarrow \text{gr}(G_2; \mathbb{k})$ and $\text{gr}(G_1/G_1^{(i)}; \mathbb{k}) \rightarrow \text{gr}(G_2/G_2^{(i)}; \mathbb{k})$. On the other hand, it is not *a priori* clear that a morphism $\text{gr}(G_1; \mathbb{k}) \rightarrow \text{gr}(G_2; \mathbb{k})$ should induce morphisms between the corresponding Chen Lie algebras. Nevertheless, as the next theorem shows, this happens for filtered-formal groups.

Theorem 5.6. *Let G_1 and G_2 be two \mathbb{k} -filtered-formal groups. Then every morphism of graded Lie algebras, $\alpha: \text{gr}(G_1; \mathbb{k}) \rightarrow \text{gr}(G_2; \mathbb{k})$, induces morphisms $\alpha_i: \text{gr}(G_1/G_1^{(i)}; \mathbb{k}) \rightarrow \text{gr}(G_2/G_2^{(i)}; \mathbb{k})$ for all $i \geq 1$. Consequently, if $\text{gr}(G_1; \mathbb{k}) \cong \text{gr}(G_2; \mathbb{k})$, then $\text{gr}(G_1/G_1^{(i)}; \mathbb{k}) \cong \text{gr}(G_2/G_2^{(i)}; \mathbb{k})$, for all i .*

Proof. Fix an index $i \geq 1$, and consider the following diagram of graded Lie algebras:

$$(26) \quad \begin{array}{ccccc} \text{gr}(G_1; \mathbb{k}) & \longrightarrow & \text{gr}(G_1; \mathbb{k}) / \text{gr}(G_1; \mathbb{k})^{(i)} & \xrightarrow{\Psi_{G_1}^{(i)}} & \text{gr}(G_1/G_1^{(i)}; \mathbb{k}) \\ \downarrow \alpha & & \downarrow \beta_i & & \downarrow \alpha_i \\ \text{gr}(G_2; \mathbb{k}) & \longrightarrow & \text{gr}(G_2; \mathbb{k}) / \text{gr}(G_2; \mathbb{k})^{(i)} & \xrightarrow{\Psi_{G_2}^{(i)}} & \text{gr}(G_2/G_2^{(i)}; \mathbb{k}) \end{array}$$

The morphism α induces a morphism β_i between the respective solvable quotients. By Theorem 5.5, the maps $\Psi_{G_1}^{(i)}$ and $\Psi_{G_2}^{(i)}$ are isomorphisms. We define the desired morphism α_i to be the composition $\Psi_{G_2}^{(i)} \circ \beta_i \circ (\Psi_{G_1}^{(i)})^{-1}$. The last claim follows at once. \square

Taking $i = 2$ in the above theorem, we obtain the following corollary.

Corollary 5.7. *Suppose G_1 and G_2 are two \mathbb{k} -filtered-formal groups. If $\theta_k(G_1) \neq \theta_k(G_2)$ for some $k \geq 1$, then $\text{gr}(G_1; \mathbb{k}) \not\cong \text{gr}(G_2; \mathbb{k})$, as graded Lie algebras.*

6. TAYLOR EXPANSIONS AND FORMALITY PROPERTIES

In this section we relate the notions of Taylor expansion and filtered-formality for a finitely generated group G .

6.1. Taylor expansions and isomorphisms of filtered Lie algebras. As the next theorem shows, Taylor expansions of G are intimately related to isomorphisms between the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ and the LCS completion of the associated graded Lie algebra $\text{gr}(G; \mathbb{k})$.

Theorem 6.1. *There is a one-to-one correspondence between Taylor expansions $G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ and filtration-preserving Lie algebra isomorphisms $\mathfrak{m}(G; \mathbb{k}) \rightarrow \widehat{\text{gr}}(G; \mathbb{k})$ inducing the identity on $\text{gr}(G; \mathbb{k})$.*

Proof. First suppose $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ is a Taylor expansion. Then, by Proposition 2.3, there is a filtration-preserving Hopf algebra isomorphism $\widehat{E}: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, inducing the identity on $\text{gr}(\mathbb{k}G)$. Recall that $\widehat{\mathbb{k}G} \cong U(\mathfrak{m}(G; \mathbb{k}))$ and $\widehat{\text{gr}}(\mathbb{k}G) \cong U(\widehat{\text{gr}}(G; \mathbb{k}))$, as filtered Hopf algebras. Taking primitives, we obtain a filtration-preserving isomorphism of complete Lie algebras, $\text{Prim}(\widehat{E}): \mathfrak{m}(G; \mathbb{k}) \rightarrow \widehat{\text{gr}}(G; \mathbb{k})$, inducing the identity on $\text{gr}(G; \mathbb{k})$.

Now suppose there is an isomorphism of filtered, complete Lie algebras, $\alpha: \mathfrak{m}(G; \mathbb{k}) \rightarrow \widehat{\text{gr}}(G; \mathbb{k})$, such that $\text{gr}(\alpha) = \text{id}$. Taking universal enveloping algebras, we obtain an isomorphism of filtered, complete Hopf algebras, $U(\alpha): \widehat{\mathbb{k}G} \xrightarrow{\cong} \widehat{\text{gr}}(\mathbb{k}G)$, such that $\text{gr}(\phi) = \text{id}$. By Proposition 2.5, the map $U(\alpha)$ induces a Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$. \square

Using this theorem, we obtain in Corollaries 6.2 and 6.4 alternate interpretations of filtered-formality and 1-formality.

Corollary 6.2. *A finitely generated group G has a Taylor expansion if and only if G is filtered-formal.*

Proof. Follows at once from Theorem 6.1 and Definition 5.2. \square

Theorem 6.3. *Let G be a finitely generated group.*

- (1) *Suppose there is a split monomorphism $\iota: K \rightarrow G$. If G has a Taylor expansion, then K also has a Taylor expansion.*
- (2) *The group G has a Taylor expansion over a field \mathbb{k} of characteristic 0 if and only if G has a Taylor expansion over \mathbb{Q} .*
- (3) *G_1 and G_2 have a Taylor expansion if and only if $G_1 * G_2$ has a Taylor expansion if and only if $G_1 \times G_2$ has a Taylor expansion.*
- (4) *If G has a Taylor expansion, then all the solvable quotients $G/G^{(i)}$ have a Taylor expansion.*

Proof. The first three claims follow from Corollary 6.2 and Theorem 5.3. Claim (4) follows from Corollary 6.2 and Theorem 5.5. \square

Corollary 6.4. *A finitely generated group G is 1-formal if and only if there is a Taylor expansion $G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ and $\text{gr}(\mathbb{k}G)$ is a quadratic algebra.*

Proof. We know that G is 1-formal if and only if G is filtered-formal and graded-formal. By Corollary 6.2, G is filtered-formal if and only if it has a Taylor expansion. On the other hand, G is graded-formal if and only if $\text{gr}(G; \mathbb{k})$ admits a quadratic presentation. As shown in [28, §2.2.3], this latter condition is equivalent to the quadraticity of $\text{gr}(\mathbb{k}G)$. This completes the proof. \square

Example 6.5. The reduced free group RF_n , introduced by J. Milnor in his study of link homotopy [37] is the quotient of the free group $F_n = \langle x_1, \dots, x_n \rangle$ by the normal subgroup generated by all elements of the form $[x_i, gx_i g^{-1}]$ with $g \in F_n$. The relations in RF_n can be reduced to multiple group commutators in x_1, \dots, x_n with some x_i appears at least twice. In [29], Lin showed that RF_n has Taylor expansions induced from certain expansions of the free group F_n (the power series expansion and the expansion arising from formal power series connections, as described in §3.2). It follows from Corollary 6.2 that the group RF_n is filtered-formal.

6.2. Taylor expansions of nilpotent groups. As before, let G be a finitely generated group. The next result shows that the Taylor expansions of G are inherited by the nilpotent quotients $G/\Gamma_i G$.

Theorem 6.6. *Suppose G admits a Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$. Then each nilpotent quotient $G/\Gamma_i G$ admits an induced Taylor expansion, $E_i: G/\Gamma_i G \rightarrow \widehat{\text{gr}}(\mathbb{k}[G/\Gamma_i G])$.*

Proof. By Theorem 6.1, the Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ determines a filtered Lie algebra isomorphism, $\alpha: \mathfrak{m}(G; \mathbb{k}) \rightarrow \widehat{\text{gr}}(G; \mathbb{k})$. From the proof of [49, Theorem 7.13], we deduce that α induces filtered Lie algebra isomorphisms, $\alpha_i: \mathfrak{m}(G/\Gamma_i G; \mathbb{k}) \rightarrow \widehat{\text{gr}}(G/\Gamma_i G; \mathbb{k})$. Using Theorem 6.1 again, we obtain the desired Taylor expansions, $E_i: G/\Gamma_i G \rightarrow \widehat{\text{gr}}(G/\Gamma_i G; \mathbb{k})$. \square

Example 6.7. As noted in §3.2, the finitely generated free group F admits Taylor expansions. By Theorem 6.6, the k -step, free nilpotent group $F/\Gamma_{k+1} F$ admits Taylor expansions for each $k \geq 1$.

Example 6.8. Let G be a finitely generated, torsion-free, 2-step nilpotent group, and suppose G_{ab} is also torsion-free. As shown in [49], the group G is filtered-formal. Thus, by Corollary 6.2, G admits a Taylor expansion.

Example 6.9. Let \mathfrak{m} be the 5-dimensional, nilpotent Lie algebra with non-zero Lie brackets given by $[e_1, e_3] = e_4$ and $[e_1, e_4] = [e_2, e_3] = e_5$. This Lie algebra may be realized as the Malcev Lie algebra of a finitely generated, torsion-free nilpotent group G . As noted in [16, 49], this group is not filtered-formal. Thus, the group G admits no Taylor expansion.

7. AUTOMORPHISMS OF FREE GROUPS AND ALMOST-DIRECT PRODUCTS

7.1. Braid groups. An automorphism of the free group $F_n = \langle x_1, \dots, x_n \rangle$ is a permutation-conjugacy automorphism if it sends each generator x_i to a conjugate of some other generator x_j . The Artin braid group B_n is the subgroup of $\text{Aut}(F_n)$ consisting of all permutation-conjugacy automorphisms which fix the product $x_1 \cdots x_n$. As shown for instance in [8], the group B_n is generated by the elementary braids $\sigma_1, \dots, \sigma_{n-1}$ (where σ_i sends x_i to x_{i+1} and x_{i+1} to $x_{i+1}^{-1}x_i x_{i+1}$ while fixing the other x_k 's), subject to the relations

$$(27) \quad \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & |i - j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \leq i \leq n - 2. \end{cases}$$

The pure braid group P_n is the kernel of the canonical projection $B_n \rightarrow S_n$ that sends a generator σ_i to the transposition $(i, i + 1)$. This group is generated by the braids

$$(28) \quad A_{ij} := (\sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1}) \sigma_i^2 (\sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1})^{-1}, \text{ for } 1 \leq i < j \leq n.$$

The pure braid group P_n decomposes as a semidirect product, $P_n = F_{n-1} \rtimes P_{n-1}$, where P_{n-1} acts on F_{n-1} by restriction of the Artin representation $B_{n-1} \subset \text{Aut}(F_{n-1})$. The group P_n is 1-formal. The associated graded Lie algebra $\text{gr}(P_n, \mathbb{k})$ is generated by $\{t_{ij} \mid 1 \leq i < j \leq n\}$ subject to the relations $[t_{ij}, t_{kl}] = 0$ and $[t_{ij}, t_{ik} + t_{jk}] = 0$ whenever i, j, k, l are distinct.

7.2. Taylor expansions for the pure braid groups. Explicit Taylor expansions for the pure braid groups P_n over $\mathbb{k} = \mathbb{C}$ can be constructed using Chen's method of iterated integrals, see e.g. [29, 4, 26]. Let $\text{Conf}_n(\mathbb{C}) = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \{z_i = z_j\}$ be the configuration space of n ordered points in \mathbb{C} , so that $P_n = \pi_1(\text{Conf}_n(\mathbb{C}), 0)$. Consider the logarithmic 1-forms on $\text{Conf}_n(\mathbb{C})$ given by

$$(29) \quad w_{ij} = w_{ji} = \frac{1}{2\pi\sqrt{-1}} \cdot d \log(z_i - z_j).$$

Clearly, these 1-forms are closed. Furthermore, as shown by Arnold [1], these 1-forms satisfy the relations $w_{ij} \wedge w_{jl} + w_{jl} \wedge w_{li} + w_{li} \wedge w_{ij} = 0$.

As shown by Kohno [24], the complete Hopf algebra $\widehat{\text{gr}}(\mathbb{k}P_n)$ admits a presentation with generators $\{X_{ij} = X_{ji}; 1 \leq i < j \leq n\}$, subject to the infinitesimal pure braid relations

$$(30) \quad \begin{cases} [X_{ij}, X_{kl}] = 0 \\ [X_{ij}, X_{il} + X_{lj}] = 0. \end{cases}$$

The formal power series connection $\omega = \sum_{1 \leq i < j \leq n} w_{ij} X_{ij}$ on $\text{Conf}_n(\mathbb{C})$ is flat. The corresponding monodromy representation yields a (faithful) Taylor expansion for the pure braid group, $J: P_n \rightarrow \widehat{\text{gr}}(\mathbb{k}P_n)$, given by (13), more explicitly, as stated in [4], (first appeared in [25])

$$(31) \quad J(g) = 1 + \sum_{k=1}^{\infty} \sum_{1 \leq i_1 < j_1, \dots, i_k < j_k \leq n} \left(\frac{1}{2\pi\sqrt{-1}} \right)^k \int_{0 \leq t_1 \leq \dots \leq t_k \leq 1} \bigwedge_{s=1}^k d \log(z_{i_s} - z_{j_s}) X_{i_1, j_1} \cdots X_{i_k, j_k},$$

where $g \in P_n$ is represented by a piecewise smooth loop $\gamma: [0, 1] \rightarrow \text{Conf}_n(\mathbb{C})$ at 0, and z_i is the i -th coordinate of the loop γ .

The Taylor expansion J is called the monodromy of the flat connection in [24], and the holonomy homomorphism in [26]. This expansion is a finite type invariant for the pure braid groups, and a prototype for the Kontsevich integral in knot theory.

7.3. Welded braid groups. The welded braid group (or, the braid-permutation group) wB_n is the subgroup of $\text{Aut}(F_n)$ consisting of all permutation-conjugacy automorphisms of F_n . The welded pure braid group (also known as the group of basis-conjugating automorphisms, or McCool group) wP_n is the kernel of the canonical projection $wP_n \rightarrow S_n$. In [36], McCool gave a finite presentation for wP_n ; the generators are the automorphisms α_{ij} ($1 \leq i \neq j \leq n$) sending x_i to $x_j x_i x_j^{-1}$.

The subgroup of wP_n generated by the elements α_{ij} with $i > j$ is called the *upper welded pure braid group* (or, upper triangular McCool group), and is denoted by wP_n^+ . As shown in [13], the upper welded pure braid group wP_n^+ also decomposes as a semidirect product, $wP_n^+ = F_{n-1} \rtimes wP_{n-1}^+$.

Work of Berceanu and Papadima from [7] establishes the 1-formality of the groups wP_n and wP_n^+ . Bar-Natan and Dancso, in [5], investigate expansions of welded braid groups. The Chen ranks of the groups P_n , wP_n , and wP_n^+ were computed in [15], [14], and [50], respectively. We summarize those results, as follows.

Theorem 7.1 ([15, 14, 50]). *The Chen ranks of P_n , wP_n , and wP_n^+ are given by*

- (1) $\theta_1(P_n) = \binom{n}{2}$, $\theta_2(P_n) = \binom{n}{3}$, and $\theta_k(P_n) = (k-1)\binom{n+1}{4}$ for $k \geq 3$.
- (2) $\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$ for $k \gg 0$.
- (3) $\theta_1(wP_n^+) = \binom{n}{2}$, $\theta_2(wP_n^+) = \binom{n}{3}$, and $\theta_k(wP_n^+) = \binom{n+1}{4} + \sum_{i=3}^k \binom{n+i-2}{i+1}$ for $k \geq 3$.

7.4. Distinguishing some related Lie algebras. Both the pure braid groups P_n and the upper McCool groups wP_n^+ are iterated semidirect products of the form $F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$. Clearly, $P_1 = wP_1^+ = \{1\}$ and $P_2 = wP_2^+ = \mathbb{Z}$; it is also known that $P_3 \cong wP_3^+ \cong F_2 \times F_1$. Furthermore, both P_n and wP_n^+ share the same LCS ranks and the same Betti numbers as the corresponding direct product of free groups, $\Pi_n = \prod_{i=1}^{n-1} F_i$, see [1, 13, 19, 24].

Proposition 7.2. *For each $n \geq 4$, the graded Lie algebras $\text{gr}(P_n, \mathbb{k})$, $\text{gr}(wP_n^+, \mathbb{k})$, and $\text{gr}(\Pi_n, \mathbb{k})$ are pairwise non-isomorphic.*

Proof. Using the computations recorded in Theorem 7.1, we find that $\theta_4(P_n) = 3\binom{n+1}{4}$ and $\theta_4(wP_n^+) = 2\binom{n+1}{4} + \binom{n+2}{5}$. Furthermore, the computation of K.-T. Chen recorded in Example 4.1 implies that $\theta_4(\Pi_n) = 3\binom{n+2}{5}$, cf. [15].

Comparing these ranks and using Corollary 5.7 shows that the graded Lie algebras $\text{gr}(P_n, \mathbb{k})$, $\text{gr}(\Pi_n, \mathbb{k})$, and $\text{gr}(wP_n^+, \mathbb{k})$ are pairwise non-isomorphic, as claimed \square

This proposition recovers (in stronger form) the following result from [50]: For each $n \geq 4$, the groups P_n , wP_n^+ , and Π_n are pairwise non-isomorphic.

7.5. Almost-direct products. A semi-direct product of groups, $H \rtimes Q$, is called an *almost-direct product* of H and Q , if the action of Q on H induces a trivial action on the abelianization H_{ab} , that is, $qhq^{-1} \equiv h$ modulo $[H, H]$ for any $q \in Q$ and $h \in H$.

Theorem 7.3. *Let $G = H \rtimes Q$ be a almost-direct product. Then,*

- (1) $\text{gr}(G; \mathbb{k}) \cong \text{gr}(H; \mathbb{k}) \rtimes \text{gr}(Q; \mathbb{k})$ as graded Lie algebras.
- (2) $\widehat{\text{gr}}(\mathbb{k}G) \cong \widehat{\text{gr}}(\mathbb{k}H) \widehat{\otimes} \widehat{\text{gr}}(\mathbb{k}Q)$ as graded vector spaces.

Proof. The first claim follows from [19, Theorem (3.1)], while the second claim follows from [40, Theorem 3.1]. \square

In general, an almost-direct product of 1-formal groups need not be 1-formal, or even filtered-formal.

Example 7.4. Let L be the link of 5 great circles in S^3 corresponding to the arrangement of transverse planes through the origin of \mathbb{R}^4 denoted as $\mathcal{A}(31425)$ in Matei–Suciu [35]. The link group $G = \pi_1(S^3 \setminus L)$ is isomorphic to the almost-direct product $F_4 \rtimes_{\alpha} F_1$, where $\alpha = A_{1,3}A_{2,3}A_{2,4} \in P_4$.

From [48], based on the work of Berceanu and Papadima [6], the group G is graded-formal. On the other hand, as noted by Dimca, Papadima, and Suciu in [18, Example 8.2], the Tangent Cone theorem does not hold for this group, and thus G is not 1-formal. Consequently, G is not filtered-formal.

8. FAITHFUL TAYLOR EXPANSIONS AND THE RTFN PROPERTY

8.1. Residually torsion free-nilpotent groups. A group G is said to be *residually torsion-free nilpotent* (for short RTFN) if for any $g \in G$, $g \neq 1$, there exists a torsion-free nilpotent group Q , and an epimorphism $\psi: G \rightarrow Q$ such that $\psi(g) \neq 1$. Equivalently, G is residually torsion-free nilpotent if and only if $\bigcap_{k \geq 1} \tau_k G = \{1\}$, where

$$(32) \quad \tau_k G = \{g \in G \mid g^n \in \Gamma_k G, \text{ for some } n \in \mathbb{N}\}.$$

For a group G , the property of being residually torsion-free nilpotent is inherited by all subgroups, and is preserved under direct products and free products.

By [43, Ch. VI, Thm. 2.26], a group G is residually torsion-free nilpotent if and only if the group-algebra $\mathbb{k}G$ is residually nilpotent, that is, $\bigcap_{k \geq 1} I^k = \{0\}$, where I is the augmentation ideal. Therefore, if G is finitely generated, the RTFN condition is equivalent to the injectivity of the canonical map to the pronilpotent completion, $\kappa: G \rightarrow \mathfrak{M}(G, \mathbb{k})$, where recall $\mathfrak{M}(G, \mathbb{k})$ is the set of group-like elements in $\widehat{\mathbb{k}G}$.

If G is residually nilpotent and $\text{gr}_k(G)$ is torsion-free for $k \geq 1$, then G is residually torsion-free nilpotent. Residually torsion-free nilpotent implies residually nilpotent, which in turn implies residually finite. Examples of residually torsion-free nilpotent groups include torsion-free nilpotent groups, free groups and surface groups; more examples will be discussed below.

8.2. Torelli groups. Let G be a finitely generated group, and let $\text{Aut}(G)$ be its group of automorphisms. The *Torelli group* of G is the subgroup of $\text{Aut}(G)$ consisting of all automorphisms inducing the identity on abelianization; that is,

$$(33) \quad \text{IA}(G) = \ker(\text{Aut}(G) \rightarrow \text{Aut}(G/[G, G])).$$

Example 8.1. Let F_n be the free group of rank n , and let \mathbb{Z}^n be its abelianization. Identify the automorphism group $\text{Aut}(\mathbb{Z}^n)$ with the general linear group $\text{GL}_n(\mathbb{Z})$. As is well-known, the map $\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$ which sends an automorphism to the induced map on the abelianization is surjective. The Torelli group $\text{IA}(F_n) = \ker(\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z}))$ is classically denoted by IA_n . Magnus showed that this group is finitely generated. Clearly, $\text{IA}_1 = \{1\}$, while, as noted by Magnus, $\text{IA}_2 = \text{Inn}(F_2) \cong F_2$. On the other hand, Krstić and McCool showed that IA_3 admits no finite presentation. It is still unknown whether IA_n admits a finite presentation for $n \geq 4$.

Example 8.2. Let Σ_g be a Riemann surface of genus g , and let $\mathcal{S}_g = \text{IA}(\pi_1(\Sigma_g))$ be the associated Torelli group. For $g \leq 1$, the group \mathcal{S}_g is trivial, while for $g = 2$, it is not finitely generated. On the other hand, it is known that \mathcal{S}_g is finitely generated for $n \geq 3$.

As noted by Hain [22] in the case of the Torelli group of a Riemann surface and proved by Berceanu and Papadima [7] in full generality, a stronger assumption on G leads to a stronger conclusion on $\text{IA}(G)$.

Theorem 8.3 ([22, 7]). *Let G be a finitely generated, residually nilpotent group, and suppose $\text{gr}_k(G)$ is torsion-free for all $k \geq 1$. Then the Torelli group $\text{IA}(G)$ is residually torsion-free nilpotent.*

As shown by Magnus, all free group F_n are residually torsion-free nilpotent. Hence, the Torelli groups $\text{IA}(F_n)$ are residually torsion-free nilpotent. Furthermore, all its subgroups, such as the pure braid group P_n , the McCool group wP_n , and the upper McCool group wP_n^+ are also residually torsion-free nilpotent. We refer to [2, 33, 47] for more details and references on this subject.

8.3. The RTFN property and Taylor expansions. The next result relates the RTFN property of a filtered-formal group to the injectivity of the corresponding Taylor expansion.

Proposition 8.4. *A finitely generated group G has a faithful Taylor expansion if and only if G is residually torsion-free nilpotent and filtered-formal.*

Proof. By Corollary 6.2, the group G is filtered-formal if and only if there is a Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$. In this case, by Propositions 2.3 and 2.5, the map $\widehat{E}: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ is an isomorphism of filtered Hopf algebras, which fits into the commuting diagram

$$(34) \quad \begin{array}{ccc} G & \xrightarrow{\kappa} & \widehat{\mathbb{k}G} \\ & \searrow E & \downarrow \widehat{E} \\ & & \widehat{\text{gr}}(\mathbb{k}G). \end{array}$$

Hence, E is injective if and only if κ is injective. That is to say, the expansion E is faithful if and only if the group G is RTFN. \square

Example 8.5. Consider the braid group B_n , with $n \geq 3$. Let us identify the complete Hopf algebra $\widehat{\text{gr}}(\mathbb{k}B_n)$ with $\mathbb{k}[[X]]$, the power series ring over \mathbb{k} in one variable. The homomorphism $L: B_n \rightarrow \mathbb{k}[[X]]$ given by $L(\sigma_i) = \exp(X)$ is a Taylor expansion of the braid group, since $\log(\exp(\sigma_i)\exp(\sigma_j)) = 2X$ is a group-like element in $\mathbb{k}[[X]]$. It is clear that this expansion is not faithful, since $L([\sigma_1, \sigma_2]) = 0$ but $[\sigma_1, \sigma_2] \neq 1 \in B_n$. In fact, it is known that the braid groups B_n ($n \geq 3$) are not RTFN, see [46].

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