Quasi-projective Bestvina-Brady groups

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Braids and their Ramifications

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Joint work with :
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* [DPS1] "Formality, Alexander invariants, and a question of Serre. math.AT/0512480
* [DPS2] "Quasi-Kähler Bestvina-Brady groups", to appear in J. Alg. Geometry ~2007 math. AG/0603446

* [PS1] "Algebraic invariants for right-angled Artin groups", Math. Ann. 2006.
* [PS2] "Algebraic invariants for Bestvina-Brady groups", to appear in J. London. Math. Soc. math.GR/0603240

* [DPS3] "Non-finiteness properties of fundamental groups of smooth projective varieties" math.AG/0609456

Artin's braid groups (1926/1947)

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} : \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \le i \le n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| > 1 \end{array} \right\rangle$$

$$1 \to P_n \to B_n \to S_n \to 1$$

Serre's problem (1958)

Which finitely-presented groups G are "quasi-projective", i.e.,

$$G = \pi_1(M)$$

M a smooth, connected, complex quasi-proj. variety ?

Fox - Neuwirth (1962)

$P_n = \pi_1(F(\mathbb{C}, n))$, where

$$F(\mathbb{C}, n) := \mathbb{C}^n \setminus \bigcup_{1 \le i < j \le n} \{z_i = z_j\}$$

$B_n = \pi_1(C(\mathbb{C}, n))$, where

$$C(\mathbb{C}, n) := F(\mathbb{C}, n)/S_n \underset{\text{Viète}}{=} \mathbb{C}^n \setminus \{\underbrace{\Delta_n}_{\text{discriminant}} = 0\}$$

$\therefore$ P_n, B_n are quasi-projective

Artin groups

Let $\Gamma = (V, E, \ell)$ be an edge-labeled finite, simple graph

$$G_\Gamma = \langle v \in V : \underbrace{vwv\cdots}_{m} = \underbrace{wvw\cdots}_{m} \quad \forall e = \{v,w\} \in E,\ \ell(e) = m \rangle$$

$$W_\Gamma = G_\Gamma / \langle\langle v^2 = 1 : v \in V \rangle\rangle$$

$$1 \to P_\Gamma \longrightarrow G_\Gamma \longrightarrow W_\Gamma \longrightarrow 1$$

Example $\Gamma = K_{n-1}$

$\rightsquigarrow G_\Gamma = B_n$

Example Right-angled Artin groups

$\Gamma = (V, E) \qquad \ell(e) = 2, \ \forall e \in E$

$G_\Gamma = \langle V : vw = wv, \ \forall \{v,w\} \in E \rangle$

- $\Gamma = K_n$  $\rightsquigarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \overline{K}_n$  $\rightsquigarrow G_\Gamma = F_n$
- $\Gamma = \Gamma' * \Gamma''$  $\rightsquigarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

- $\Gamma \cong \Gamma' \iff G_\Gamma \cong G_{\Gamma'}$

[Kim, Makar-Limanov, Neggers, Roush 1980
Droms 1987]

Brieskorn (1971) $\quad \Gamma = (V, E, \ell)$

W_Γ finite $\Rightarrow$ G_Γ quasi-proj.

sketch: $\mathcal{A}_\Gamma$ = reflection arrangement of type W_Γ (over $\mathbb{C}$)

$M_\Gamma = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}_\Gamma} H \qquad (n = |V|)$

$P_\Gamma = \pi_1(M_\Gamma)$

$G_\Gamma = \pi_1(M_\Gamma / W_\Gamma) = \pi_1(\mathbb{C}^n \setminus \{\underbrace{\delta_\Gamma}_{\text{discriminant}} = 0\})$

Kapovich - Millson (1998)

$\exists$ infinitely many $\Gamma = (V, E, \ell)$ s.t.
G_Γ not quasi-proj

DPS1 (2005) $\quad \Gamma = (V, E)$

G_Γ quasi-proj. $\Leftrightarrow$ $\Gamma = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r} =: \underbrace{K_{n_1 \ldots n_r}}_{\text{complete multipartite graph}}$

$\Leftrightarrow G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

Obstructions to quasi-projectivity

Deligne (~1972) M smooth, quasi-proj. variety
$\Longrightarrow H^*(M;\mathbb{C})$ has Mixed Hodge Structure (W,F)

Definition (Quillen 1969) Malcev Lie algebra of a group G: $E_G := \mathrm{Prim}(\widehat{\mathbb{Q}G})$

– a filtered Lie alg. with $gr(E_G) \cong gr(G)\otimes\mathbb{Q}$.

Morgan's test (1978) If G quasi-proj:

$$E_G \cong \widehat{\mathbb{L}}/J$$

$\mathbb{L}$: free Lie alg. on gens in degree 1 & 2
J : closed Lie ideal, gens in deg 2, 3, 4

Example $G = \langle x, y \mid [x,[x,[x,[x,y]]]] = 1 \rangle$ not q-proj.

Definition (Sullivan 1977)

A finitely presented group G is <u>1-formal</u> if

$$E_G \cong \widehat{\mathbb{L}}/J$$

L free on gens in deg 1
J generated in deg 2

M formal space $\Longrightarrow G=\pi_1(M)$ is 1-formal

$\Uparrow$ Deligne-Griffiths Morgan-Sullivan

M projective manifold

Kapovich – Millson :

- G quasi-proj $\Rightarrow$ Hom (G, Lie group) has only certain types of singularities
- $\forall\, \Gamma = (V, E, \ell)$, G_Γ is 1-formal
- $\exists\, \Gamma$ s.t. G_Γ does not pass the K-M test

Characteristic varieties

$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_s \rangle$ fin. pres. group

$T_G = \mathrm{Hom}(G, \mathbb{C}^*) = \mathrm{Hom}(G_{ab}, \mathbb{C}^*)$

T_G° = identity component $\cong (\mathbb{C}^*)^n$, $n = b_1(G)$

$$V_1(G) := \{ \rho \in T_G : \underbrace{H_1(G; \mathbb{C}_\rho)}_{= H_1(C_*(\widetilde{K(G,1)}) \otimes_{\mathbb{Z}G} \mathbb{C}_\rho)} \neq 0 \}$$

Eg: $G = \mathbb{Z} = \langle t \rangle$, $K(G,1) = S^1$, $T_G = \mathbb{C}^*$

$$C_*(\widetilde{K(G,1)}) : \quad C_1 \xrightarrow{\partial_1} C_0, \qquad C_1 = \mathbb{Z}\mathbb{Z} \xrightarrow{t-1} \mathbb{Z}\mathbb{Z} = C_0$$

$$C_*(\widetilde{K(G,1)}) \otimes_{\mathbb{Z}\mathbb{Z}} \mathbb{C}_\rho : \quad \mathbb{C} \xrightarrow{\rho - 1} \mathbb{C}$$

$H_1 \neq 0 \iff \rho = 1$ $\quad \therefore V_1(\mathbb{Z}) = \{1\}$

For $\rho \in T_G^\circ$, $\rho \neq 1$:

$$\rho \in V_1(G) \iff \mathrm{rank} \underbrace{\left(\frac{\partial r_i}{\partial x_j} \right)^{ab}}_{A_G \,=\, \text{Alexander matrix}} (\rho) < m - 1$$

Arapura's test (1997)

G quasi-proj. $\Rightarrow V_1(G) = \bigcup_\alpha \rho_\alpha \cdot T_\alpha$

root of 1 — subtorus of T_G^0

Resonance varieties

$A = H^*(G;\mathbb{C})$

$a \in A^1 = \mathrm{Hom}(G,\mathbb{C}) = \mathbb{C}^n$ $\qquad a \cdot a = 0$

$\rightsquigarrow (A,a): \quad 0 \to A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \to \cdots$

$$R_1(G) = \{a \in A^1 : H^1(A,a) \neq 0\}$$

$$= \{a : \text{rank } A_G^{\ell in}(a) < m-1\}$$

linearized Alex. matrix

$(t_i \mapsto 1-a_i$, take linear part$)$

Resonance test (DPS 2005)

Let G be a quasi-proj., 1-formal group.
Write $R_1(G) = \bigcup_\alpha R^\alpha$ (irred. components). Then:

(R1) R^α linear subspace

(R2) $R^\alpha \cap R^\beta = 0$, $\forall \alpha \neq \beta$

(R3) R^α is p-isotropic, of dim $\geq 2p+2$ $(p = 0$ or $1)$

Right-angled Artin groups

$\Gamma = (V, E) \rightsquigarrow G_\Gamma$

$K(G_\Gamma, 1) =$ subcomplex of $(S^1)^{|V|}$ with

k cells $\longleftrightarrow$ k cliques of Γ

$\longleftrightarrow$ $(k+1)$-simplices in $\Delta(\Gamma)$ — flag complex

$\shortparallel$ $Z_{\Delta(\Gamma)}(S^1, *)$

$C_*(\widetilde{K(G_\Gamma, 1)})$: Salvetti complex

$H^*(G_\Gamma) = \Lambda / J_\Gamma$ $\quad$ Λ : exterior alg on v^*, $v \in V$

J_Γ : ideal $\langle v^* w^* : \{v, w\} \notin E_\Gamma \rangle$

G_Γ 1-formal $\quad$ (so passes Morgan's test)

$K(G_\Gamma, 1)$ formal $\quad$ [follows from Papadima-Yuzvinsky]

— . —

Let $T_V = \mathrm{Hom}(G_\Gamma, \mathbb{C}^*) = (\mathbb{C}^*)^{|V|}$

$$V_1(G_\Gamma) = \bigcup_{\substack{W \subset V \\ \Gamma_W \text{ max disconnected}}} T_W$$

[DPS1]

T_W coord. subtorus supported on W

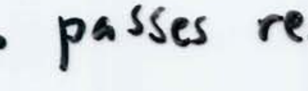
$\rightarrow G_\Gamma$ passes Arapura's test

$$R_1(G_\Gamma) = \bigcup_{\substack{W \subset V \\ \Gamma_W \text{ max disc.}}} H_W$$

[PS1]

H_W coord subspace supported on W

G_Γ passes resonance tests $\iff \Gamma = K_{n_1, \ldots, n_r}$

Ex $\Gamma =$ 

$R_1(G_\Gamma) = H_{134} \cup H_{124}$

intersect in $H_{12} \neq 0$

Artin groups

$\Gamma=(V,E,\ell) \rightsquigarrow \Gamma^{\text{odd}}=(V, E^{\text{odd}})$

$E^{\text{odd}} = \{e\in E;\ \ell(e) \text{ odd}\}$

$\rightsquigarrow \widetilde{\Gamma}=(\widetilde{V},\widetilde{E})$ $\qquad$ $\widetilde{V}$ = components of Γ^{odd}

$\widetilde{E} = \{(c,c') : \exists v\in c,\ v'\in c',\ \{v,v'\}\in E\}$

ex

Γ $\qquad\qquad$ Γ^{odd} $\qquad\qquad$ $\widetilde{\Gamma}$

Theorem [DPS11] For any $\Gamma=(V,E,\ell)$, TFAE:

- $E_{G_\Gamma} \cong E_G$, for some quasi-proj. G
- Resonance test (R3) satisfied by G_Γ
- $\widetilde{\Gamma}$ complete multipartite
- $E_{G_\Gamma} \cong E_{F_{n_1}\times\cdots\times F_{n_r}}$

Bestvina-Brady groups

$$N_\Gamma = \ker(\nu : G_\Gamma \longrightarrow \mathbb{Z}) \qquad \Gamma = (V, E)$$
$$v \longmapsto 1$$

$$1 \longrightarrow N_\Gamma \xrightarrow{\iota} G_\Gamma \xrightarrow{\nu} \mathbb{Z} \longrightarrow 1$$

- $\Gamma = \overline{K_2}$ • • $\rightsquigarrow G_\Gamma = F_2 \rightsquigarrow N_\Gamma = F_\infty$

* Γ connected $\iff$ N_Γ finitely generated [Meier, Van Wyk 1995]

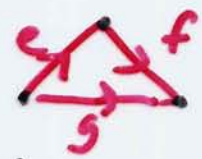
- $\Gamma = K_{2,2}$ $\square$ $\rightsquigarrow G_\Gamma = F_2 \times F_2 \rightsquigarrow N_\Gamma$ fin. gen. but not fin pres

reason: $H_2(N_\Gamma) = \mathbb{Z}^\infty$ [Stallings 1963]

* $\pi_1(\Delta(\Gamma)) = 0 \iff N_\Gamma$ finitely presented [Bestvina-Brady 1997]

in this case:

$$N_\Gamma = \langle e \in E \mid ef = fe, \; ef = g \rangle$$

$\forall$ 

in Γ

and $\iota(e) = uv^{-1}$ if $e = \{u, v\}$

[Dicks-Leary 1999]

Example Γ tree on n vertices $\Rightarrow N_\Gamma = F_{n-1}$

[$\Delta(\Gamma) = \Gamma \simeq *$, Γ has no triangles, Γ has $n-1$ edges]

In particular, N_Γ's not classified by Γ's

Theorem [DPS2]

N_Γ quasi-projective $\iff$ Γ is either

- a tree
- a complete multipartite graph $K_{n_1,\dots,n_r}$ with $\begin{cases} \text{some } n_i = 1 \\ \text{all } n_i > 1 \ \& \ r \geq 3 \end{cases}$

in which case, N_Γ belongs to precisely one of:

(1) $\mathbb{Z}^r$, $r \geq 0$

(2) $F_{n_1} \times \cdots \times F_{n_r}$, $n_i > 1$

(3) $\mathbb{Z}^r \times F_{n_1} \times \cdots \times F_{n_s}$, $r > 0$, $n_i > 1$

(4) $N_{K_{n_1,\dots,n_r}}$, $r \geq 3$, $n_i > 1$

N_Γ projective $\iff$ $\Gamma = K_n$, n odd

$\iff$ $N_\Gamma = \mathbb{Z}^{2r}$

Example $N = N_{K_{2,2,2}}$ (Stallings group)

$N = \pi_1\big(\mathbb{C}^2 \setminus$  $\big)$

Question (Kollár 1995)

Given a (quasi-) projective group G, is there a group π, commensurable to G up to finite kernels,

$$G \leftarrow H_1 \rightarrow G_1 \leftarrow H_2 \rightarrow \cdots \leftarrow H_n \rightarrow \pi$$

each $\rightarrow$ of finite ker & cofinite im

such that π admits a $K(\pi, 1)$ which is a smooth, (quasi-) projective variety?

Theorem [DPS 2]

Let N_Γ quasi-proj. Then:

- N_Γ of type 1, 2, or 3 $\Longrightarrow$ YES ✓
- N_Γ of type 4 $\Longrightarrow$ NO

since: type 4 $\Rightarrow$ $\tilde{H}_*(\Delta(\Gamma)) \neq 0$

$\Rightarrow$ N_Γ not FP_∞

$\xRightarrow{\text{Bieri}}$ N_Γ not commens. (up to finite ker) to any FP_∞ gp.

$\Rightarrow$ N_Γ " " " " any π with finite $K(\pi,1)$