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The Chen Groups of the Pure Braid Group

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Abstract. The Chen groups of a group are the lower central series quotients of its maximal metabelian quotient. We show that the Chen groups of the pure braid group P_n are free abelian, and we compute their ranks. The computation of these Chen groups reduces to the computation of the Hilbert series of a certain graded module over a polynomial ring, and the latter is carried out by means of a Groebner basis algorithm. This result shows that, for $n \geq 4$, the group P_n is not a direct product of free groups.

1. Introduction

The Chen groups of a group G are the lower central series quotients of G modulo its second commutator subgroup G''. These groups were introduced by Chen in [**Ch**], so as to provide a computable approximation to the lower central series quotients of a link group. Although apparently weaker invariants than the lower central series quotients of G itself, the lower central series quotients of G/G'' sometimes provide more subtle information about the structure of the group G. In this paper, we illustrate this point by computing the Chen groups of the pure braid group P_n, for every $n \geq 2$. Our results show that, unlike the lower central series quotients of P_n, the Chen groups are not determined by the exponents of the braid arrangement.

Theorem 1.1. *The Chen groups of the pure braid group P_n are free abelian. The rank, θ_k, of the k^{th} Chen group of P_n is given by*

$$\theta_1 = \binom{n}{2}, \qquad \theta_2 = \binom{n}{3}, \qquad \textit{and} \quad \theta_k = (k-1)\cdot\binom{n+1}{4} \quad \textit{for } k \geq 3.$$

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Remark. The ranks of the Chen groups of P_n (for $k \geq 2$) are given by the generating series

$$\sum_{k=2}^{\infty} \theta_k t^{k-2} = \binom{n+1}{4} \cdot \frac{1}{(1-t)^2} - \binom{n}{4}.$$

Throughout the paper, we use the convention $\binom{p}{q} = 0$ if $p < q$.

For any group G, let $\Gamma_k(G)$ denote its k^{th} lower central series subgroup, defined inductively by $\Gamma_1(G) = G$ and $\Gamma_{k+1}(G) = [\Gamma_k(G), G]$ for $k \geq 1$. The projection of G onto its maximal metabelian quotient G/G'' induces an epimorphism

$$\frac{\Gamma_k(G)}{\Gamma_{k+1}(G)} \twoheadrightarrow \frac{\Gamma_k(G/G'')}{\Gamma_{k+1}(G/G'')}$$

from the k^{th} lower central series quotient of G to the k^{th} Chen group of G. For $1 \leq k \leq 3$, it is easy to see that this epimorphism is in fact an isomorphism. In particular, the first three Chen groups of P_n and the first three lower central series quotients of P_n are identical. The lower central series quotients of the pure braid group were computed by Kohno, using Sullivan's minimal models technique.

Theorem 1.2 ([**K**]). *The lower central series quotients of the pure braid group P_n are free abelian. Their ranks, ϕ_k, are given by the equality*

$$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k} = \prod_{i=1}^{n-1} (1-it) = \sum_{j \geq 0} (-1)^j b_j(P_n) t^j \quad \textit{in } \mathbb{Z}[[t]],$$

where $b_j(P_n)$ denotes the j^{th} betti number of P_n.

The relation between the ranks of the lower central series quotients and the betti numbers described above holds in greater generality (see [**KO**] for one generalization). For instance, if $G = G(\mathcal{A})$ is the fundamental group of the complement of a *fiber-type* hyperplane arrangement $\mathcal{A}$ (see [**FR1**], [**OT**]), we have the following.

Theorem 1.3 ([**FR1**], [**FR2**]). *Let $\mathcal{A}$ be a fiber-type hyperplane arrangement with exponents $\{d_1, d_2, \dots, d_\ell\}$, let G denote the fundamental group of the complement of $\mathcal{A}$, and let ϕ_k denote the rank of the k^{th} lower central series quotient of G. Then the lower central series quotients of G are free abelian and, in $\mathbb{Z}[[t]]$, we have*

$$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k} = \prod_{i=1}^{\ell} (1-d_i t) = \sum_{j \geq 0} (-1)^j b_j(G) t^j.$$

Implicit in the above result are the computation of the betti numbers of the group G, and the factorization of the Poincaré polynomial of G. In the case where $G = P_n$, these computations were carried out by Arnol'd [**A**], who also determined the structure of the cohomology ring of P_n. For a generalization to the configuration spaces of n points in $\mathbb{R}^m$, see Cohen [**Co**]. The cohomology of the complement of an arbitrary arrangement was found by Brieskorn [**Br**]; and Orlik and Solomon [**OS**] subsequently provided a combinatorial description of the cohomology ring. These results suffice for the computation of the cohomology of $G(\mathcal{A})$, since the complement of any fiber-type arrangement $\mathcal{A}$ is an Eilenberg-MacLane space. The factorization of the Poincaré polynomial of the group of a fiber-type arrangement may be deduced from the fact that fiber-type arrangements are *supersolvable*, see [**T**].

Alternatively, one can make explicit use of the structure of the group G to compute its betti numbers and Poincaré polynomial. The group of any fiber-type arrangement may be realized as an iterated semi-direct product of free groups (see e.g. [**OT**]). In particular, the group of the braid arrangement $\mathcal{P}_n = \{H_{i,j} = \ker(z_i - z_j)\}$ in $\mathbb{C}^n$ may be realized as $P_n = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, see [**Bi**], [**Ha**]. For a detailed discussion of the homology of iterated semi-direct products of free groups, see [**CS1**].

Both the direct product $\Pi_n = F_{n-1} \times \cdots \times F_2 \times F_1$, and the semi-direct product P_n, may be realized as the fundamental groups of the complements of (different) fiber-type arrangements with exponents $\{1, 2, \ldots, n-1\}$. The above results show that the betti numbers, and the ranks of the lower central series quotients of P_n are equal to those of Π_n. Thus neither homology nor the lower central series can distinguish between the direct product Π_n and the semi-direct product P_n. (For $n \leq 3$ this is not surprising, as $P_2 \cong F_1$ and $P_3 \cong F_2 \times F_1$.)

However, these groups **can** be distinguished by means of their respective Chen groups. First, it should be noted that the Chen groups of a group G are invariants of isomorphism type for G, since the derived series and the lower central series subgroups of a group are characteristic. The Chen groups of a (single) free group are known ([**Ch**], [**Mu**]), and the Chen groups of a direct product of free groups can therefore be easily computed, see [**CS3**].

THEOREM 1.4. *Let $G = F_{d_1} \times \cdots \times F_{d_\ell}$ be a direct product of free groups. Then the Chen groups of G are free abelian. The rank, θ_k, of the k^{th} Chen group of G is given by*

$$\theta_1 = \sum_{i=1}^{\ell} d_i, \quad \textit{and by} \quad \theta_k = (k-1) \cdot \sum_{i=1}^{\ell} \binom{k + d_i - 2}{k} \quad \textit{for } k \geq 2.$$

In particular, the ranks of the Chen groups of $\Pi_n = F_{n-1} \times \cdots \times F_1$ are

$$\theta_1 = \binom{n}{2}, \qquad \textit{and} \quad \theta_k = (k-1) \cdot \binom{k+n-2}{k+1} \quad \textit{for } k \geq 2.$$

Together, Theorem 1.1 and Theorem 1.4 yield

COROLLARY 1.5. *For $n \geq 4$, the groups P_n/P_n'' and Π_n/Π_n'' are not isomorphic.*

Corollary 1.6. *For $n \geq 4$, the groups P_n and Π_n are not isomorphic.*

Corollary 1.6 may also be obtained by analyzing the cohomology rings of the groups P_n and Π_n. This is not an obvious task, for these graded rings are typically given in terms of generators and relations; nevertheless, Falk [**Fa**] managed to define a new invariant that does distinguish these rings. The corollary may also be deduced from the Tits conjecture for P_5, proved in [**DLS**].

The techniques we employ (see below and [**CS3**]) may be generalized so as to yield an algorithm for computing the Chen groups of the group of an arbitrary hyperplane arrangement. We have carried out this algorithm for a number of arrangements. The most striking examples we have encountered are the following.

Example ([**CS3**]). Consider the 3-arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with defining polynomials

$$Q(\mathcal{A}_1) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-z)$$

and

$$Q(\mathcal{A}_2) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-2z).$$

Both these arrangements are fiber-type, with exponents $\{1,4,4\}$. Thus, $G_1 = G(\mathcal{A}_1)$ and $G_2 = G(\mathcal{A}_2)$ have isomorphic homology groups and lower central series quotients. These arrangements were introduced in [**Fa**], where it is shown that an invariant even finer than the ranks of the lower central series quotients cannot distinguish between (the cohomology algebras of) these arrangements.

Since the lower central series quotients of G_1 and G_2 are isomorphic, so are the first three Chen groups. Explicitly, we have $\theta_1(G_1) = \theta_1(G_2) = 9$, $\theta_2(G_1) = \theta_2(G_2) = 12$, and $\theta_3(G_1) = \theta_3(G_2) = 40$ by Theorem 1.3. However, for $k \geq 4$, we have

$$\theta_k(G_1) = \frac{1}{2}(k-1)(k^2+3k+24) \qquad \text{and} \qquad \theta_k(G_2) = \frac{1}{2}(k-1)(k^2+3k+22).$$

Therefore $G_1 \not\cong G_2$, and the arrangements are topologically distinct.

The ranks of the Chen groups of all the arrangements we have considered are given by a pleasant combinatorial formula which we now describe. For an arbitrary central arrangement $\mathcal{A}$ with group $G = G(\mathcal{A})$, let $L = L(\mathcal{A})$ denote the intersection lattice of $\mathcal{A}$, let $\mu : L \to \mathbb{Z}$ be the Möbius functions of L, and let L_2 be the set of rank 2 elements in L (see [**OT**] for standards definitions and notational conventions concerning hyperplane arrangements). Define $\beta(\mathcal{A})$ to be the number of subarrangements $\mathcal{B} \subseteq \mathcal{A}$ that are lattice isomorphic to $\mathcal{P}_4$, the braid arrangement with group P_4.

Conjecture. Let $\mathcal{A}$ be a central arrangement. For $k \geq \max_{X \in L_2}\{\mu(X)+1\}$, the rank of the k^{th} Chen group of $G(\mathcal{A})$ is given by

$$\theta_k(G(\mathcal{A})) = \sum_{X \in L_2} (k-1)\binom{k+\mu(X)-2}{k} + \beta(\mathcal{A})(k-1).$$

We leave it to the reader to verify that this conjecture holds for all arrangements considered in this paper.

Implicit in the above conjecture is the assertion that, for k sufficiently large, the rank of the k^{th} Chen group of $G(\mathcal{A})$ is given by a polynomial in k of degree $\max_{X\in L_2}\{\mu(X)-1\}$. For the braid arrangement, this degree is 1, while for the arrangement with group $G=\Pi_n$, the degree is $n-2$. An arbitrary degree may be realized by the group of a central 2-arrangement of the appropriate number of hyperplanes. For a more general context in which this assertion fits (the determination of the "lower central dimension" of a certain class of metabelian groups), see Baumslag [**Ba**].

The structure of this paper is as follows:

In section 2, we show how to reduce the computation of the Chen groups of an arrangement to a problem in commutative algebra. This follows the approach taken by Massey in [**Ma**] for the computation of the Chen groups of a classical link: For any group G with $G/G'=\mathbb{Z}^N$, the nilpotent completion of G/G'' corresponds to the I-adic completion of the Λ-module $B=G'/G''$, where I is the augmentation ideal of the group ring $\Lambda=\mathbb{Z}\mathbb{Z}^N$. Thus, the computation of the Chen groups reduces to the computation of $\operatorname{gr}(\widehat{B})$, the associated graded module over the polynomial ring $\operatorname{gr}(\widehat{\Lambda})\cong\mathbb{Z}[x_1,\dots,x_N]$.

In section 3, we find a presentation for $B=P_n'/P_n''$. This is accomplished topologically as follows. The Λ-module B is the first homology group of $\widetilde{M}_n$, the maximal abelian cover of the complement of the braid arrangement. A free Λ-presentation of B is obtained by comparing the chain complex of $\widetilde{M}_n$ with that of the universal (abelian) cover of the N-torus, where $N=\binom{n}{2}$ is the number of generators of P_n (the cardinality of the braid arrangement.)

In section 4, a presentation for $\operatorname{gr}(\widehat{B})$ is obtained. This is done by finding a Groebner basis for the module generated by the rows of the presentation matrix for B. The computation is much easier than one might expect: The theoretical upper bound for the degrees of the polynomial entries in this Groebner basis is doubly exponential in N, whereas the actual polynomials we find are at most cubics.

Finally, in section 5, the ranks of the Chen groups of P_n are computed from the coefficients of the Hilbert series, $\sum_{k\geq 0}\operatorname{rank}\left(\operatorname{gr}(\widehat{B})_{(k)}\right)t^k$, of the graded module $\operatorname{gr}(\widehat{B})$.

2. Outline of the Proof

Our approach to the computation of the Chen groups of P_n follows that of Massey, who studied an analogous problem for link groups in [**Ma**] (see also [**MT**].) We start by outlining Massey's setup in a context which covers both link groups and hyperplane arrangements groups.

First, consider the free abelian group $\mathbb{Z}^N$, and fix generators $A_1,\dots,A_N$. We then can identify $\Lambda=\mathbb{Z}\mathbb{Z}^N$, the group ring of $\mathbb{Z}^N$, with $\mathbb{Z}[A_i,A_i^{-1}]$, the ring of

Laurent polynomials in the variables A_i. This ring can be viewed as a subring of $\mathbb{Z}[[x_i]]$, the ring of formal power series in the variables $x_1, \dots, x_N$. The "Magnus embedding" (see [**Ma**], and compare [**MKS**]), $\mathbb{Z}[A_i^{\pm 1}] \hookrightarrow \mathbb{Z}[[x_i]]$, is given by $A_i \mapsto 1 - x_i$ and $A_i^{-1} \mapsto \sum_{k=0}^{\infty} x_i^k$. Notice that the image of the polynomial subring $\mathbb{Z}[A_i]$ under this map is contained in $\mathbb{Z}[x_i]$, the polynomial ring in the variables $x_1, \dots, x_N$.

Next, let $\epsilon : \mathbb{Z}\mathbb{Z}^N \to \mathbb{Z}$ be the augmentation map, and $I = \ker \epsilon$ be the augmentation ideal of Λ. Consider the completion of Λ relative to the I-adic topology, $\widehat{\Lambda} = \varprojlim \Lambda / I^k$. Then, the Magnus embedding extends to a ring isomorphism $\widehat{\Lambda} \xrightarrow{\sim} \mathbb{Z}[[x_i]]$.

Finally, consider the I-adic filtration on Λ, and its associated graded ring, $\operatorname{gr}(\Lambda) = \bigoplus_{k \geq 0} I^k / I^{k+1}$. Also, consider the $\mathfrak{m}$-adic filtration on $\mathbb{Z}[[x_i]]$, where $\mathfrak{m} = \langle x_1, \dots, x_N \rangle$, and its associated graded ring, $\operatorname{gr}(\mathbb{Z}[[x_i]]) = \bigoplus_{k \geq 0} \mathfrak{m}^k / \mathfrak{m}^{k+1} \cong \mathbb{Z}[x_i]$. Then, the Magnus embedding induces a graded ring isomorphism $\operatorname{gr}(\Lambda) \xrightarrow{\sim} \operatorname{gr}(\mathbb{Z}[[x_i]])$.

Now let G be a group with abelianization $G/G' \cong \mathbb{Z}^N$. Then each homology group of G' supports the structure of a module over the ring $\Lambda = \mathbb{Z}\mathbb{Z}^N$. The main object of our study is the first homology group of G',

$$B = B(G) = G'/G'',$$

viewed as a Λ-module, called the *Alexander invariant* of G, see [**Ma**], [**MT**], [**Hi**], [**L**].

Let $\widehat{B}$ be the I-adic completion of B. Let

$$\operatorname{gr}(B) = \bigoplus_{k \geq 0} I^k B / I^{k+1} B \quad \text{and} \quad \operatorname{gr}(\widehat{B}) = \bigoplus_{k \geq 0} \mathfrak{m}^k \widehat{B} / \mathfrak{m}^{k+1} \widehat{B}$$

be the graded modules associated to the I-adic filtration on B and the $\mathfrak{m}$-adic filtration on $\widehat{B}$, respectively. Then, the canonical map $B \to \widehat{B}$ induces an isomorphism $\operatorname{gr}(B) \xrightarrow{\sim} \operatorname{gr}(\widehat{B})$ of graded modules over the graded polynomial ring $\mathbb{Z}[x_i]$. Massey [**Ma**] observed that

$$\frac{I^k B}{I^{k+1} B} = \frac{\Gamma_{k+2}(G/G'')}{\Gamma_{k+3}(G/G'')}.$$

Combining these facts, we can restate Massey's result as follows:

Theorem 2.1 ([**Ma**]). *The generating series for the ranks of the Chen groups of G,*

$$\sum_{k=0}^{\infty} \theta_{k+2} t^k,$$

equals the Hilbert series of the graded module associated to the I-adic completion of $B(G)$,

$$H(\operatorname{gr}(\widehat{B}), t) = \sum_{k=0}^{\infty} \operatorname{rank}(\mathfrak{m}^k \widehat{B} / \mathfrak{m}^{k+1} \widehat{B}) t^k.$$

An immediate consequence of this theorem is that, for k sufficiently large, the rank θ_k of the k^{th} Chen group of G is given by a polynomial in k. Indeed, this is just the Hilbert-Serre polynomial of $\operatorname{gr}(\widehat{B})$, see [**ZS**].

Thus, the determination of the Chen groups of G has been reduced to the determination of the graded module $\operatorname{gr}(\widehat{B})$. In order to achieve this, we first need a finite presentation,

$$\Lambda^a \xrightarrow{\Omega} \Lambda^b \to B \to 0,$$

for the Λ-module B. In the case where G is the group of a complexified real hyperplane arrangement, we have an algorithm for doing this based on the "Randell presentation" ([**R**]) of G, see [**CS2**]. In the case where $G = P_n$, we find a simplified presentation for B in section 3. A useful feature of the matrix of Ω is that all its entries are actual polynomials in the variables A_i. This can also be done in general, by replacing the generators e_s of the free module Λ^b by suitable multiples $\lambda_s e_s$, if necessary. Denote by $J = \operatorname{Im}(\Omega)$ the submodule of Λ^b generated by the rows of the matrix of Ω.

It is now an easy task to find a presentation for the I-adic completion of B. Simply take $\widehat{\Lambda}^a \xrightarrow{\Omega} \widehat{\Lambda}^b \to \widehat{B} \to 0$, where $\widehat{\Omega}$ is obtained from Ω via the Magnus embedding. Clearly, $\operatorname{Im}(\widehat{\Omega}) = \widehat{J}$, the completion of J. Since all the entries of the matrix for $\widehat{\Omega}$ belong to the subring $\mathbb{Z}[x_i] \subset \mathbb{Z}[[x_i]]$, we may restrict $\widehat{\Omega}$ to a map $\mathbf{\Omega} : \mathbb{Z}[x_i]^a \to \mathbb{Z}[x_i]^b$, whose image, $\widehat{J} \cap \mathbb{Z}[x_i]^b$, we will denote by $\mathbf{J}$.

Next, we must find a presentation for the associated graded module $\operatorname{gr}(\widehat{B}) = \operatorname{gr}\left(\mathbb{Z}[[x_i]]^b/\widehat{J}\right)$. This module is known to be isomorphic to $\mathbb{Z}[x_i]^b/\textsc{lt}(\widehat{J})$, where $\textsc{lt}(\widehat{J})$ is the submodule of $\mathbb{Z}[x_i]^b$ consisting of lowest degree homogeneous forms of elements in $\widehat{J}$ (the "leading terms" of elements in $\widehat{J}$), see e.g. [**ZS**]. From the definitions, we have $\textsc{lt}(\widehat{J}) = \textsc{lt}(\mathbf{J})$. Thus, $\operatorname{gr}(\widehat{B}) = \mathbb{Z}[x_i]^b/\textsc{lt}(\mathbf{J})$, and we are left with finding a finite generating set for the module $\textsc{lt}(\mathbf{J})$.

Such a set is provided by Mora's algorithm ([**Mo**]) for finding the tangent cone of an affine variety at the origin, see [**CLO**], [**BW**] for detailed explanations. A script that implements this algorithm using the symbolic algebra package *Macaulay* was written by Michael Stillman. We gratefully acknowledge the use of this script, and thank Tony Iarrobino for guiding us to it. Essentially, we must find a (minimal) *Groebner basis* $\mathcal{G} = \{g_1, \dots, g_c\}$ for the module $\mathbf{J}$, with respect to a suitable monomial ordering. Then, $\textsc{lt}(\mathbf{J})$ has Groebner basis $\textsc{lt}(\mathcal{G}) = \{\textsc{lt}(g_1), \dots, \textsc{lt}(g_c)\}$, out of which we can extract a minimal Groebner basis $\mathcal{H} = \{h_1, \dots, h_d\}$. Putting all these facts together, we obtain the following.

Theorem 2.2. *The* $\mathbb{Z}[x_i]$*-module* $\operatorname{gr}(\widehat{B})$ *has a finite presentation given by*

$$\mathbb{Z}[x_i]^d \xrightarrow{\mathbf{\Theta}} \mathbb{Z}[x_i]^b \to \operatorname{gr}(\widehat{B}) \to 0,$$

where the matrix of $\mathbf{\Theta}$ *has rows* $h_1, \dots, h_d$ *defined above.*

In the case where $G = P_n$, we use Buchberger's algorithm to find a Groebner basis $\mathcal{G}$ for $\mathbf{J}$ in section 4. The determination of the Groebner basis $\mathcal{H}$ for $\textsc{lt}(\mathbf{J})$

and the computation of the Hilbert series for $\mathrm{gr}(\widehat{B})$ is carried out in section 5. It follows from Theorems 2.1 and 5.6 that the ranks of the Chen groups of P_n are as stated in Theorem 1.1.

To finish the proof of Theorem 1.1 we only have to show that the Chen groups of P_n are free abelian. But this follows from the discussion above and a careful look at the various changes of bases that we perform. Indeed, we always work over $\mathbb{Z}$, and never divide by non-zero integers different from ± 1.

The Chen groups of a link group may have torsion, as the examples in [**Ma**] illustrate. We do not know whether the Chen groups of an arrangement group are always torsion free.

3. The Presentation Matrix

We briefly sketch our algorithm for finding a presentation of the module $B = B(G)$ in the case where $G = P_n$ is the group of pure braids on n strings. For a detailed discussion of this algorithm in the case where G is the fundamental group of the complement of an arbitrary complexified real hyperplane arrangement, see [**CS2**].

Recall that the pure braid group P_n may be realized as the fundamental group of the complement M_n of the hyperplane arrangement $\mathcal{P}_n = \{H_{i,j} = \ker(z_i - z_j)\}$ in $\mathbb{C}^n$. It is easy to show that the complement M_n may be realized as a linear slice of the complex N-torus, $T = (\mathbb{C}^*)^N$, where $N = \binom{n}{2}$ is the cardinality of $\mathcal{P}_n$ (the number of generators of P_n). Let $\widetilde{M}_n$ and $\widetilde{T}$ denote the universal abelian covers of M_n and T respectively. Let $\Lambda = \mathbb{Z}\mathbb{Z}^N$ denote the group ring of $\mathbb{Z}^N = H_1(M_n) = H_1(T)$. We wish to identify the Λ-module $B = H_1(\widetilde{M}_n)$.

For that, let $C = C_\bullet(\widetilde{M}_n)$ denote the augmented chain complex of $\widetilde{M}_n$. This complex is of the form

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $C_0 = \Lambda$, $C_1 = \Lambda^N$, and $C_2 = \Lambda^\rho$ (here $\rho = b_2(M_n)$ is the number of relations in P_n). We specify the first several differentials of this complex using the "Burau presentation" of P_n (which may be obtained from the standard presentation of P_n found in [**Bi**], [**MKS**], [**Ha**]). The pure braid group P_n has generators $A_{i,j}$, $1 \le i < j \le n$, and relations

$$\begin{aligned} A_{i,k}A_{j,k}A_{i,j} = A_{i,j}A_{i,k}A_{j,k} = A_{j,k}A_{i,j}A_{i,k} \qquad & i<j<k, \\ A_{k,l}A_{i,j} = A_{i,j}A_{k,l},\ A_{i,l}A_{j,k} = A_{j,k}A_{i,l},\ A_{j,l}A_{i,k}^{A_{j,k}} = A_{i,k}^{A_{j,k}}A_{j,l} \quad & i<j<k<l, \end{aligned}$$

where $u^v = v^{-1}uv$. The first two differentials of this complex are

$$\partial_1 = \begin{pmatrix} 1 - A_{1,2} & 1 - A_{1,3} & 1 - A_{2,3} & \dots & 1 - A_{n-1,n} \end{pmatrix}^T$$

(where $(\bullet)^T$ denotes the transpose), and $\partial_2 = \left(\dfrac{\partial R}{\partial A_{i,j}}\right)$, the abelianization of the Jacobian matrix of Fox derivatives of the relations $\{R\}$ of P_n, where R runs through the relations specified above.

At this point, it is an easy task to find a presentation for the *Alexander module* $A = A(P_n)$, which is defined as $H_1(\widetilde{M}_n, p^{-1}(*))$, where $p : \widetilde{M}_n \to M_n$ is the covering map, and $*$ is the basepoint of M_n, see [**Ma**], [**Hi**]. Indeed, A is the cokernel of $\partial_2 : \Lambda^\rho \to \Lambda^N$. This module, the Alexander invariant, and the augmentation ideal comprise the Crowell exact sequence $0 \to B \to A \to I \to 0$. We now find a presentation for B by comparing the chain complexes $C_\bullet(\widetilde{M}_n)$ and $C_\bullet(\widetilde{T})$.

The chain complex $C' = C_\bullet(\widetilde{T})$ is the "standard" Λ-free resolution of $\mathbb{Z}$. The terms of this resolution may be described as follows: $C'_0 = \Lambda$, $C'_1 = \Lambda^N$ is a free Λ-module with basis $\{e_{i,j} \mid 1 \le i < j \le n\}$, and for $k > 1$, $C'_k = \bigwedge^k C'_1 = \Lambda^{\binom{N}{k}}$. The differentials are given by

$$d_k(e_{i_1,j_1} \wedge \cdots \wedge e_{i_k,j_k}) = \sum_{r=1}^{k} (-1)^{r-1}(1 - A_{i_r,j_r})(e_{i_1,j_1} \wedge \cdots \wedge \hat{e}_{i_r,j_r} \wedge \cdots \wedge e_{i_k,j_k}).$$

The natural inclusion $M_n \hookrightarrow T$ induces a chain map $\phi : C \to C'$ covering the identity map $\mathbb{Z} \to \mathbb{Z}$. The map ϕ is not unique, but its chain homotopy class is unique. Clearly we may identify C_0 and C'_0, so set $\phi_0 = \mathrm{id} : \Lambda \to \Lambda$. Furthermore, since $H_1(M_n) = H_1(T)$, we identify C_1 and C'_1, and set $\phi_1 = \mathrm{id} : \Lambda^N \to \Lambda^N$.

Letting R denote a generic relation in P_n, we define $\phi_2 : C_2 \to C'_2$ by

$$\phi_2(R) = \begin{cases} A_{j,k}e_{i,j}e_{i,k} + e_{j,k}e_{i,k} & \text{if } R \text{ is } A_{i,k}A_{j,k}A_{i,j} = A_{j,k}A_{i,j}A_{i,k}, \\ A_{i,k}e_{i,j}e_{j,k} + e_{i,j}e_{i,k} & \text{if } R \text{ is } A_{i,k}A_{j,k}A_{i,j} = A_{i,j}A_{i,k}A_{j,k}, \\ e_{i,j}e_{k,l} & \text{if } R \text{ is } A_{k,l}A_{i,j} = A_{i,j}A_{k,l}, \\ e_{j,k}e_{i,l} & \text{if } R \text{ is } A_{i,l}A_{j,k} = A_{j,k}A_{i,l}, \\ A_{j,k}^{-1}(A_{j,l}-1)e_{j,k}e_{i,k} + e_{i,k}e_{j,l} & \text{if } R \text{ is } A_{j,l}A_{i,k}^{A_{j,k}} = A_{i,k}^{A_{j,k}}A_{j,l}. \end{cases}$$

Proposition 3.1. *We have $d_2 \circ \phi_2 = \partial_2$.*

The proof of this result is an exercise in Fox calculus which we omit.

Now define a map $\psi : C'_2 \to C'_2$ by

$$\begin{aligned} \psi(e_{i,k}e_{j,l}) &= e_{i,k}e_{j,l} + (1 - A_{j,l})(A_{j,k}^{-1}e_{j,k}e_{i,k} - e_{i,j}e_{i,k} + A_{i,k}e_{i,j}e_{j,k}), \\ \psi(e_{j,k}e_{i,k}) &= e_{j,k}e_{i,k} - A_{j,k}e_{i,j}e_{i,k} + A_{i,k}A_{j,k}e_{i,j}e_{j,k}, \\ \psi(e_{i,j}e_{i,k}) &= e_{i,j}e_{i,k} - A_{i,k}e_{i,j}e_{j,k}, \end{aligned}$$

for $i < j < k < l$, and $\psi(e_{p,q}e_{r,s}) = e_{p,q}e_{r,s}$ otherwise.

Proposition 3.2. *The map ψ is an isomorphism, and we have*

$$\psi \circ \phi_2(R) = \begin{cases} e_{j,k}e_{i,k} & \textit{if } R \textit{ denotes } A_{i,k}A_{j,k}A_{i,j} = A_{j,k}A_{i,j}A_{i,k}, \\ e_{i,j}e_{i,k} & \textit{if } R \textit{ denotes } A_{i,k}A_{j,k}A_{i,j} = A_{i,j}A_{i,k}A_{j,k}, \\ e_{i,j}e_{k,l} & \textit{if } R \textit{ denotes } A_{k,l}A_{i,j} = A_{i,j}A_{k,l}, \\ e_{j,k}e_{i,l} & \textit{if } R \textit{ denotes } A_{i,l}A_{j,k} = A_{j,k}A_{i,l}, \\ e_{i,k}e_{j,l} & \textit{if } R \textit{ denotes } A_{j,l}A_{i,k}^{A_{j,k}} = A_{i,k}^{A_{j,k}}A_{j,l}. \end{cases}$$

Proof. With respect to the ordering

$$e_{1,2}e_{2,3} < e_{1,2}e_{1,3} < e_{2,3}e_{1,3} < \cdots$$
$$\cdots < e_{1,2}e_{1,n} < e_{2,3}e_{1,n} < e_{1,3}e_{1,n} < \cdots < e_{2,n}e_{1,n}$$

of the basis elements of C_2', the matrix of ψ is lower triangular, with ones on the diagonal, hence is an isomorphism. The second assertion of the proposition is a routine computation. □

Corollary 3.3. *The map $\phi_2 : C_2 \to C_2'$ is injective.*

Restricting our attention to the truncated complex $C_2 \to C_1 \to C_0$ (which we continue to denote by C), we observe that the chain map $\phi : C \to C'$ is injective. Letting Q denote the quotient, we have an exact sequence of complexes $C \to C' \to Q$, with C' acyclic. Note that since $C_i = C_i'$ for $i = 0$ and $i = 1$, we have $Q_0 = 0$ and $Q_1 = 0$. Passing to homology, we obtain $H_i(C) = H_{i+1}(Q)$ since $H_i(C') = 0$. In particular, we have $B = H_1(C) = H_2(Q)$. We obtain our presentation of B by modifying the standard Λ-free resolution of $\mathbb{Z}$ (i.e. the complex C') as indicated in the commutative diagram

$$\begin{array}{ccccccccccccc}
 & & & & 0 & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} \\
 & & & & \downarrow & & \downarrow{\scriptstyle \psi\circ\phi_2} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
\cdots & \xrightarrow{d_5} & C_4' & \xrightarrow{d_4} & C_3' & \xrightarrow{\tilde{d}_3} & C_2' & \xrightarrow{\tilde{d}_2} & C_1' & \xrightarrow{d_1} & C_0' & \xrightarrow{\epsilon} & \mathbb{Z} \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \pi} & & \downarrow & & & & \\
\cdots & \xrightarrow{d_5} & Q_4 & \xrightarrow{d_4} & Q_3 & \xrightarrow{\Delta} & Q_2 & \longrightarrow & 0 & & & &
\end{array}$$

where ϵ denotes the augmentation map, $\tilde{d}_2 = d_2 \circ \psi^{-1}$, $\tilde{d}_3 = \psi \circ d_3$, $\pi : C_2' \to Q_2 = C_2' / \operatorname{Im}(\psi \circ \phi_2)$ denotes the natural projection, and $\Delta = \pi \circ \tilde{d}_3$. Since $\operatorname{Im}(\psi \circ \phi_2)$ is a direct summand of C_2', the quotient $Q_2 = \operatorname{span}\{e_{i,j}e_{j,k} \mid 1 \le i < j < k \le n\}$ is a free Λ-module, and $\Delta : Q_3 \to Q_2$ provides a presentation for $B = H_2(Q)$.

Carrying out the various steps indicated above, we obtain

Theorem 3.4. *The Alexander invariant $B = B(P_n)$ of the pure braid group P_n has a presentation $\Lambda^a \xrightarrow{\Delta} \Lambda^b \to B \to 0$, with $b = \binom{N}{2} - b_2(P_n) = \binom{n}{3}$ generators, and $a = \binom{N}{3}$ relations. This presentation is given by $\Delta = \pi \circ \psi \circ d_3$, where*

$$d_3(e_{r,s}e_{i,j}e_{k,l}) = (1 - A_{k,l})e_{r,s}e_{i,j} - (1 - A_{i,j})e_{r,s}e_{k,l} + (1 - A_{r,s})e_{i,j}e_{k,l}$$

and

$$\pi \circ \psi(e_{r,s}e_{i,j}) = \begin{cases} (A_{r,s} - 1)A_{i,j}e_{i,s}e_{s,j} & \text{if } r < i < s < j, \\ -A_{r,j}e_{r,s}e_{s,j} & \text{if } r = i < s < j, \\ A_{i,s}A_{r,s}e_{i,r}e_{r,s} & \text{if } i < r < s = j, \\ e_{r,s}e_{s,j} & \text{if } r < s = i < j, \\ 0 & \text{if } r < s < i < j \text{ or } i < r < s < j. \end{cases}$$

A long sequence of elementary row operations (which we suppress) has the effect of replacing the map Δ by the map $\Omega = \Delta \circ \Xi$, where Ξ is a Λ-linear automorphism of Λ^a. This yields the following simplified presentation for B.

THEOREM 3.5. *The Alexander invariant B of the pure braid group P_n has a presentation $\Lambda^{\binom{N}{3}} \xrightarrow{\Omega} \Lambda^{\binom{n}{3}} \to B \to 0$, where $N = \binom{n}{2}$ and*

$$\Omega(e_{r,s}e_{i,j}e_{k,l}) = \begin{cases} (1 - A_{r,s}A_{r,j}A_{s,j})e_{r,s}e_{s,j} & \text{if } r = k < s = i < j = l, \\ (1 - A_{j,l}A_{r,l}A_{s,l})e_{r,s}e_{s,j} & \text{if } r = k < s = i < j < l, \\ (1 - A_{k,j})e_{k,r}e_{r,s} + (1 - A_{k,j})e_{r,s}e_{s,j} & \text{if } k < r = i < s < j = l, \\ (1 - A_{r,l})e_{i,r}e_{r,s} + (1 - A_{i,s})e_{r,s}e_{s,l} & \text{if } i < r < s = j = k < l, \\ (1 - A_{j,l})e_{r,s}e_{s,j} + (1 - A_{r,s})e_{s,j}e_{j,l} & \text{if } r < s = i < j = k < l, \\ A_{s,l}(A_{r,l} - 1)e_{r,s}e_{s,j} + (1 - A_{s,j})e_{r,j}e_{j,l} & \text{if } r < s = i = k < j < l, \\ A_{k,j}(A_{k,j} - 1)e_{r,k}e_{k,s} + (1 - A_{k,j})e_{r,s}e_{s,j} & \text{if } r < k < s = i < j = l, \\ A_{s,l}(A_{j,l} - 1)e_{r,s}e_{s,j} + (1 - A_{r,s})e_{r,j}e_{j,l} & \text{if } r = i = k < s < j < l, \\ A_{r,l}A_{s,l}(1 - A_{i,l})e_{i,r}e_{r,s} + (1 - A_{r,s})e_{i,r}e_{r,l} & \text{if } i < r = k < s = j < l, \\ A_{s,l}A_{j,l}(1 - A_{s,l})e_{r,s}e_{s,j} + (1 - A_{r,j})e_{r,s}e_{s,l} & \text{if } r = i < s = k < j < l, \\ A_{s,j}A_{i,j}(1 - A_{i,j})e_{r,s}e_{s,i} + (1 - A_{i,j})e_{r,s}e_{s,j} & \text{if } r < s = k < i < j = l, \\ (1 - A_{k,l})e_{r,s}e_{s,j} & \text{if } r < s = i < j < k < l, \\ (1 - A_{k,l})e_{r,s}e_{s,j} & \text{if } r < s = i < k < j < l, \\ (1 - A_{k,l})e_{r,s}e_{s,j} & \text{if } r = i < k < s < j < l, \\ (1 - A_{k,l})e_{r,s}e_{s,j} & \text{if } k < r < s = i < j < l, \\ (1 - A_{r,s})e_{i,j}e_{j,l} & \text{if } r < s < i < j = k < l, \\ (1 - A_{r,s})e_{i,j}e_{j,l} & \text{if } r < i = k < s < j < l, \\ (1 - A_{r,s})e_{i,j}e_{j,l} & \text{if } i < r < s < j = k < l, \\ (1 - A_{i,j})e_{r,s}e_{s,l} & \text{if } r < s = k < i < j < l, \\ (1 - A_{i,j})e_{r,s}e_{s,l} & \text{if } r = k < i < s < j < l, \\ (1 - A_{r,s})e_{k,i}e_{i,j} & \text{if } r < k < i < s < j = l, \\ 0 & \text{otherwise.} \end{cases}$$

When written in a suitably ordered basis, the matrix of Ω takes a particularly nice "block lower-triangular form." For example, the presentation matrix for $B(P_4)$ is:

$$\Omega = \begin{pmatrix} 1 - A_{1,2}A_{1,3}A_{2,3} & 0 & 0 & 0 \\ 1 - A_{1,4}A_{2,4}A_{3,4} & 0 & 0 & 0 \\ 0 & 1 - A_{2,3}A_{2,4}A_{3,4} & 0 & 0 \\ 1 - A_{3,4} & 1 - A_{1,2} & 0 & 0 \\ 1 - A_{2,4} & 1 - A_{1,3} & 0 & 0 \\ 1 - A_{1,4} & 1 - A_{1,4} & 0 & 0 \\ 0 & 0 & 1 - A_{1,3}A_{1,4}A_{3,4} & 0 \\ A_{2,4}(A_{3,4} - 1) & 0 & 1 - A_{1,2} & 0 \\ A_{2,4}(A_{2,4} - 1) & 0 & 1 - A_{2,4} & 0 \\ A_{2,4}(A_{1,4} - 1) & 0 & 1 - A_{2,3} & 0 \\ 0 & 0 & 0 & 1 - A_{1,2}A_{1,4}A_{2,4} \\ A_{2,4}A_{3,4}(1 - A_{3,4}) & 0 & 0 & 1 - A_{3,4} \\ A_{2,4}A_{3,4}(1 - A_{2,4}) & 0 & 0 & 1 - A_{1,3} \\ A_{2,4}A_{3,4}(1 - A_{1,4}) & 0 & 0 & 1 - A_{2,3} \end{pmatrix}.$$

Let $J = \operatorname{Im}(\Omega)$ denote the submodule of relations of Ω. In other words, J is the submodule of $\Lambda^{\binom{n}{3}}$ generated by the rows of the presentation matrix of Ω.

Corollary 3.6. *The module J is generated by*

$$G_1 = \left\{ \begin{array}{ll} \gamma_{r,s,k} = (1 - A_{r,s}A_{r,k}A_{s,k})e_{r,s}e_{s,k}, & \\ \gamma^q_{r,s,k} = (1 - A_{r,q}A_{s,q}A_{k,q})e_{r,s}e_{s,k} & \text{for } k < q \leq n \end{array} \right\},$$

$$G_2 = \left\{ \begin{array}{rl} (1 - A_{k,q})e_{r,s}e_{s,k} + (1 - A_{r,s})e_{s,k}e_{k,q}, & \\ (1 - A_{s,q})e_{r,s}e_{s,k} + (1 - A_{r,k})e_{s,k}e_{k,q}, & \\ (1 - A_{r,q})e_{r,s}e_{s,k} + (1 - A_{r,q})e_{s,k}e_{k,q}, & \\ -A_{s,q}(1 - A_{k,q})e_{r,s}e_{s,k} + (1 - A_{r,s})e_{r,k}e_{k,q}, & \\ -A_{s,q}(1 - A_{s,q})e_{r,s}e_{s,k} + (1 - A_{s,q})e_{r,k}e_{k,q}, & \\ -A_{s,q}(1 - A_{r,q})e_{r,s}e_{s,k} + (1 - A_{s,k})e_{r,k}e_{k,q}, & \\ A_{s,q}A_{k,q}(1 - A_{k,q})e_{r,s}e_{s,k} + (1 - A_{k,q})e_{r,s}e_{s,q}, & \\ A_{s,q}A_{k,q}(1 - A_{s,q})e_{r,s}e_{s,k} + (1 - A_{r,k})e_{r,s}e_{s,q}, & \\ A_{s,q}A_{k,q}(1 - A_{r,q})e_{r,s}e_{s,k} + (1 - A_{s,k})e_{r,s}e_{s,q} & \text{for } k < q \leq n \end{array} \right\},$$

$$G_3 = \left\{ \gamma^{p,q}_{r,s,k} = (1 - A_{p,q})e_{r,s}e_{s,k} \quad \text{for } p, q \notin \{r, s\} \text{ and } k < p < q \leq n \right\},$$

for $1 \leq r < s < k \leq n$.

The elements $\gamma_{r,s,k}$, $\gamma^q_{r,s,k}$, $\gamma^{p,q}_{r,s,k}$ belong to the submodule $J^k_{r,s} = J \cap \Lambda \cdot e_{r,s}e_{s,k}$, but they do not generate this module. Consider the following sets:

$$G_4 = \left\{ \begin{array}{l} \gamma^{i,j,q}_{r,s,k} = (1 - A_{i,q})(-A_{t,u} + A_{j,q})e_{r,s}e_{s,k} \\ \quad \text{for } i \in \{r, s, k\},\ \{j, t, u\} = \{r, s, k\} \text{ and } k < q \leq n \end{array} \right\},$$

$$G_5 = \left\{ \begin{array}{l} \gamma^{i,j,p,q}_{r,s,k} = (1 - A_{i,p})(1 - A_{j,q})e_{r,s}e_{s,k} \\ \quad \text{for } i, j \in \{r, s, k\} \text{ and } k < p < q \leq n \end{array} \right\},$$

where $1 \leq r < s < k \leq n$. It is readily seen that the elements $\gamma^{i,j,k}_{r,s,k}$, $\gamma^{i,j,p,q}_{r,s,k}$ also belong to the submodule $J^k_{r,s}$. The reason for introducing these sets will become apparent in the next section, where we shall make use of the following fact.

Proposition 3.7. *Let d be a 2×2 minor of Ω that involves the column $e_{r,s}e_{s,k}$ and one of the columns $e_{s,k}e_{k,q}$, $e_{r,k}e_{k,q}$, or $e_{r,s}e_{s,q}$. Then $d \cdot e_{r,s}e_{s,k}$ belongs to the submodule $J^k_{r,s}$, and can be expressed as a Λ-combination of elements from $G^k_{r,s} = (G_1 \cup G_3 \cup G_4 \cup G_5) \cap \Lambda \cdot e_{r,s}e_{s,k}$.*

Proof. This is a straightforward computation. For example,

$$(1 - A_{k,q})(1 - A_{s,k}A_{s,q}A_{k,q})e_{r,s}e_{s,k} = (1 - A_{k,q}) \cdot \gamma^q_{r,s,k} + A_{s,q}A_{k,q} \cdot \gamma^{k,r,s}_{r,s,k},$$

and

$$((1 - A_{r,k})(1 - A_{k,q}) - (1 - A_{r,s})(1 - A_{s,q}))e_{r,s}e_{s,k} = \gamma^{k,s,q}_{r,s,k} - \gamma^{s,k,q}_{r,s,k}.$$

The other 2×2 minors are handled similarly. $\square$

4. The Groebner Basis

Let $\widehat{B}$ be the I-adic completion of B, viewed as a module over $\widehat{\Lambda} \cong \mathbb{Z}[[x_{i,j}]]$, $1 \le i < j \le n$. This module has a presentation $\widehat{\Lambda}^{\binom{N}{3}} \xrightarrow{\widehat{\Omega}} \widehat{\Lambda}^{\binom{n}{3}} \to \widehat{B} \to 0$, where $\widehat{\Omega}$ is obtained from Ω via the Magnus embedding. Let $\widehat{J} = \operatorname{Im}(\widehat{\Omega})$ be the submodule of $\widehat{\Lambda}^{\binom{n}{3}}$ generated by the rows of the matrix of $\widehat{\Omega}$. As noted before, the entries of this matrix belong to the polynomial subring $R = \mathbb{Z}[x_{i,j}]$, and so we may restrict $\widehat{\Omega}$ to a map $\mathbf{\Omega} : R^{\binom{N}{3}} \to R^{\binom{n}{3}}$. Let $\mathbf{J} = \operatorname{Im}(\mathbf{\Omega}) = \widehat{J} \cap R^{\binom{n}{3}}$. We can view the elements of $\mathbf{J}$ as linear forms in the variables $e_{r,s}e_{s,k}$, $1 \le r < s < k \le n$, with coefficients in R. The purpose of this section is to identify a (minimal) Groebner basis $\mathcal{G} = \{g_1, \dots, g_c\}$ for the module $\mathbf{J}$, with respect to a suitable monomial ordering on $R[e_{r,s}e_{s,k}]$. This means that $\langle \text{IN}(\mathbf{J}) \rangle$, the module generated by the initial terms of elements in $\mathbf{J}$, has generating set $\{\text{IN}(g_1), \dots, \text{IN}(g_c)\}$, see e.g. [**CLO**], [**BW**].

It follows from Corollary 3.6 that the R-module $\mathbf{J}$ is generated by $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, where $\mathcal{G}_i$ is the image of G_i in $\mathbb{Z}[x_{i,j}]$ under the assignment $A_{i,j} \mapsto 1 - x_{i,j}$. It will turn out that the required Groebner basis is the larger generating set, $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_2$, where

$$\mathcal{G}_1 = \left\{ \begin{array}{ll} g_{r,s,k} = (1 - (1 - x_{r,s})(1 - x_{r,k})(1 - x_{s,k}))e_{r,s}e_{s,k}, & \\ g^q_{r,s,k} = (1 - (1 - x_{r,q})(1 - x_{s,q})(1 - x_{k,q}))e_{r,s}e_{s,k} & \text{for } k < q \le n \end{array} \right\},$$

$$\mathcal{G}_3 = \{\, g^{p,q}_{r,s,k} = x_{p,q}e_{r,s}e_{s,k} \quad \text{for } p, q \notin \{r, s\} \text{ and } k < p < q \le n \,\},$$

$$\mathcal{G}_4 = \left\{ \begin{array}{l} g^{i,j,q}_{r,s,k} = x_{i,q}(x_{t,u} - x_{j,q})e_{r,s}e_{s,k} \\ \quad \text{for } i \in \{r, s, k\},\ \{j, t, u\} = \{r, s, k\} \text{ and } k < q \le n \end{array} \right\},$$

$$\mathcal{G}_5 = \{\, g^{i,j,p,q}_{r,s,k} = x_{i,p}x_{j,q}e_{r,s}e_{s,k} \quad \text{for } i, j \in \{r, s, k\} \text{ and } k < p < q \le n \,\},$$

$$\mathcal{G}_2 = \left\{ \begin{array}{rl} x_{k,q}e_{r,s}e_{s,k} + x_{r,s}e_{s,k}e_{k,q}, & \\ x_{s,q}e_{r,s}e_{s,k} + x_{r,k}e_{s,k}e_{k,q}, & \\ x_{r,q}e_{r,s}e_{s,k} + x_{r,q}e_{s,k}e_{k,q}, & \\ -x_{k,q}(1 - x_{s,q})e_{r,s}e_{s,k} + x_{r,s}e_{r,k}e_{k,q}, & \\ -x_{s,q}(1 - x_{s,q})e_{r,s}e_{s,k} + x_{s,q}e_{r,k}e_{k,q}, & \\ -x_{r,q}(1 - x_{s,q})e_{r,s}e_{s,k} + x_{s,k}e_{r,k}e_{k,q}, & \\ x_{k,q}(1 - x_{s,q})(1 - x_{k,q})e_{r,s}e_{s,k} + x_{k,q}e_{r,s}e_{s,q}, & \\ x_{s,q}(1 - x_{s,q})(1 - x_{k,q})e_{r,s}e_{s,k} + x_{r,k}e_{r,s}e_{s,q}, & \\ x_{r,q}(1 - x_{s,q})(1 - x_{k,q})e_{r,s}e_{s,k} + x_{s,k}e_{r,s}e_{s,q} & \text{for } k < q \le n \end{array} \right\},$$

for $1 \le r < s < k \le n$.

We first have to define a monomial ordering on our polynomial ring. Start by ordering the variables in $R = \mathbb{Z}[x_{i,j}]$ as follows:

$$x_{1,2} > x_{2,3} > x_{1,3} > x_{3,4} > x_{2,4} > x_{1,4} > \cdots\cdots > x_{n-1,n} > \cdots > x_{1,n},$$

and extend this ordering to the graded reverse lexicographic ("grevlex") order on the set of monomials in R. Recall briefly how this is done (see [**CLO**] for details). Let x^α denote an arbitrary monomial in the variables $x_{i,j}$, and let $\deg(x^\alpha)$ be its multidegree. We say that $x^\alpha > x^\beta$ if $\deg(\alpha) > \deg(\beta)$, or $\deg(\alpha) = \deg(\beta)$ and, in $\alpha - \beta$, the right-most entry is negative. For a polynomial $f \in R$, we will denote by $\text{IN}(f)$ its highest term $c_\alpha x^\alpha$, and by $\deg(f)$ its multidegree α.

Finally, extend the grevlex ordering on R to one on $R[e_{r,s}e_{s,k}]$ by requiring that $x_{i,j} < e_{r,s}e_{s,k}$ and

$$e_{1,2}e_{2,3} < e_{2,3}e_{3,4} < e_{1,3}e_{3,4} < e_{1,2}e_{2,4} < \cdots < e_{n-2,n-1}e_{n-1,n} < \cdots < e_{1,2}e_{2,n}.$$

Theorem 4.1. *The elements of $\mathcal{G}$ constitute a minimal Groebner basis for the module* $\mathbf{J}$ *with respect to the above monomial ordering.*

Proof. To prove that $\mathcal{G} = \{g_1, \dots, g_c\}$ is a Groebner basis for $\mathbf{J}$, it is enough to check that all the S-polynomials $S(g_i, g_j)$ either vanish or reduce to zero modulo $\mathcal{G}$, see [**CLO**]. Recall that the syzygy polynomial of f and g is

$$S(f,g) = \frac{\text{LCM}(\text{IN}(f), \text{IN}(g))}{\text{IN}(f)} \cdot f - \frac{\text{LCM}(\text{IN}(f), \text{IN}(g))}{\text{IN}(g)} \cdot g,$$

and that f reduces to zero modulo $\mathcal{G}$ if there is f can be written as $f = a_1 g_1 + a_2 g_2 + \cdots + a_c g_c$, with $\deg(a_l g_l) \le \deg f$. There is an even stronger criterion (see [**BW**]): The set $\mathcal{G} = \{g_1, \dots, g_c\}$ is a Groebner basis if, for all i, j, either $S(g_i, g_j) = 0$, or $S(g_i, g_j) = a_1 g_1 + \cdots + a_c g_c$ and $\text{IN}(a_l g_l) < \text{LCM}(\text{IN}(g_i), \text{IN}(g_j))$.

If g_i and g_j are monomials, then clearly $S(g_i, g_j) = 0$. Thus, all the S-polynomials from $\mathcal{G}_3 \cup \mathcal{G}_5$ vanish.

If the initial terms of g_i and g_j have no variables in common, then $S(g_i, g_j)$ reduces to $0 \bmod \mathcal{G}$ ("Buchberger's First Criterion"). Thus, all the following S-polynomials vanish modulo $\mathcal{G}$: those from $\mathcal{G}_1$, those involving elements from $\mathcal{G}_3$, those of the form $S(g_{r,s,k}, g^{i,j,p,q}_{r,s,k})$, and those of the form $S(g^{i,j,q}_{r,s,k}, g^{l,l,q}_{r,s,k})$ with $l \neq i, l \neq j$.

Next, we find standard representations for the S-polynomials from $\mathcal{G}_1 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5$ not covered by the arguments above.

$$\begin{aligned}
S(g_{r,s,k}, g^{r,k,q}_{r,s,k}) &= -x_{s,k} \cdot g^{q}_{r,s,k} + (x_{r,s} - x_{k,q}) \cdot g^{r,r,q}_{r,s,k} + (1 - x_{k,q}) \cdot g^{s,r,q}_{r,s,k} \\
&\quad + (-1 + x_{r,s} + x_{s,k} - x_{s,k}x_{k,q}) \cdot g^{r,s,q}_{r,s,k} + g^{k,r,q}_{r,s,k} \\
&\quad + (-1 + x_{r,q} + x_{s,q}) \cdot g^{r,k,q}_{r,s,k}, \\
S(g^{q}_{r,s,k}, g^{r,k,q}_{r,s,k}) &= x_{k,q} \cdot g^{q}_{r,s,k} + (1 - x_{k,q}) \cdot g^{s,k,q}_{r,s,k} + g^{k,k,q}_{r,s,k} \\
&\quad + (1 - x_{s,q} - x_{k,q}) \cdot g^{r,k,q}_{r,s,k}, \\
S(g^{q}_{r,s,k}, g^{k,r,p,q}_{r,s,k}) &= (1 - x_{s,q} - x_{k,q}) \cdot g^{k,r,p,q}_{r,s,k} + (1 - x_{k,q}) \cdot g^{k,s,p,q}_{r,s,k} + g^{k,k,p,q}_{r,s,k},
\end{aligned}$$

$$S(g_{r,s,k}^{r,k,q}, g_{r,s,k}^{s,k,q}) = x_{r,q} \cdot (g_{r,s,k}^{s,k,q} - g_{r,s,k}^{k,s,q}),$$
$$S(g_{r,s,k}^{r,k,q}, g_{r,s,k}^{k,r,p,q}) = -x_{r,q} \cdot g_{r,s,k}^{k,k,p,q}.$$

It is readily verified that all these polynomials reduce to 0 modulo $\mathcal{G}$. The remaining S-polynomials are obtained from the ones above by permuting indices.

We are left with computing the S-polynomials involving at least one element from $\mathcal{G}_2$. For simplicity, let us write a typical element in $\mathcal{G}_2$ as $ge_1 + xe_2$, where $e_1 = e_{r,s}e_{s,k}$, $e_2 = e_{t,u}e_{u,q}$, with $\{t,u\} \subset \{r,s,k\}$, and $x = x_{i,j}$ is a (linear) monomial with indices $\{i,j\} \subset \{r,s,k,q\}$, but $\{i,j\} \not\subset \{t,u,q\}$. Since $e_1 < e_2$, we have

$$S(ge_1 + xe_2, g'e_1 + x'e_2) = (x'g - xg')e_1,$$

and this clearly reduces to 0 mod $\mathcal{G}$ by Proposition 3.7.

Now let fe_2 be an arbitrary element from $\mathcal{G} \cap R \cdot e_{t,u}e_{u,q}$. Notice that the polynomial f does not involve the variable x. Hence:

$$\begin{aligned} S(ge_1 + xe_2, fe_2) &= \text{IN}(f) \cdot (ge_1 + xe_2) - x \cdot fe_2 \\ &= fge_1 - (f - \text{IN}(f)) \cdot (ge_1 + xe_2), \end{aligned}$$

where fge_1 can be written as an R-combination of elements in $\mathcal{G}$ by Proposition 3.7. We claim that $S(ge_1+xe_2, fe_2)$ satisfies the criterion mentioned above. There are two possibilities to consider: If $\text{IN}(ge_1 + xe_2) = ge_1$, then we are done, since ge_1 and fe_2 involve separate variables, as can be easily checked. If $\text{IN}(ge_1 + xe_2) = xe_2$, then $\text{LCM}(\text{IN}(ge_1 + xe_2), \text{IN}(fe_2)) = \text{IN}(f)xe_2$, $\text{IN}(fge_1) = \text{IN}(f)\text{IN}(g)e_1 < \text{IN}(f)xe_2$, and $\text{IN}(f - \text{IN}(f)) \cdot (ge_1 + xe_2) = \text{IN}(f - \text{IN}(f))xe_2 < \text{IN}(f)xe_2$. This proves the claim.

The Groebner basis $\mathcal{G}$ is minimal in the sense that each of its elements has leading coefficient 1, and, for all $g_i \in \mathcal{G}$, $\text{IN}(g_i) \notin \langle \text{IN}(\mathcal{G} - \{g_i\}) \rangle$. □

5. The Hilbert Series

We now pass from the completion $\widehat{\Lambda} \cong \mathbb{Z}[[x_{i,j}]]$ to the associated graded ring $\text{gr}(\widehat{\Lambda}) \cong \mathbb{Z}[x_{i,j}]$, and compute the Hilbert series of the graded module $\text{gr}(\widehat{B})$ associated to $\widehat{B}$. We first find a minimal Groebner basis $\mathcal{H}$ for $\mathcal{J} := \text{LT}(\mathbf{J})$, the module over $R = \mathbb{Z}[x_{i,j}]$ of leading forms of $\mathbf{J}$.

Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_3 \cup \mathcal{H}_4 \cup \mathcal{H}_5 \cup \mathcal{H}_2$, where

$$\mathcal{H}_1 = \left\{ \begin{array}{l} h_{r,s,k} = \text{LT}(g_{r,s,k}) = (x_{r,s} + x_{r,k} + x_{s,k})e_{r,s}e_{s,k}, \\ h_{r,s,k}^q = \text{LT}(g_{r,s,k}^q) = (x_{r,q} + x_{s,q} + x_{k,q})e_{r,s}e_{s,k} \quad \text{for } k < q \leq n \end{array} \right\},$$

$$\mathcal{H}_3 = \{ h_{r,s,k}^{p,q} = \text{LT}(g_{r,s,k}^{p,q}) = x_{p,q}e_{r,s}e_{s,k} \quad \text{for } k < p < q \leq n \},$$

$$\mathcal{H}_4 = \left\{ \begin{array}{l} h_{r,s,k}^{i,i,q} = \text{LT}(g_{r,s,k}^{i,j,q}) = x_{i,q}(x_{t,k} - x_{j,q})e_{r,s}e_{s,k} \\ \quad \text{for } i,j \in \{r,s\},\ \{j,t\} = \{r,s\}, \text{ and } k < q \leq n \end{array} \right\},$$

$$\mathcal{H}_5 = \left\{ \begin{array}{c} h^{i,j,p,q}_{r,s,k} = \text{LT}(g^{i,j,p,q}_{r,s,k}) = x_{i,p}x_{j,q}e_{r,s}e_{s,k} \\ \text{for } i,j \in \{r,s\} \text{ and } k < p < q \leq n \end{array} \right\},$$

$$\mathcal{H}_2 = \left\{ \begin{array}{ll} x_{s,k}e_{p,r}e_{r,s} + x_{p,r}e_{r,s}e_{s,k}, & -x_{s,k}e_{r,p}e_{p,s} + x_{r,p}e_{r,s}e_{s,k}, \\ x_{r,k}e_{p,r}e_{r,s} + x_{p,s}e_{r,s}e_{s,k}, & -x_{p,k}e_{r,p}e_{p,s} + x_{p,s}e_{r,s}e_{s,k}, \\ x_{p,k}e_{p,r}e_{r,s} + x_{p,k}e_{r,s}e_{s,k}, & -x_{r,k}e_{r,p}e_{p,s} + x_{p,k}e_{r,s}e_{s,k}, \\ x_{p,k}e_{r,s}e_{s,p} + x_{p,k}e_{r,s}e_{s,k}, & x_{s,k}e_{r,s}e_{s,p} + x_{r,p}e_{r,s}e_{s,k}, \\ x_{r,k}e_{r,s}e_{s,p} + x_{s,p}e_{r,s}e_{s,k} & \text{for } 1 \leq p < k,\ p \notin \{r,s\} \end{array} \right\},$$

for $1 \leq r < s < k \leq n$. Note that $\mathcal{H}_1 = \text{LT}(\mathcal{G}_1)$, $\mathcal{H}_3 = \text{LT}(\mathcal{G}_3)$, $\mathcal{H}_4 \subset \text{LT}(\mathcal{G}_4)$, and $\mathcal{H}_5 \subset \text{LT}(\mathcal{G}_5)$. Also observe that $\mathcal{H}_2 = \text{LT}(\mathcal{G}_2)$ consists of the leading terms of $\mathcal{G}_2$, but with slightly different indexing.

Proposition 5.1. *The elements of $\mathcal{H}$ constitute a minimal Groebner basis for the module $\mathcal{J} = \text{LT}(\mathbf{J})$.*

Proof. By Theorem 4.1 and the remarks in section 2, the elements of the set $\text{LT}(\mathcal{G})$ form a Groebner basis for $\mathcal{J}$. However, this basis is not minimal. Compute:

$$\begin{aligned}
\text{LT}(g^{r,k,q}_{r,s,k}) &= x_{r,q} \cdot h_{r,s,k} - x_{r,q} \cdot h^{q}_{r,s,k} - h^{r,s,q}_{r,s,k} - h^{r,r,q}_{r,s,k}, \\
\text{LT}(g^{s,k,q}_{r,s,k}) &= x_{s,q} \cdot h_{r,s,k} - x_{s,q} \cdot h^{q}_{r,s,k} - h^{s,r,q}_{r,s,k} - h^{s,s,q}_{r,s,k}, \\
\text{LT}(g^{k,s,q}_{r,s,k}) &= (x_{r,k} - x_{s,q}) \cdot h^{q}_{r,s,k} - h^{r,s,q}_{r,s,k} - h^{s,s,q}_{r,s,k}, \\
\text{LT}(g^{k,r,q}_{r,s,k}) &= (x_{s,k} - x_{r,q}) \cdot h^{q}_{r,s,k} - h^{s,r,q}_{r,s,k} - h^{r,r,q}_{r,s,k}, \\
\text{LT}(g^{k,k,q}_{r,s,k}) &= x_{k,q} \cdot h_{r,s,k} + h^{r,r,q}_{r,s,k} + h^{s,s,q}_{r,s,k} + h^{r,s,q}_{r,s,k} + h^{s,r,q}_{r,s,k} \\
&\quad + (x_{r,q} + x_{s,q} - x_{r,k} - x_{s,k} - x_{k,q}) \cdot h^{q}_{r,s,k}, \\
\text{LT}(g^{k,s,p,q}_{r,s,k}) &= x_{s,q} \cdot h^{p}_{r,s,k} - h^{r,s,p,q}_{r,s,k} - h^{s,s,p,q}_{r,s,k}, \\
\text{LT}(g^{k,r,p,q}_{r,s,k}) &= x_{r,q} \cdot h^{p}_{r,s,k} - h^{r,r,p,q}_{r,s,k} - h^{s,r,p,q}_{r,s,k}, \\
\text{LT}(g^{s,k,p,q}_{r,s,k}) &= x_{s,p} \cdot h^{q}_{r,s,k} - h^{s,r,p,q}_{r,s,k} - h^{s,s,p,q}_{r,s,k}, \\
\text{LT}(g^{r,k,p,q}_{r,s,k}) &= x_{r,p} \cdot h^{q}_{r,s,k} - h^{r,r,p,q}_{r,s,k} - h^{r,s,p,q}_{r,s,k}, \\
\text{LT}(g^{k,k,p,q}_{r,s,k}) &= x_{k,p} \cdot h^{q}_{r,s,k} + h^{r,s,p,q}_{r,s,k} + h^{s,r,p,q}_{r,s,k} + h^{r,r,p,q}_{r,s,k} + h^{s,s,p,q}_{r,s,k} \\
&\quad - (x_{r,q} + x_{s,q}) \cdot h^{p}_{r,s,k}.
\end{aligned}$$

Since the elements of $\mathcal{H}$ have pairwise distinct initial terms, and each has leading coefficient 1, they form a minimal Groebner basis. □

In order to facilitate the Hilbert series computation, we now define a filtration of the module $\mathcal{J} = \text{LT}(\mathbf{J})$. For each k, $3 \leq k \leq n$, let $\mathcal{J}^k = \mathcal{J} \cap R\{e_{r,s}e_{s,j} \mid 1 \leq r < s < j \leq k\}$, and let $Q^k = \mathcal{J}^k/\mathcal{J}^{k-1}$. Note that $\mathcal{J}^k \cap R[k-1] = \mathcal{J}^{k-1}$, where $R[\ell] = R\{e_{r,s}e_{s,j} \mid 1 \leq r < s < j \leq \ell\}$. A routine diagram chase yields an inclusion $Q^k \hookrightarrow R[k]/R[k-1]$, so we view Q^k as a submodule of $R\{e_{r,s}e_{s,k} \mid 1 \leq r < s < k\}$.

For each fixed $\{r,s\}$ with $r<s<k$, let $Q^k_{r,s} = Q^k \cap R\cdot e_{r,s}e_{s,k}$. We view the module $Q^k_{r,s}$ as an ideal in R in the obvious manner. We then have

Proposition 5.2.
(i) $Q^k = \bigoplus_{1\le r<s<k} Q^k_{r,s}$.
(ii) *The Hilbert series,* $H(\mathrm{gr}(\widehat{B}),t)$*, of the graded module* $\mathrm{gr}(\widehat{B})$ *is given by*

$$H(\mathrm{gr}(\widehat{B}),t) = \sum_{k=3}^{n} H(Q^k,t).$$

Furthermore, the Hilbert series of the module Q^k *is given by*

$$H(Q^k,t) = \sum_{1\le r<s<k} H(Q^k_{r,s},t),$$

the sum of the Hilbert series of the **ideals** $Q^k_{r,s}$.

Thus the problem is reduced to computing the Hilbert series of the ideals $Q^k_{r,s}$. First we find minimal Groebner bases for these ideals. Suppressing the index $e_{r,s}e_{s,k}$, let

$$\mathcal{H}^k_{r,s} = \left\{ \begin{array}{l} h_{r,s,k} = x_{r,s}+x_{r,k}+x_{s,k}, \\ h^q_{r,s,k} = x_{r,q}+x_{s,q}+x_{k,q} \quad \text{for } k<q\le n, \\ h^{i,j,q}_{r,s,k} = x_{i,q}(x_{t,k}-x_{j,q}) \quad \text{for } i\in\{r,s\},\ \{j,t\}=\{r,s\},\ k<q\le n, \\ h^{i,j,p,q}_{r,s,k} = x_{i,p}x_{j,q} \quad \text{for } i,j\in\{r,s\} \text{ and } k<p<q\le n, \\ h^{p,q}_{r,s,k} = x_{p,q} \quad \text{for } p\notin\{r,s,k\},\ p<q, \text{ and } k\le q\le n \end{array} \right\}.$$

Proposition 5.3. *The elements of* $\mathcal{H}^k_{r,s}$ *constitute a minimal Groebner basis for the ideal* $Q^k_{r,s}$.

Proof. This follows from Proposition 5.1 and the definition of the ideal $Q^k_{r,s}$. □

Proposition 5.4. *For* $1\le r<s<k$ *and* $1\le u<v<k$*, we have* $H(Q^k_{r,s},t) = H(Q^k_{u,v},t)$.

Proof. The automorphism of the polynomial ring $R=\mathbb{Z}[x_{i,j}]$, $1\le i<j\le n$, defined by interchanging the pairs of indeterminates $\{x_{r,s},x_{u,v}\}$, $\{x_{r,l},x_{u,l}\}$, and $\{x_{s,l},x_{v,l}\}$ for each l, $k\le l\le n$, maps $Q^k_{r,s}$ isomorphically onto $Q^k_{u,v}$. □

Proposition 5.5. *For* $1\le r<s<k$*, the Hilbert series of the ideal* $Q^k_{r,s}$ *is given by*

$$H(Q^k_{r,s},t) = \frac{1+2(n-k)t-(n-k)t^2}{(1-t)^2}.$$

Proof. By the above result, it suffices to consider $Q^k_{1,2}$. This ideal is generated by the elements of the set $\mathcal{H}^k_{1,2}$.

Define a linear coordinate change by

$$y_{i,j} = \begin{cases} x_{1,2} + x_{1,k} + x_{2,k} & \text{if } (i,j) = (1,2), \\ x_{1,q} + x_{2,q} + x_{k,q} & \text{if } (i,j) = (k,q),\ k < q \leq n, \\ x_{i,j} & \text{otherwise.} \end{cases}$$

In $\mathbb{Z}[y_{i,j}]$, $1 \leq i < j \leq n$, the indeterminates $y_{p,q}$, for $(p,q) \neq (1,l), (2,l)$ and $k \leq l \leq n$, are generators of the ideal $Q^k_{1,2}$. Thus, we may restrict our attention to the ring $\mathbb{Z}[y_{1,k}, y_{2,k}, \dots, y_{1,n}, y_{2,n}]$, and consider the ideal $\mu^k_{1,2}$ generated by

$$\begin{array}{ll} y_{1,p}y_{1,q},\ y_{1,p}y_{2,q},\ y_{2,p}y_{1,q},\ y_{2,p}y_{2,q} & \text{for } k < p < q \leq n, \\ (y_{1,k} - y_{2,q})y_{1,q},\ (y_{1,k} - y_{2,q})y_{2,q} & \text{for } k < q \leq n, \\ (y_{2,k} - y_{1,q})y_{1,q},\ (y_{2,k} - y_{1,q})y_{2,q} & \text{for } k < q \leq n. \end{array}$$

Define another linear coordinate change by

$$z_{1,k} = y_{1,k} - \sum_{p=k+1}^{n} y_{2,p}, \quad z_{2,k} = y_{2,k} - \sum_{p=k+1}^{n} y_{1,p}, \quad \text{and} \quad z_{i,j} = y_{i,j} \quad \text{otherwise.}$$

Then in $\mathbb{Z}[z_{1,k}, z_{2,k}, \dots, z_{1,n}, z_{2,n}]$, the ideal $\mu^k_{1,2}$ is generated by

$$\begin{array}{ll} z_{1,p}z_{1,q},\ z_{1,p}z_{2,q},\ z_{2,p}z_{1,q},\ z_{2,p}z_{2,q} & \text{for } k < p < q \leq n, \\ z_{1,q}(z_{1,k} + \sum\limits_{\substack{p=k+1 \\ p\neq q}}^{n} z_{2,p}),\ z_{1,q}(z_{2,k} + \sum\limits_{\substack{p=k+1 \\ p\neq q}}^{n} z_{1,p}) & \text{for } k < q \leq n, \\ z_{2,q}(z_{1,k} + \sum\limits_{\substack{p=k+1 \\ p\neq q}}^{n} z_{2,p}),\ z_{2,q}(z_{2,k} + \sum\limits_{\substack{p=k+1 \\ p\neq q}}^{n} z_{1,p}) & \text{for } k < q \leq n. \end{array}$$

Since $z_{1,q}z_{1,p}$, $z_{1,q}z_{2,p}$, $z_{2,q}z_{1,p}$, $z_{2,q}z_{2,p}$ are generators of $\mu^k_{1,2}$ for $k < p < q \leq n$, we observe that the ideal $\mu^k_{1,2}$ is in fact generated by

$$z_{1,p}z_{1,q},\ z_{1,p}z_{2,q},\ z_{2,p}z_{1,q},\ z_{2,p}z_{2,q} \quad \text{for } k \leq p < q \leq n.$$

We now find the Hilbert series of $\mu^k_{1,2}$. By the above remarks, we have $H(\mu^k_{1,2}, t) = H(Q^k_{1,2}, t)$. As $\mu^k_{1,2}$ is a monomial ideal, the m^{th} term of the Hilbert series of $\mu^k_{1,2}$ is given by number of monomials in $\mathbb{Z}[z_{1,k}, z_{2,k}, \dots, z_{1,n}, z_{2,n}]$ of total degree m not in $\mu^k_{1,2}$. Evidently, these monomials are of the form $z^{\alpha}_{1,q}z^{m-\alpha}_{2,q}$, $k \leq q \leq n$, $0 \leq \alpha \leq m$. Hence we have

$$H(\mu^k_{1,2}, t) = 1 + \sum_{m\geq 1} (m+1)(n-k+1)t^m = \frac{1 + 2(n-k)t - (n-k)t^2}{(1-t)^2}. \quad \square$$

Combining the above results, we obtain

THEOREM 5.6. *The Hilbert series of the graded module* $\mathrm{gr}(\widehat{B})$ *is given by*

$$H(\mathrm{gr}(\widehat{B}),t) = \sum_{k=3}^{n} \binom{k-1}{2} \cdot \frac{1+2(n-k)t-(n-k)t^2}{(1-t)^2} = \binom{n+1}{4} \cdot \frac{1}{(1-t)^2} - \binom{n}{4}. \qquad \square$$

Note added in proof. We have recently found a counterexample to the conjecture stated in the Introduction, see [**CS3**].

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