

SIGMA-INVARIANTS AND TROPICAL VARIETIES

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ABSTRACT. The Bieri–Neumann–Strebel–Renz invariants $\Sigma^q(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ of a connected, finite-type CW-complex X are the vanishing loci for Novikov–Sikorav homology in degrees up to q , while the characteristic varieties $\mathcal{V}^q(X) \subset H^1(X, \mathbb{C}^\times)$ are the nonvanishing loci for homology with coefficients in rank 1 local systems in degree q . We show that each BNSR invariant $\Sigma^q(X, \mathbb{Z})$ is contained in the complement of the tropical variety associated to the algebraic variety $\mathcal{V}^{\leq q}(X)$, and provide applications to several classes of groups and spaces.

1. INTRODUCTION

1.1. The Sigma-invariants. In [6], Bieri, Neumann, and Strebel introduced powerful new invariants of finitely generated groups. To each such group G they associated a subset $\Sigma^1(G)$ of the unit sphere $S(G)$ in the real vector space $\text{Hom}(G, \mathbb{R})$. This “geometric” invariant of the group is cut out of the sphere by open cones, and is independent of a finite generating set for G . In [7], Bieri and Renz recast the BNS invariant in homological terms, and defined a nested family of higher-order invariants, $\{\Sigma^q(G, \mathbb{Z})\}_{q \geq 1}$, which record the finiteness properties of normal subgroups of G with abelian quotients.

In [28], Farber, Geoghegan, and Schütz further extended these notions, from groups to spaces. The BNSR invariants of a connected, finite-type CW-complex X form a nested sequence of subsets, $\{\Sigma^q(X, \mathbb{Z})\}_{q \geq 1}$, inside the unit sphere $S(X) \subset H^1(X, \mathbb{R})$. The sphere $S(X)$ can be thought of as parametrizing all free abelian covers of X , while the Σ -invariants (which are again open subsets), keep track of the geometric finiteness properties of those covers. The BNSR invariants $\Sigma^q(X, \mathbb{Z})$ are the vanishing loci for Novikov–Sikorav homology in degrees up to q . The significance of these invariants lies in the fact that they control the finiteness properties of kernels of projections to abelian quotients; in particular, they encode information about algebraic fiberings of the group $G = \pi_1(X)$, i.e., homomorphisms $G \rightarrow \mathbb{Z}$ with finitely generated kernels.

The actual computation of the BNS invariant is enormously complicated, and has been achieved so far only for some special classes of groups, such as metabelian groups [5], one-relator groups [9], right-angled Artin groups [42], Kähler groups [15], and the pure braid groups [37]. If G is the fundamental group of a compact, connected 3-manifold M , the BNS invariant is the projection onto the unit sphere of the fibered faces of the unit ball in Thurston’s norm, introduced in [63] in order to understand all the ways in which M may fiber over the circle. In all these examples, and also for

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the groups studied by Kielak in [36], the set $\Sigma^1(G)$ is the intersection of $S(G)$ with a finite union of open rational polyhedral cones. As shown in [6], though, this is not always the case.

It thus makes sense to look for approximations to the BNSR invariants which (1) are more computable, and (2) are finite unions of open rational polyhedral cones. In a special case, such an approximation was found by McMullen [41], who showed that, for a compact, connected, orientable 3-manifold with empty or toroidal boundary, the Thurston norm unit ball B_T is contained in the Alexander norm unit ball B_A , which is defined in terms of the Newton polytope of the multi-variable Alexander polynomial Δ_G . Using the characteristic varieties (a generalization of the zero-locus of Δ_G), Papadima and the author gave in [48] a much more general upper bound—valid for all BNSR invariants $\Sigma^q(X, \mathbb{Z})$ —albeit not as sharp as McMullen’s in the case of 3-manifolds. Our goal here is to strengthen the bound from [48] so as to largely recover McMullen’s result (at least at the level of the fibered faces of B_T), and also to recast Delzant’s theorem from [15] in this general framework. The tools we use come mostly from the theory of cohomology jump loci, and draw in an essential way on ideas and methods from tropical geometry.

1.2. Tropicalized characteristic varieties. Let X be a space as above, and let $G = \pi_1(X)$ be its fundamental group. The group of complex-valued characters, $\mathbb{T}_G := \text{Hom}(G, \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times)$, is a complex algebraic group, which may be thought of as the moduli space of rank 1 local systems on X . Taking homology with coefficients in such local systems carves out subvarieties $\mathcal{V}^q(X) \subset \mathbb{T}_G$ where the homology in degree q does not vanish. These *characteristic varieties*, which control the Betti numbers of regular abelian covers of X (see e.g. [57, 58, 62]), provide the initial input towards finding computable bounds for the BNSR invariants.

For the next step, we enlarge our moduli space to \mathbb{K} -valued characters, where $\mathbb{K} = \mathbb{C}\{\{t\}\}$ is the field of Puiseux series with complex coefficients. Notably, this is an algebraically closed field, which comes endowed with a non-Archimedean valuation, $\mathbb{K}^\times \rightarrow \mathbb{Q} \subset \mathbb{R}$, see e.g. [40]. Let $\nu_X: H^1(X, \mathbb{K}^\times) \rightarrow H^1(X, \mathbb{R})$ be the homomorphism induced by the coefficient map. Given a subvariety W of the algebraic \mathbb{K} -group $H^1(X, \mathbb{K}^\times)$, we define its *tropicalization*, $\text{Trop}(W)$, to be the closure of $\nu_X(W \times_{\mathbb{C}} \mathbb{K})$ inside the Euclidean space $H^1(X, \mathbb{R})$.

Our main result (proved in Theorem 6.5), reads as follows.

Theorem 1.1. *Let $\mathcal{V}^{\leq q}(X)$ be the union of the characteristic varieties of X in degrees up to q . Then $\Sigma^q(X, \mathbb{Z}) \subseteq S(\text{Trop}(\mathcal{V}^{\leq q}(X)))^c$.*

This upper bound is a finite union of open polyhedral cones in $H^1(X, \mathbb{R})$, intersected with the unit sphere $S(X)$. The theorem recovers—in a much stronger form—the main result of [48], which asserts that $\Sigma^q(X, \mathbb{Z})$ is contained in $S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)))^c$, where $\tau_1(W) \subset \mathbb{C}^n$ is the *exponential tangent cone* at the identity to a subvariety $W \subset (\mathbb{C}^\times)^n$, and $\tau_1^{\mathbb{R}}(W)$ are the real points on it. Indeed, as we show in Proposition 2.7, we always have an inclusion $\tau_1^{\mathbb{R}}(W) \subseteq \text{Trop}(W)$. Nevertheless, this inclusion is oftentimes strict, since, for instance, $\tau_1(W)$ is a union of rationally defined linear subspaces, whereas $\text{Trop}(W)$ is a (not necessarily symmetric about the origin) polyhedral complex. Furthermore, tropicalization may detect components of $\mathcal{V}^q(X)$ that do not pass through the origin, or even do not lie in \mathbb{T}_G^0 (the identity component of \mathbb{T}_G), whereas the exponential tangent cone only depends on the analytic germ at 1 of $\mathcal{V}^q(X)$.

1.3. Bounding the BNS invariant. For a finitely generated group G , Theorem 1.1 yields the following tropical bound on the BNS invariant:

$$(1) \quad \Sigma^1(G) \subseteq -S(\text{Trop}(\mathcal{V}^1(G)))^c,$$

where $-\Sigma$ denotes the image of a subset $\Sigma \subset S(G)$ under the antipodal map. The minus sign arises due to the conventions we follow here, according to which $\Sigma^1(G) = -\Sigma^1(G, \mathbb{Z})$, as in [7]; for a detailed account of our choices of signs in several definitions, we refer to Remark 5.7. We single out two classes of groups where the bound from (1) can be refined.

The first class (which includes the 3-manifold groups mentioned previously) consists of groups G for which the Alexander polynomial Δ_G is symmetric and satisfies a certain condition which guarantees that $\mathcal{V}^1(G) \cap \mathbb{T}_G^0$ coincides (at least away from 1) with the zero-locus of Δ_G . For such groups, we prove in Theorem 7.4 that

$$(2) \quad \Sigma^1(G) \subseteq \bigcup_{F \text{ an open facet of } B_A} S(F).$$

For closed, orientable, 3-manifolds with empty or toroidal boundary this inclusion also follows from the aforementioned work of Thurston, Bieri–Neumann–Strebel, and McMullen.

The second class (which includes Kähler groups, arrangement groups, and certain Seifert manifold groups), consists of groups G for which there are homomorphisms f_α from G onto groups G_α so that $\mathcal{V}^1(G_\alpha)$ contains a component of \mathbb{T}_{G_α} . In Theorem 7.5, we prove that

$$(3) \quad \Sigma^1(G) \subseteq \left(\bigcup_{\alpha} S(f_\alpha^*(H^1(G_\alpha, \mathbb{R}))) \right)^c.$$

When $G = \pi_1(M)$ is the fundamental group of a compact Kähler manifold and the maps f_α are induced by suitable orbifold fibrations from M to orbifold Riemann surfaces, work of Delzant [15] shows that this inclusion holds as equality, and so $\Sigma^1(M) = S(\text{Trop}(\mathcal{V}^1(M)))^c$. When M is the complement of an arrangement of hyperplanes in \mathbb{C}^ℓ and $G = \pi_1(M)$, the maps f_α arise from orbifold fibrations with base a punctured, genus 0 orbifold. The bound (3) takes into account the translated tori in $\mathcal{V}^1(M)$ —which the previous bounds did not—but it remains to be seen whether the bound is sharp in this setting.

1.4. Organization. The paper is organized in two, roughly equal parts. Part one develops the general theory relating the Sigma-invariants to the tropicalization of the characteristic varieties. In §2 we review some basic concepts from tropical geometry, and establish a connection with the construction of the exponential tangent cone. In §3 we recall needed facts about the characteristic varieties and the Alexander polynomial, while in §4 we define and study the tropicalization of the characteristic varieties. After a quick review of the BNSR invariants in §5, we prove in §6 our main result. Finally, in §7 we establish formulas (2) and (3).

The second part outlines applications of this theory to several classes of spaces and groups. We start in §8 with 1-relator groups, where Brown’s algorithm provides a quick way for computing the Σ -invariants, and continue in §9 with 3-manifold groups. In preparation for the final three sections, we discuss in §10 the cohomology jump loci and the Σ -invariants of 2-dimensional orbifolds. In §11 we identify a large class of Seifert fibered manifolds for which the BNS invariant is empty. We conclude in §§12–13 with the interplay between the tropical characteristic varieties and the BNS invariants of compact Kähler manifolds and complements of hyperplane arrangements.

2. TROPICAL VARIETIES AND EXPONENTIAL TANGENT CONES

We start by reviewing the basics of tropical geometry, following the treatment in the monograph of Maclagan and Sturmfels [40], as well as [8, 13, 24, 46, 51, 52].

2.1. Valuations. Let $\mathbb{K} := \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}\langle\langle t^{1/n} \rangle\rangle$ be the field of Puiseux series with complex coefficients. A nonzero element of \mathbb{K} has the form $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots$, where $c_i \in \mathbb{C}^\times$ and $a_1 < a_2 < \dots$ are rational numbers that have a common denominator.

The field \mathbb{K} is algebraically closed, and admits a valuation, $\text{val}: \mathbb{K}^\times \rightarrow \mathbb{Q}$, given by $\text{val}(c(t)) = a_1$. This makes \mathbb{K} into a valued field, with value group \mathbb{Q} , and defines a non-Archimedean absolute value on \mathbb{K} , by setting $|c| = \exp(-\text{val}(c))$ for $c \neq 0$ and $|0| = 0$. The valuation ring, $R = \{x \in \mathbb{K} \mid \text{val}(x) \geq 0\}$, is equal to $\bigcup_{n \geq 1} \mathbb{C}\langle\langle t^{1/n} \rangle\rangle$; this is a local ring with maximal ideal $\mathfrak{m} = \{x \in \mathbb{K} \mid \text{val}(x) > 0\}$ and residue field $R/\mathfrak{m} = \mathbb{C}$.

For an element $h \in R$, we denote by $h|_{t=0}$ its image in the residue field. A Laurent polynomial $f \in \mathbb{K}[\mathbf{x}^\pm] := \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has the form $f = \sum_{\mathbf{u} \in A} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$, with $a_{\mathbf{u}} \in \mathbb{K}^\times$, for some finite subset $A = \text{supp}(f) \subset \mathbb{Z}^n$. For $w \in \mathbb{Q}^n$, write $f(t^{w_1} x_1, \dots, t^{w_n} x_n) = t^b g(x_1, \dots, x_n)$, where $g \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and no positive power of t divides g . The initial form of f with respect to w is then given by $\text{in}_w f = g|_{t=0}$.

2.2. Tropical varieties. Let $(\mathbb{K}^\times)^n = \text{Spec}(\mathbb{K}[\mathbf{x}^\pm])$ be the algebraic torus of dimension n over \mathbb{K} . Taking the n -fold product of the valuation map $\mathbb{K}^\times \rightarrow \mathbb{Q}$, we obtain the map $\text{val}: (\mathbb{K}^\times)^n \rightarrow \mathbb{Q}^n \subset \mathbb{R}^n$. For an ideal $I \subset \mathbb{K}[\mathbf{x}^\pm]$, let $W = V(I)$ be the subvariety of $(\mathbb{K}^\times)^n$ cut out by I . The *tropicalization* of $W \subset (\mathbb{K}^\times)^n$ is the closure in \mathbb{R}^n (with the Euclidean metric topology) of the image of W under the valuation map,

$$(4) \quad \text{Trop}(W) := \overline{\text{val}(W)}.$$

It follows directly from the definition that

$$(5) \quad \text{Trop}(V \cup W) = \text{Trop}(V) \cup \text{Trop}(W).$$

An important result in tropical geometry is that a point $w \in \mathbb{Q}^n$ belongs to $\text{Trop}(W)$ if and only if $V(\text{in}_w I) \neq \emptyset$, where $\text{in}_w I$ is the ideal of $\mathbb{C}[\mathbf{x}^\pm]$ spanned by $\{\text{in}_w f \mid f \in I\}$, see [51, 8, 40]. Moreover, as shown in [52, Theorem 7.11] in a more general context,

$$(6) \quad \text{Trop}(W) \cap \mathbb{Q}^n = \text{val}(W).$$

The tropicalization of a hypersurface in $(\mathbb{K}^\times)^n$ can be described as follows. For a Laurent polynomial $f = \sum_{\mathbf{u} \in A} a_{\mathbf{u}} \mathbf{t}^{\mathbf{u}}$, one defines the *tropical polynomial* $\text{trop}(f)$ by

$$(7) \quad \text{trop}(f)(w) = \min_{\mathbf{u} \in A} (w \cdot \mathbf{u} + \text{val}(a_{\mathbf{u}})).$$

This is a piecewise linear, concave function $\mathbb{R}^n \rightarrow \mathbb{R}$. As a consequence of Kapranov's theorem [24, Thm. 2.1.1], the set $\text{Trop}(V(f))$ coincides with the corner locus of $\text{trop}(f)$: that is, the subset of \mathbb{R}^n on which the minimum is achieved for at least two values of \mathbf{u} . In other words, the tropicalization of $V(f)$ is the locus in \mathbb{R}^n where $\text{trop}(f)$ fails to be linear.

As shown in [40, Prop. 3.1.6], the tropical hypersurface $\text{Trop}(V(f))$ is the support of a pure, rational polyhedral complex of dimension $n - 1$ in \mathbb{R}^n . This complex may be described as the $(n - 1)$ -skeleton of the polyhedral complex dual to a regular subdivision of the Newton polytope of f given by the weights $\text{val}(a_{\mathbf{u}})$ on the lattice points in $\text{Newt}(f) := \text{conv}\{\mathbf{u} \mid a_{\mathbf{u}} \neq 0\} \subset \mathbb{R}^n$.

Finally, if $W = V(I)$ is an arbitrary subvariety in $(\mathbb{K}^\times)^n$, then $\text{Trop}(W) = \bigcap_{f \in I} \text{Trop}(V(f))$. As shown in [40, Cor. 3.2.4], this set is the support of a rational polyhedral complex in \mathbb{R}^n , a fact which explains once again why all the rational points on $\text{Trop}(W)$ are contained in $\text{val}(W)$, as stated in (6). For instance, if W is a curve, then $\text{Trop}(W)$ is a graph with rational edge directions.

Tropicalization enjoys the following naturality property with respect to morphisms of algebraic tori. Let $\alpha: (\mathbb{K}^\times)^m \rightarrow (\mathbb{K}^\times)^n$ be a monomial map, so that the induced morphism on coordinate rings, $\alpha^*: \mathbb{K}[\mathbf{y}^\pm] \rightarrow \mathbb{K}[\mathbf{x}^\pm]$ is given by the $n \times m$ integral matrix A associated to the homomorphism

$$(8) \quad \alpha^\vee: \text{Hom}_{\text{alg-gp}}(\mathbb{K}^\times, (\mathbb{K}^\times)^m) = \mathbb{Z}^m \longrightarrow \text{Hom}_{\text{alg-gp}}(\mathbb{K}^\times, (\mathbb{K}^\times)^n) = \mathbb{Z}^n.$$

Let $\text{trop}(\alpha) := \alpha^\vee \otimes \text{id}_\mathbb{R}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be the \mathbb{R} -linear map given by the transpose matrix A^\top . Then, as shown in [40, Cor. 3.2.13], for every subvariety $W \subset (\mathbb{K}^\times)^m$,

$$(9) \quad \text{Trop}(\overline{\alpha(W)}) = \text{trop}(\alpha)(\text{Trop}(W)).$$

Remark 2.1. If V and W are irreducible subvarieties of \mathbb{C}^n , then $\text{Trop}(V \times W) = \text{Trop}(V) \times \text{Trop}(W)$, see [13]. On the other hand, tropicalization does not always respect intersections: if V and W are subvarieties of $(\mathbb{K}^\times)^n$, then $\text{Trop}(V \cap W) \subseteq \text{Trop}(V) \cap \text{Trop}(W)$, but the inclusion may be strict. Nevertheless, as shown in [46], this inclusion holds as equality, provided that the tropicalizations intersect in the expected dimension.

2.3. Tropicalizing subvarieties in a complex torus. The “constant coefficient case” is that of varieties defined over the field \mathbb{C} with trivial valuation. Let W be a subvariety of $(\mathbb{C}^\times)^n$ —also known as a complex, very affine variety—and let $W \times_{\mathbb{C}} \mathbb{K}$ be the subvariety of $(\mathbb{K}^\times)^n$ obtained by extension of the base field. The tropicalization of W is then defined as

$$(10) \quad \text{Trop}(W) := \text{Trop}(W \times_{\mathbb{C}} \mathbb{K}).$$

For such varieties, the tropicalization is a rational polyhedral fan in \mathbb{R}^n . When $W = V(f)$ is a hypersurface in $(\mathbb{C}^\times)^n$ defined by a Laurent polynomial $f \in \mathbb{C}[\mathbf{x}^\pm]$, there is a very concrete, geometric interpretation of $\text{Trop}(W)$.

Given a polytope P , we denote by $\mathcal{F}(P)$ its face fan, i.e., the set of cones spanned by the faces of P , and by $\mathcal{N}(P)$ its (inner) normal fan. If 0 is in the interior of P , then the normal fan to P coincides with the face fan of the (polar) dual polytope,

$$(11) \quad \mathcal{N}(P) = \mathcal{F}(P^\Delta),$$

see [69, Exercise 7.1]. As in §2.2, let $\text{Newt}(f)$ be the Newton polytope of f , that is, the convex hull in \mathbb{R}^n of the set $\text{supp}(f) \subset \mathbb{Z}^n$. Then, as noted in [8],

$$(12) \quad \text{Trop}(V(f)) = \mathcal{N}(\text{Newt}(f))^{\text{codim}>0},$$

the positive-codimension skeleton of the normal fan to that polytope. In particular, a top-dimensional cone in $\text{Trop}(V(f))$ corresponds to an edge of $\text{Newt}(f)$, or to a ridge in the normal fan.

2.4. Tropicalized translated subtori. Let $\mathbb{k} = \mathbb{C}$ or \mathbb{K} , and let $(\mathbb{k}^\times)^n$ be an algebraic torus over \mathbb{k} . By definition, an *algebraic subtorus* of $(\mathbb{k}^\times)^n$ is the image, $T = \text{im}(\alpha)$, of a monomial inclusion, $\alpha: (\mathbb{k}^\times)^m \hookrightarrow (\mathbb{k}^\times)^n$, for which the image of the dual map, $\alpha^\vee: \mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$, is a primitive sublattice of \mathbb{Z}^n ; see for instance [8, 61]. It follows from formula (9) (see also [8]) that the tropicalization of such an algebraic subtorus is the m -dimensional linear subspace

$$(13) \quad \text{Trop}(T) = \text{im}(\text{trop}(\alpha)) = \alpha^\vee(\mathbb{Z}^m) \otimes \mathbb{R} \subset \mathbb{R}^n.$$

A *translated subtorus* in $(\mathbb{k}^\times)^n$ is a subvariety of the form $z \cdot T$, where T is an algebraic subtorus and $z \in (\mathbb{k}^\times)^n$. The tropicalization of such a variety is an affine subspace, given by

$$(14) \quad \text{Trop}(z \cdot T) = \text{Trop}(T) + \text{val}(z).$$

In the constant coefficient case ($\mathbb{k} = \mathbb{C}$), this formula reduces to

$$(15) \quad \text{Trop}(z \cdot T) = \text{Trop}(T).$$

Remark 2.2. A partial converse to formula (15) was established in [8, Lem. 3.1]: Let $W \subset (\mathbb{C}^\times)^n$ be an irreducible subvariety, let T be an algebraic subtorus, and suppose $\text{Trop}(W) \subset \text{Trop}(T)$; then there exists a point $z \in (\mathbb{C}^\times)^n$ such that $W \subset z \cdot T$.

2.5. The exponential map. Let $\exp: \mathbb{C}^n \rightarrow (\mathbb{C}^\times)^n$ be the n -fold product of the map $\mathbb{C} \rightarrow \mathbb{C}^\times$, $z \mapsto e^z$. The next lemma is based on work from [58, 61, 62]; since we will need the explicit construction from the lemma, we give a quick, mostly self-contained proof.

Lemma 2.3. *If L is a linear subspace of \mathbb{Q}^n , then $\exp(L \otimes_{\mathbb{Q}} \mathbb{C})$ is an algebraic subtorus of $(\mathbb{C}^\times)^n$. Conversely, every algebraic subtorus $T \subset (\mathbb{C}^\times)^n$ can be realized as $T = \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$, for some linear subspace $L \subset \mathbb{Q}^n$.*

Proof. Let $\Lambda = L \cap \mathbb{Z}^n$; then Λ is a primitive sublattice of \mathbb{Z}^n and $L = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The inclusion map, $\Lambda \hookrightarrow \mathbb{Z}^n$, is the Pontryagin dual to a monomial inclusion map, $T \hookrightarrow (\mathbb{C}^\times)^n$, where T is a complex algebraic subtorus. Moreover, as shown in [61, Lemma 6.1],

$$(16) \quad T = \exp(L \otimes_{\mathbb{Q}} \mathbb{C}).$$

Now suppose T is an algebraic subtorus of $(\mathbb{C}^\times)^n$. If $\alpha: T \hookrightarrow (\mathbb{C}^\times)^n$ is the corresponding monomial inclusion map, and if we let $L = \text{im}(\alpha^\vee) \otimes \mathbb{Q}$, then $T = \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$, by (16). \square

As an application, we obtain the following corollary.

Corollary 2.4. *Let V be a subvariety of $(\mathbb{C}^\times)^n$, all of whose irreducible components are translated subtori. Write $V = \bigcup_i \rho_i \cdot \exp(L_i \otimes_{\mathbb{Q}} \mathbb{C})$, for some (finitely many) linear subspaces $L_i \subset \mathbb{Q}^n$ and some $\rho_i \in (\mathbb{C}^\times)^n$. Then $\text{Trop}(V) = \bigcup_i L_i \otimes_{\mathbb{Q}} \mathbb{R}$.*

Proof. By Lemma 2.3, it is indeed possible to write each irreducible component of V in the stated form. We then have

$$(17) \quad \begin{aligned} \text{Trop}(V) &= \text{Trop}\left(\bigcup_i \rho_i \cdot \exp(L_i \otimes_{\mathbb{Q}} \mathbb{C})\right) \\ &= \bigcup_i \text{Trop}(\rho_i \cdot \exp(L_i \otimes_{\mathbb{Q}} \mathbb{C})) && \text{by (5)} \\ &= \bigcup_i \text{Trop}(\exp(L_i \otimes_{\mathbb{Q}} \mathbb{C})) && \text{by (15)} \\ &= \bigcup_i L_i \otimes_{\mathbb{Q}} \mathbb{R} && \text{by (13),} \end{aligned}$$

where at the last step we also used the proof of Lemma 2.3. This completes the proof. \square

2.6. The exponential tangent cone. The following notion was introduced by Dimca, Papadima, and the author in [21], and further studied in [48, 58, 62].

Definition 2.5. Let $W \subset (\mathbb{C}^\times)^n$ be an algebraic subvariety. The *exponential tangent cone* of W at 1 is the homogeneous subvariety $\tau_1(W) \subset \mathbb{C}^n$, defined by

$$(18) \quad \tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}.$$

This set depends only on the analytic germ of W at the identity; in particular, $\tau_1(W) \neq \emptyset$ if and only if $1 \in W$. Furthermore, τ_1 commutes with finite unions, as well as arbitrary intersections. The main features of this construction are encapsulated in the following result.

Lemma 2.6 ([21, 58]). *For a subvariety $W \subset (\mathbb{C}^\times)^n$, the set $\tau_1(W)$ is a finite union of rationally defined linear subspaces of \mathbb{C}^n . Moreover, $\tau_1(W)$ is contained in $\text{TC}_1(W)$, the tangent cone at 1 to W .*

Let $\tau_1^{\mathbb{R}}(W) := \tau_1(W) \cap \mathbb{R}^n$ be the set of real points on the exponential tangent cone. The next proposition—which may be thought of as the starting point of our investigation—establishes a noteworthy relationship between this set and the tropicalization of W .

Proposition 2.7. *Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic subvariety. Then $\tau_1^{\mathbb{R}}(W) \subseteq \text{Trop}(W)$.*

Proof. As a consequence of Lemma 2.6, the set $\tau_1^{\mathbb{R}}(W)$ is a finite union of rationally defined linear subspaces of \mathbb{R}^n . Let $L \otimes_{\mathbb{Q}} \mathbb{R}$ be one of those subspaces, with L as its rational points.

By the definition of $\tau_1(W)$, the set $T = \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$ lies inside W ; thus, $\text{Trop}(T) \subset \text{Trop}(W)$. On the other hand, by Corollary 2.4, $\text{Trop}(T) = L \otimes_{\mathbb{Q}} \mathbb{R}$. This shows that $L \otimes_{\mathbb{Q}} \mathbb{R} \subset \text{Trop}(W)$, thereby proving the claim. \square

If W is an algebraic subtorus of $(\mathbb{C}^\times)^n$, the above inclusion is of course an equality. In general, though, the inclusion is strict. For instance, if $W = \rho T$, with T a subtorus and $\rho \notin T$, then $\tau_1^{\mathbb{R}}(W) = \emptyset$, whereas $\text{Trop}(W) = \text{Trop}(T)$ is nonempty. More generally, we have the following result.

Proposition 2.8. *Let W be an algebraic subvariety of $(\mathbb{C}^\times)^n$. Suppose there is a subtorus $T \subset (\mathbb{C}^\times)^n$ such that $T \not\subset W$, yet $\rho T \subset W$ for some $\rho \in (\mathbb{C}^\times)^n$. Then $\tau_1^{\mathbb{R}}(W) \subsetneq \text{Trop}(W)$.*

Proof. By Lemma 2.3, there is a linear subspace $L \subset \mathbb{Q}^n$ such that $T = \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$. By Corollary 2.4, $\text{Trop}(\rho T) = L \otimes_{\mathbb{Q}} \mathbb{R}$. Since, by assumption, $\rho T \subset W$, we must have

$$(19) \quad L \otimes_{\mathbb{Q}} \mathbb{R} \subset \text{Trop}(W).$$

On the other hand, the assumption that $T \not\subset W$ together with definition (18) imply that $L \otimes_{\mathbb{Q}} \mathbb{C} \not\subset \tau_1(W)$; thus,

$$(20) \quad L \otimes_{\mathbb{Q}} \mathbb{R} \not\subset \tau_1^{\mathbb{R}}(W).$$

Putting together (19) and (20), we conclude that the set $\text{Trop}(W) \setminus \tau_1^{\mathbb{R}}(W)$ is nonempty. \square

3. CHARACTERISTIC VARIETIES AND ALEXANDER POLYNOMIALS

We give in this section a brief review of the theory of characteristic varieties and Alexander polynomials. For details and further references we refer to [21, 48, 49, 58, 60] for the former, and [20, 25, 34, 41, 65] for the latter.

3.1. Jump loci for twisted homology. Let X be a connected CW-complex with finite q -skeleton, for some $q \geq 1$. Without loss of generality, we may assume X has a single 0-cell, call it x_0 . Let $G = \pi_1(X, x_0)$ be the fundamental group of X based at x_0 , and let

$$(21) \quad \text{Hom}(G, \mathbb{C}^\times) = H^1(X, \mathbb{C}^\times)$$

be the group of complex-valued, multiplicative characters of G , which we shall denote at times as \mathbb{T}_X or \mathbb{T}_G . Since \mathbb{C}^\times is abelian, the abelianization map, $\text{ab}: G \twoheadrightarrow G_{\text{ab}}$, induces an isomorphism, $\text{ab}^*: \mathbb{T}_{G_{\text{ab}}} \xrightarrow{\cong} \mathbb{T}_G$. The character group of G is a complex algebraic group, with coordinate ring $\mathbb{C}[G_{\text{ab}}]$, and with identity 1 corresponding to the trivial representation. The identity component, \mathbb{T}_G^0 , is an algebraic torus of dimension $n = b_1(X)$; the connected components of \mathbb{T}_G are translates of this torus by characters indexed by the torsion subgroup of $G_{\text{ab}} = H_1(X, \mathbb{Z})$.

For each character $\rho: G \rightarrow \mathbb{C}^\times$, we let \mathbb{C}_ρ be the corresponding rank 1 local system on X , i.e., the vector space \mathbb{C} , viewed as a module over $\mathbb{C}[G]$ via the action $g \cdot a = \rho(g)a$. The *characteristic varieties* of X (in degree $i \leq q$) are the jump loci for homology with such twisted coefficients,

$$(22) \quad \mathcal{V}^i(X) = \{\rho \in H^1(X, \mathbb{C}^\times) \mid H_i(X, \mathbb{C}_\rho) \neq 0\}.$$

Here is an alternative description of these loci, which makes it clear that they are Zariski closed subsets of the character group, at least for $i < q$. Let $X^{\text{ab}} \rightarrow X$ be the maximal abelian cover, corresponding to the projection $\text{ab}: G \twoheadrightarrow G_{\text{ab}}$. Upon lifting the cell structure of X to this cover, we obtain a chain complex of free $\mathbb{Z}[G_{\text{ab}}]$ -modules, $(C_*(X^{\text{ab}}, \mathbb{Z}), \partial^{\text{ab}})$. By definition, a character $\rho \in \mathbb{T}_X = \mathbb{T}_{G_{\text{ab}}}$ belongs to $\mathcal{V}^i(X)$ precisely when $\text{rank } \partial_{i+1}^{\text{ab}}(\rho) + \text{rank } \partial_i^{\text{ab}}(\rho) < c_i$, where c_i is the number of i -cells of X and the evaluation of ∂_i^{ab} at ρ is obtained by applying the ring morphism $\mathbb{C}[G_{\text{ab}}] \rightarrow \mathbb{C}$, $g \mapsto \rho(g)$ to each entry. Hence, $\mathcal{V}^i(X)$ is the zero-set of the ideal of minors of size c_i of the block-matrix $\partial_{i+1}^{\text{ab}} \oplus \partial_i^{\text{ab}}$. The case $i = q$ is more delicate, requiring the replacement of X with a CW-complex having finite $(q + 1)$ -skeleton, yet the same jump loci; see [49, Prop. 4.1].

We can also define the characteristic varieties of X with coefficients in an arbitrary field \mathbb{k} as the subvarieties $\mathcal{V}^i(X, \mathbb{k})$ of the algebraic group $H^1(X, \mathbb{k}^\times)$ consisting of those characters $\rho: \pi_1(X) \rightarrow \mathbb{k}^\times$ for which $H^i(X, \mathbb{k}_\rho) \neq 0$. The argument outlined above shows that the jump loci $\mathcal{V}^i(X, \mathbb{k})$ are determinantal varieties of matrices defined over \mathbb{Z} . Consequently, these constructions are compatible with restriction and extension of the base field; that is, if $\mathbb{k} \subset \mathbb{L}$ is a field extension, then

$$(23) \quad \mathcal{V}^i(X, \mathbb{k}) = \mathcal{V}^i(X, \mathbb{L}) \cap H^1(X, \mathbb{k}^\times),$$

$$(24) \quad \mathcal{V}^i(X, \mathbb{L}) = \mathcal{V}^i(X, \mathbb{k}) \times_{\mathbb{k}} \mathbb{L}.$$

3.2. Properties of the characteristic varieties. The characteristic varieties of X are homotopy-type invariants. Indeed, if $f: X \rightarrow Y$ is a homotopy equivalence, then the induced morphism on character groups, $f^*: H^1(Y, \mathbb{C}^\times) \rightarrow H^1(X, \mathbb{C}^\times)$, restricts to isomorphisms $\mathcal{V}^i(Y) \xrightarrow{\cong} \mathcal{V}^i(X)$.

Clearly, $1 \in \mathcal{V}^i(X)$ if and only if $b_i(X) \neq 0$; moreover, $\mathcal{V}^0(X) = \{1\}$. The set $\mathcal{V}^1(X)$ depends only on the fundamental group $G = \pi_1(X)$, and, in fact, only on its maximal metabelian quotient, G/G'' ; thus, we shall sometimes write this set as $\mathcal{V}^1(G)$. We also have the following (partial) functoriality property.

Proposition 3.1 ([58]). *Let G be a finitely generated group, and let $\varphi: G \twoheadrightarrow Q$ be a surjective homomorphism. Then the induced morphism between character groups, $\varphi^*: \mathbb{T}_Q \rightarrow \mathbb{T}_G$, $\varphi^*(\rho)(g) = \varphi(\rho(g))$, is injective, and restricts to an embedding $\mathcal{V}^1(Q) \hookrightarrow \mathcal{V}^1(G)$.*

Each homology group $H_i(X^{\text{ab}}, \mathbb{C})$ naturally acquires the structure of a $\mathbb{C}[G_{\text{ab}}]$ -module. The *Alexander varieties* of X are the zero-sets of the annihilators of these modules,

$$(25) \quad \mathcal{W}^i(X) = V(\text{ann}(H_i(X^{\text{ab}}, \mathbb{C}))).$$

As such, these algebraic sets are subvarieties of the character group $\mathbb{T}_G = \text{Spec}(\mathbb{C}[G_{\text{ab}}])$. We will mainly be interested here in the unions of the characteristic varieties up to a fixed degree, $\mathcal{V}^{\leq q}(X) = \bigcup_{i \leq q} \mathcal{V}^i(X)$. If X has finite q -skeleton, then

$$(26) \quad \mathcal{V}^{\leq q}(X) = \mathcal{W}^{\leq q}(X),$$

where $\mathcal{W}^{\leq q}(X) = \bigcup_{i \leq q} \mathcal{W}^i(X)$, see [48, 49]. Moreover, if $b_1(X) > 0$, then $\mathcal{V}^{\leq 1}(X) = \mathcal{V}^1(X)$.

3.3. The Alexander polynomial. Let $H = G_{\text{ab}}/\text{Tors}(G_{\text{ab}})$ be the maximal torsion-free abelian quotient of the group $G = \pi_1(X, x_0)$. It is readily seen that the group ring $\mathbb{Z}H$ is a commutative Noetherian ring and a unique factorization domain. Let $q: X^H \rightarrow X$ be the regular cover corresponding to the projection $G \rightarrow H$, i.e., the maximal torsion-free abelian cover of X . The *Alexander module* of X is defined as the relative homology group

$$(27) \quad A_X = H_1(X^H, q^{-1}(x_0), \mathbb{Z}),$$

viewed as a $\mathbb{Z}H$ -module. This module depends only on the group G : if $I_G = \ker(\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z})$ is the augmentation ideal, then $A_X \cong A_G$, where $A_G := \mathbb{Z}H \otimes_{\mathbb{Z}G} I_G$. To see why, fix a basepoint $\tilde{x}_0 \in q^{-1}(x_0)$; sending each element $g - 1 \in I_G$ to the path in X^H from \tilde{x}_0 to $g\tilde{x}_0$ obtained by lifting the loop g at \tilde{x}_0 induces an isomorphism $A_G \xrightarrow{\cong} A_X$.

Now let $E_1(A_X) \subseteq \mathbb{Z}H$ be the ideal of codimension 1 minors in a $\mathbb{Z}H$ -presentation for A_X . The *Alexander polynomial* of X is then defined as the greatest common divisor of the elements in this determinantal ideal,

$$(28) \quad \Delta_X = \text{gcd}(E_1(A_X)).$$

When G admits a finite presentation, say $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_s \rangle$, we can be more explicit. Let J_G be the $s \times m$ matrix whose entries are the Fox derivatives $\partial_j r_i$ of the relators, viewed as elements in $\mathbb{Z}G$. Then $\mathcal{V}^1(G)$ coincides (away from 1) with the subvariety of \mathbb{T}_G defined by the minors of size $m - 1$ of J_G^{ab} , the matrix obtained from J_G by applying the morphism $\text{ab}: \mathbb{Z}G \rightarrow \mathbb{Z}G_{\text{ab}}$ to its entries. Likewise, Δ_X is the greatest common divisor of the minors of size $m - 1$ of J_G^α , where $\alpha: \mathbb{Z}G \rightarrow \mathbb{Z}H$ is defined by the projection map $G \rightarrow H$.

Set $n = b_1(G)$ and assume that $n > 0$. Upon fixing a basis for $H \cong \mathbb{Z}^n$, we may identify $\mathbb{T}_G^0 = \mathbb{T}_H$ with $(\mathbb{C}^\times)^n$ and $\mathbb{Z}H$ with the ring of Laurent polynomials in $t_1^{\pm 1}, \dots, t_n^{\pm 1}$. The Alexander polynomial $\Delta_G \in \mathbb{Z}H$ is well-defined up to multiplication by units in this ring, which are all of the form $\pm g$, for some $g \in H$ (we write $\Delta \doteq \Delta'$ if $\Delta = \pm g \cdot \Delta'$ in $\mathbb{Z}H$).

Let $\text{Newt}(\Delta_G) \subset H_1(G, \mathbb{R}) = \mathbb{R}^n$ be the Newton polytope of Δ_G . Every cohomology class $\phi \in H^1(G; \mathbb{Z}) \cong \text{Hom}(H, \mathbb{Z})$ defines a linear functional, $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$; we let $\phi(\text{Newt}(\Delta_G))$ be the image of the Newton polytope under this map. In [41], McMullen defined the *Alexander norm* of a class $\phi \in H^1(G; \mathbb{Z})$, denoted $\|\phi\|_A$, as the length of the interval $\phi(\text{Newt}(\Delta_G)) \subset \mathbb{R}$. Clearly, the function $\phi \mapsto \|\phi\|_A$ is convex and linear on rays, making it a semi-norm on $H^1(G; \mathbb{Z})$, which extends to a semi-norm on $H^1(G, \mathbb{R})$. We let $B_A \subset H^1(G, \mathbb{R})$ be the unit ball in this semi-norm.

If the Alexander polynomial of G is symmetric, i.e., invariant up to units in $\mathbb{Z}H$ under the involution $g \mapsto g^{-1}$, then the Alexander norm ball, B_A , is, up to a scale factor of $1/2$, the polar dual of the Newton polytope of Δ_G (see [39] for a proof):

$$(29) \quad B_A = \frac{1}{2} \text{Newt}(\Delta_G)^\Delta.$$

We conclude with a result that relates $V(\Delta_G)$, the algebraic hypersurface in \mathbb{T}_G^0 defined by the vanishing of Δ_G , to the first characteristic variety of G . This is done under a certain hypothesis

relating the ideal of $\mathbb{Z}H$ generated by Δ_G , the augmentation ideal $I_H \subset \mathbb{Z}H$, and the Alexander module, $A_G = \mathbb{Z}H \otimes_{\mathbb{Z}G} I_G$.

Proposition 3.2 ([20]). *Suppose that $I_H^p \cdot (\Delta_G) = E_1(A_G)$, for some $p \geq 0$. Then $\mathcal{V}^1(G) \cap \mathbb{T}_G^0 = V(\Delta_G) \cup \{1\}$.*

Groups with first Betti number equal to 1, finitely presented groups with more generators than relators (such as the 1-relator groups from §8), and the 3-manifold groups from §9 all satisfy the hypothesis of Proposition 3.2.

Now suppose G satisfies this hypothesis, and also G_{ab} is torsion-free (so that $G_{\text{ab}} = H$). Then $\mathcal{V}^1(G)$ itself coincides with $V(\Delta_G)$, at least away from 1: for instance, if $\Delta_G \doteq 1$ then $1 \in \mathcal{V}^1(G)$, though $V(\Delta_G) = \emptyset$. On the other hand, if $\Delta_G(1) = 0$, then $\mathcal{V}^1(G) = V(\Delta_G)$.

4. TROPICALIZING THE CHARACTERISTIC VARIETIES

Throughout this section, X will be a space having the homotopy type of a connected CW-complex with finite q -skeleton, for some $q \geq 1$. Set $n = b_1(X)$; to avoid trivialities, we will assume that $n > 0$.

4.1. Tropicalizing affine varieties. As in §2, we let $\mathbb{K} = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with complex coefficients. Recall this is an algebraically closed field which supports a non-Archimedean valuation, $\text{val}: \mathbb{K}^\times \rightarrow \mathbb{Q}$. Letting $\text{val}^{\times n}: (\mathbb{K}^\times)^n \rightarrow \mathbb{Q}^n \subset \mathbb{R}^n$ be the n -fold product of this map, we define a “valuation map” on the whole \mathbb{K} -character variety of $\pi_1(X)$,

$$(30) \quad \nu_X: H^1(X, \mathbb{K}^\times) \longrightarrow \mathbb{Q}^n \subset \mathbb{R}^n,$$

by requiring that ν_X restricts to $\text{val}^{\times n}$ on each connected component of $H^1(X, \mathbb{K}^\times) = \coprod (\mathbb{K}^\times)^n$. In other words, if $\rho: \pi_1(X) \rightarrow \mathbb{K}^\times$ is a \mathbb{K} -valued multiplicative character, then the additive character $\text{val} \circ \rho: \pi_1(X) \rightarrow \mathbb{Q}$ defines the element $\nu_X(\rho) \in H^1(X, \mathbb{Q}) = \mathbb{Q}^n \subset \mathbb{R}^n$.

Alternatively, we may view the map ν_X as the composite

$$(31) \quad H^1(X, \mathbb{K}^\times) \xrightarrow{\text{val}_*} H^1(X, \mathbb{Q}) \longrightarrow H^1(X, \mathbb{R}),$$

where the first arrow is the coefficient homomorphism induced by the map $\text{val}: \mathbb{K}^\times \rightarrow \mathbb{Q}$, and the second arrow is the coefficient homomorphism $H^1(X, \mathbb{Q}) \hookrightarrow H^1(X, \mathbb{R})$. The map ν_X depends only on the abelianization of the group $G = \pi_1(X)$, so we may also denote it by ν_G . This map enjoys the following naturality property. Let $\varphi: G \rightarrow K$ be a homomorphism between finitely generated groups. We then have a commuting diagram,

$$(32) \quad \begin{array}{ccc} H^1(K, \mathbb{K}^\times) & \xrightarrow{\varphi^*} & H^1(G, \mathbb{K}^\times) \\ \downarrow \nu_K & & \downarrow \nu_G \\ H^1(K, \mathbb{R}) & \xrightarrow{\varphi^*} & H^1(G, \mathbb{R}). \end{array}$$

Definition 4.1. The *tropicalization* of an algebraic subvariety $W \subset H^1(X, \mathbb{C}^\times)$ is the closure in $H^1(X, \mathbb{R}) \cong \mathbb{R}^n$ of the image of $W \times_{\mathbb{C}} \mathbb{K} \subset H^1(X, \mathbb{K}^\times)$ under the map ν_X ,

$$(33) \quad \text{Trop}(W) := \overline{\nu_X(W \times_{\mathbb{C}} \mathbb{K})}.$$

An extreme case is worth singling out.

Lemma 4.2. *If W contains a connected component of $H^1(X, \mathbb{C}^\times)$, then $\text{Trop}(W) = H^1(X, \mathbb{R})$.*

Proof. Let $\mathbb{T} = H^1(X, \mathbb{C}^\times)$, and let $\mathbb{T}^0 = (\mathbb{C}^\times)^n$ be the identity component. By assumption, W contains $\rho\mathbb{T}^0$, for some $\rho \in \mathbb{T}$. Therefore, $\text{Trop}(W)$ contains $\text{Trop}(\rho\mathbb{T}^0) = \text{Trop}(\mathbb{T}^0) = \mathbb{R}^n$, and we're done. \square

4.2. Tropicalized characteristic varieties. Recall from (24) that $\mathcal{V}^i(X, \mathbb{K}) = \mathcal{V}^i(X) \times_{\mathbb{C}} \mathbb{K}$. Applying Definition 4.1 to the characteristic varieties $\mathcal{V}^i(X)$, where $i \leq q$, we have that

$$(34) \quad \text{Trop}(\mathcal{V}^i(X)) = \overline{\nu_X(\mathcal{V}^i(X, \mathbb{K}))}.$$

As noted in (31), the map ν_X factors through the coefficient homomorphism $\text{val}_* : H^1(X, \mathbb{K}^\times) \rightarrow H^1(X, \mathbb{Q})$. By (6), the set of rational points on $\text{Trop}(\mathcal{V}^i(X))$ consists of all elements of the form $\nu_X(\rho)$, for some character $\rho : \pi_1(X) \rightarrow \mathbb{K}^\times$ which belongs to $\mathcal{V}^i(X, \mathbb{K})$; that is,

$$(35) \quad \text{Trop}(\mathcal{V}^i(X)) \cap H^1(X, \mathbb{Q}) = \nu_X(\mathcal{V}^i(X, \mathbb{K})).$$

These tropical varieties are homotopy-type invariants. Indeed, suppose $f : X \rightarrow Y$ is a homotopy equivalence. Then the induced isomorphism, $f^* : H^1(Y, \mathbb{Z}) \rightarrow H^1(X, \mathbb{Z})$, defines both a monomial isomorphism $\alpha = f^*$ on $H^1(-, \mathbb{C}^\times)$ and an \mathbb{R} -linear isomorphism $A = f^*$ on $H^1(-, \mathbb{R})$. As we noted in §3.2, the map α takes $\mathcal{V}^1(Y)$ to $\mathcal{V}^1(X)$. Using now formula (9) in this slightly more general context, we conclude that $A(\text{Trop}(\mathcal{V}^1(Y))) = \text{Trop}(\mathcal{V}^1(X))$.

In degree $i = 1$, the tropicalized characteristic varieties enjoy the following naturality property.

Proposition 4.3. *Let G be a finitely generated group, and let $\varphi : G \twoheadrightarrow Q$ be a surjective homomorphism. Then the induced \mathbb{R} -linear map, $\varphi^* : H^1(Q, \mathbb{R}) \hookrightarrow H^1(G, \mathbb{R})$, restricts to an embedding $\text{Trop}(\mathcal{V}^1(Q)) \hookrightarrow \text{Trop}(\mathcal{V}^1(G))$.*

Proof. It follows from Proposition 3.1 that the induced morphism between \mathbb{K} -character groups, $\varphi^* : H^1(Q, \mathbb{K}^\times) \hookrightarrow H^1(G, \mathbb{K}^\times)$, restricts to an embedding $\mathcal{V}^1(Q, \mathbb{K}) \hookrightarrow \mathcal{V}^1(G, \mathbb{K})$. Applying the valuation map from (30) and using the commutativity of diagram (32) yields the claim. \square

The next result details the relationship between the exponential tangent cones and the tropicalizations of the characteristic varieties.

Proposition 4.4. *Let X be a space as above. Then,*

- (1) $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subseteq \text{Trop}(\mathcal{V}^i(X))$, for all $i \leq q$.
- (2) Suppose there is a subtorus $T \subset \mathbb{T}_X^0$ such that $T \not\subset \mathcal{V}^i(X)$, yet $\rho T \subset \mathcal{V}^i(X)$ for some $\rho \in \mathbb{T}_X$. Then $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subsetneq \text{Trop}(\mathcal{V}^i(X))$.

Proof. The first claim follows at once from Proposition 2.7, while the second claim follows from a slight modification of the proof of Proposition 2.8. \square

Finally, let G be a finitely generated group with $b_1(G) > 0$, and let $H = G_{\text{ab}} / \text{Tors}(G_{\text{ab}})$. Under some additional conditions on the Alexander polynomial Δ_G , we can say more.

Proposition 4.5. *Suppose Δ_G is symmetric and $I_H^p \cdot (\Delta_G) = E_1(A_G)$, for some $p \geq 0$. Then the following hold for the tropical variety $Y = \text{Trop}(\mathcal{V}^1(G) \cap \mathbb{T}_G^0)$.*

- (1) $Y = \text{Trop}(V(\Delta_G)) \cup \{0\}$.
- (2) $Y = -Y$.

- (3) Y coincides with the positive-codimension skeleton of $\mathcal{F}(B_A)$, the face fan of the unit ball in the Alexander norm.

Proof. Since the Alexander ideal (Δ_G) satisfies the assumption of Proposition 3.2, we infer that $\mathcal{V}^1(G) \cap \mathbb{T}_G^0 = V(\Delta_G) \cup \{1\}$; the first claim follows at once.

Let $P = \text{Newt}(\Delta_G)$ be the Newton polytope of Δ_G inside $H_1(G, \mathbb{R}) = H \otimes \mathbb{R}$. From (12), we know that $\text{Trop}(V(\Delta_G))$ coincides with $\mathcal{N}(P)^{\text{codim}>0}$, the positive-codimension skeleton of the inner normal fan to P . Now, since Δ_G is symmetric, we have that $P = -P$, and the second claim follows.

Finally, we know from (29) that P is twice the polar dual of B_A . Moreover, $0 \in \text{int}(P)$; thus, by (11), the inner normal fan to P is the face fan of B_A . The last claim follows. \square

5. BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

In this section, we review the definition of the Sigma-invariants of a group G and, more generally, of a space X , following the approach from [4, 28, 48, 57].

5.1. The Σ -invariants of a chain complex. Let $C = (C_i, \partial_i)_{i \geq 0}$ be a chain complex over a ring R , and let q be a positive integer. We say C is of *finite q -type* if there is a chain complex C' of finitely generated, projective (left) R -modules and a chain map $C' \rightarrow C$ inducing isomorphisms $H_i(C') \rightarrow H_i(C)$ for $i < q$ and an epimorphism $H_q(C') \rightarrow H_q(C)$. For a free chain complex C , this is equivalent to being chain-homotopy equivalent to a free chain complex D for which D_i is finitely generated for all $i \leq q$.

Now let G be a finitely generated group. The character sphere

$$(36) \quad S(G) := (\text{Hom}(G, \mathbb{R}) \setminus \{0\}) / \mathbb{R}^+$$

is the set of nonzero homomorphisms $G \rightarrow \mathbb{R}$ modulo homothety. To simplify notation, we will usually denote both a nonzero homomorphism $\chi: G \rightarrow \mathbb{R}$ and its equivalence class, $[\chi] \in S(G)$, by the same symbol, χ . The character sphere may be identified with the unit sphere S^{n-1} in the real vector space $\text{Hom}(G, \mathbb{R}) \cong \mathbb{R}^n$, where $n = b_1(G)$.

Given a nonzero homomorphism $\chi: G \rightarrow \mathbb{R}$, the set $G_\chi := \{g \in G \mid \chi(g) \geq 0\}$ is a submonoid of G , which depends only on $[\chi] \in S(G)$. The monoid ring $\mathbb{Z}G_\chi$ is a subring of the group ring $\mathbb{Z}G$; thus, any $\mathbb{Z}G$ -module naturally acquires the structure of a $\mathbb{Z}G_\chi$ -module, by restriction of scalars.

Definition 5.1 ([28]). Let C be a chain complex over $\mathbb{Z}G$. For each integer $q \geq 0$, the q -th Bieri–Neumann–Strebel–Renz invariant of C is the set

$$(37) \quad \Sigma^q(C) = \{\chi \in S(G) \mid C \text{ is of finite } q\text{-type over } \mathbb{Z}G_\chi\}.$$

Suppose now that $N \triangleleft G$ is a normal subgroup such that the quotient group G/N is abelian. Let $S(G, N)$ be the set of homomorphisms $\chi \in S(G)$ for which $N \leq \ker(\chi)$. Then $S(G, N)$ is a great subsphere of $S(G)$, obtained by intersecting the unit sphere $S(G) \subset H^1(G, \mathbb{R})$ with the image of the linear map $\kappa^*: H^1(G/N, \mathbb{R}) \hookrightarrow H^1(G, \mathbb{R})$, where $\kappa: G \twoheadrightarrow G/N$ is the canonical projection. Finally, let C be a chain complex of free $\mathbb{Z}G$ -modules, with C_i finitely generated for $i \leq q$, and let N be a normal subgroup of G such that G/N is abelian. Then, as shown in [28], C is of finite q -type when restricted to $\mathbb{Z}N$ if and only if $S(G, N) \subset \Sigma^q(C)$.

5.2. The Σ -invariants of a CW-complex. Let X be a connected CW-complex with finite 1-skeleton, and let $G = \pi_1(X, x_0)$ be its fundamental group. A choice of classifying map $X \rightarrow K(G, 1)$ yields an induced isomorphism, $H^1(G, \mathbb{R}) \xrightarrow{\cong} H^1(X, \mathbb{R})$, which identifies the respective unit spheres, $S(G)$ and $S(X)$. The cell structure on X lifts to a cell structure on the universal cover \widetilde{X} , invariant under the action of G by deck transformations. Thus, the cellular chain complex $C_*(\widetilde{X}, \mathbb{Z})$ is a chain complex of free $\mathbb{Z}G$ -modules.

Definition 5.2. For each $q > 0$, the q -th Bieri–Neumann–Strebel–Renz invariant of X is the subset of $S(X)$ given by $\Sigma^q(X, \mathbb{Z}) = \Sigma^q(C_*(\widetilde{X}, \mathbb{Z}))$.

We will denote by $\Sigma^q(X, \mathbb{Z})^c$ the complement of $\Sigma^q(X, \mathbb{Z})$ in $S(X)$. It is shown in [28] that $\Sigma^q(X, \mathbb{Z})$ is an open subset of $S(X)$, which depends only on the homotopy type of X . The Σ -invariants enjoy the following naturality property.

Lemma 5.3 ([28]). *Let $f: X \rightarrow Y$ be a map between two finite CW-complexes, and assume there is a map $g: Y \rightarrow X$ such that $f \circ g \simeq \text{id}_Y$. If $f^*: H^1(Y, \mathbb{R}) \rightarrow H^1(X, \mathbb{R})$ is the induced homomorphism and $f^*: S(Y) \rightarrow S(X)$ is its restriction to character spheres, then $(f^*)^{-1}(\Sigma^q(X, \mathbb{Z})) \subseteq \Sigma^q(Y, \mathbb{Z})$, for all $q > 0$.*

Consequently, if f is a homotopy equivalence, then $f^*(\Sigma^q(Y, \mathbb{Z})) = \Sigma^q(X, \mathbb{Z})$.

We say that a subset $\Sigma \subset S^{n-1}$ is *symmetric* if it is invariant under the antipodal map, i.e., $\Sigma = -\Sigma$. In general, the BNSR invariants are not symmetric; we will illustrate this phenomenon in Examples 8.2, 8.4, and 8.5. Nevertheless, as a consequence of Lemma 5.3, we have the following symmetry criterion.

Proposition 5.4. *Suppose there is a homotopy equivalence $f: X \rightarrow X$ such that $f_*: H_1(X, \mathbb{R}) \rightarrow H_1(X, \mathbb{R})$ is equal to $-\text{id}_{H_1(X, \mathbb{R})}$. Then $\Sigma^q(X, \mathbb{Z}) = -\Sigma^q(X, \mathbb{Z})$, for all $q \geq 1$.*

5.3. The Σ -invariants of a group. Let G be a finitely generated group, and let $\text{Cay}(G)$ be the Cayley graph associated to a fixed finite generating set. The invariant $\Sigma^1(G)$ of Bieri, Neumann, and Strebel [6] is the set of homomorphisms $\chi \in S(G)$ for which the induced subgraph of $\text{Cay}(G)$ on vertex set G_χ is connected. The BNS set is an open subset of $S(G)$, which does not depend on the choice of finite generating set for G . The rational points on $S(G)$ correspond to epimorphisms $\chi: G \rightarrow \mathbb{Z}$; the kernel of χ is finitely generated if and only if both χ and $-\chi$ belong to $\Sigma^1(G)$. Additionally, the complements of the Σ -invariants enjoy the following naturality property.

Proposition 5.5 ([6]). *Suppose $\varphi: G \twoheadrightarrow Q$ is a surjective group homomorphism. Then the induced embedding, $\varphi^*: S(Q) \hookrightarrow S(G)$, restricts to an injective map between the complements of the respective BNS-invariants, $\varphi^*: \Sigma^1(Q)^c \hookrightarrow \Sigma^1(G)^c$.*

The BNS invariant was generalized by Bieri and Renz [7], as follows. For a $\mathbb{Z}G$ -module M , define $\Sigma^q(G, M) := \Sigma^q(F_\bullet)$, where $F_\bullet \rightarrow M$ is a projective $\mathbb{Z}G$ -resolution of M . In particular, this yields invariants $\Sigma^q(G, \mathbb{Z})$, where \mathbb{Z} is viewed as a trivial $\mathbb{Z}G$ -module; clearly, $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G, 1), \mathbb{Z})$. Likewise, we have the invariants $\Sigma^q(G, \mathbb{k})$, where \mathbb{k} is a field. There is always an inclusion $\Sigma^q(G, \mathbb{Z}) \subseteq \Sigma^q(G, \mathbb{k})$, but this inclusion may be strict.

As noted in [7, §1.3], $\Sigma^1(G) = -\Sigma^1(G, \mathbb{Z})$. It follows from Proposition 5.4 that $\Sigma^1(G)$ is symmetric whenever G admits an automorphism inducing minus the identity on $G_{\text{ab}} \otimes \mathbb{R}$.

5.4. Novikov–Sikorav completion. Let G be a group. The *Novikov–Sikorav completion* of the group ring $\mathbb{Z}G$ with respect to a homomorphism $\chi: G \rightarrow \mathbb{R}$ consists of all formal sums $\sum n_g g \in \mathbb{Z}^G$, having the property that, for each $c \in \mathbb{R}$, the set

$$(38) \quad \{g \in G \mid n_g \neq 0 \text{ and } \chi(g) \geq c\}$$

is finite, see [45, 54, 27]. With the usual addition and with multiplication defined by $(\sum n_g g) \cdot (\sum m_h h) = \sum (n_g m_h) gh$, the Novikov–Sikorav completion, $\widehat{\mathbb{Z}G}_\chi$, is a ring containing $\mathbb{Z}G$ as a subring. Consequently, $\widehat{\mathbb{Z}G}_\chi$ carries a natural structure of left $\mathbb{Z}G$ -module (we will also view it as a right $\widehat{\mathbb{Z}G}_\chi$ -module). For instance, if $G = \mathbb{Z} = \langle t \rangle$ and $\chi(t) = 1$, then $\widehat{\mathbb{Z}G}_\chi = \{\sum_{i \leq k} n_i t^i \mid n_i \in \mathbb{Z}, \text{ for some } k \in \mathbb{Z}\}$.

To see why $\widehat{\mathbb{Z}G}_\chi$ is a ring completion, let U_m be the additive subgroup of $\mathbb{Z}G$ (freely) generated by the set $\{g \in G \mid \chi(g) \geq m\}$. Requiring the decreasing filtration $\{U_m\}_{m \in \mathbb{Z}}$ to form a basis of open neighborhoods of 0 defines a topology on $\mathbb{Z}G$, compatible with the ring structure. Then, as noted in [4, §4.2], the Novikov–Sikorav ring is the completion of $\mathbb{Z}G$ with respect to this filtration:

$$(39) \quad \widehat{\mathbb{Z}G}_\chi = \varprojlim_m \mathbb{Z}G/U_m.$$

Moreover, the Novikov–Sikorav completion enjoys the following functoriality property. Let $\phi: G \rightarrow K$ be a homomorphism, and let $\bar{\phi}: \mathbb{Z}^G \rightarrow \mathbb{Z}^K$ be its linear extension to formal sums. If $\chi: K \rightarrow \mathbb{R}$ is a character, then $\bar{\phi}$ restricts to a morphism of topological rings between the corresponding completions, $\hat{\phi}: \widehat{\mathbb{Z}G}_{\chi \circ \phi} \rightarrow \widehat{\mathbb{Z}K}_\chi$.

5.5. Novikov–Sikorav homology. In his thesis [54], J.-Cl. Sikorav reinterpreted the BNS invariant of a finitely generated group G in terms of Novikov homology. This interpretation was extended to all BNSR invariants by Bieri [4], and later to the BNSR invariants of CW-complexes by Farber, Geoghegan and Schütz [28].

Let X be a connected CW-complex and let $G = \pi_1(X)$. Recall that the homology groups of X with coefficients in a left $\mathbb{Z}G$ -module M are given by $H_i(X, M) := H_i(C_*(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}G} M)$, where \tilde{X} is the universal cover of X and $C_*(\tilde{X}, \mathbb{Z})$ is its cellular chain complex, viewed as a right $\mathbb{Z}G$ -module via the cellular action of G on the chains of \tilde{X} . We will use in an essential way the following theorem, which expresses the BNSR invariants of X as the vanishing loci for homology with coefficients in the Novikov–Sikorav completions of G .

Theorem 5.6 ([28]). *If X is a connected CW-complex with finite q -skeleton, then*

$$(40) \quad \Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \text{ for all } i \leq q\}.$$

In particular, the BNS set $\Sigma^1(G) = -\Sigma^1(G, \mathbb{Z})$ consists of those characters $\chi \in S(G)$ for which both $H_0(G, \widehat{\mathbb{Z}G}_\chi)$ and $H_1(G, \widehat{\mathbb{Z}G}_\chi)$ vanish.

Remark 5.7. There were several steps along the way where we had to make a choice of sign in the various definitions. As much as possible, we used the original sign conventions, but the literature varies in many of the particulars, so much care must be taken; see for instance the discussions in [7, §1.3], [28, §3], and [30, §4.4]. The sign discrepancy between the BNS invariant $\Sigma^1(G)$ and the BNSR invariant $\Sigma^1(G, \mathbb{Z})$ is as noted by Bieri and Renz in [7]. The inequality from (38) used in the definition of the Novikov–Sikorav completion is as in [45, 27, 28, 48, 30], but the opposite of [4, 36], whence the minus signs in (39) and (40). In the end, as done in [30], the best way to check that all those signs and inequalities are consistent is to verify the computations on the

Baumslag–Solitar group $BS_{1,2}$ (the simplest group G for which $\Sigma^1(G)$ is not symmetric); we will also do this in Example 8.2.

6. Σ -INVARIANTS, COHOMOLOGY JUMP LOCI, AND TROPICALIZATION

In this section we prove our main result, relating the BNSR invariants of a space X to the tropicalization of its characteristic varieties. As before, we will assume that X has the homotopy type of a connected CW-complex with finite q -skeleton, for some $q \geq 1$, and $b_1(X) > 0$.

6.1. Σ -invariants and exponential tangent cones. We start by recalling the main result from [48], which establishes a bridge between the Σ -invariants of a space X as above and the real points on the exponential tangent cones to the respective characteristic varieties. As in the previous section, $S(V)$ denotes the intersection of the unit sphere $S(X) \subset H^1(X, \mathbb{R})$ with a subset $V \subset H^1(X, \mathbb{R})$; in particular, if $V = \{0\}$, then $S(V) = \emptyset$.

Theorem 6.1 ([48]). *Let X be as above. Then,*

$$(41) \quad \Sigma^q(X, \mathbb{Z}) \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)))^c.$$

Qualitatively, this theorem says that each BNSR set $\Sigma^q(X, \mathbb{Z})$ is contained in the complement of a union of rationally defined great subspheres. The proof of this result, given in [48, Prop. 8.5 and Thm. 9.1], makes use of the Novikov Betti numbers, $b_i(X, \chi)$, associated to an additive character $\chi \in S(X)$, and involves showing that

$$(42) \quad -\chi \in \Sigma^q(X, \mathbb{Z}) \Rightarrow b_{\leq q}(X, \chi) = 0 \Leftrightarrow \chi \notin \tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)).$$

We will give in Corollary 6.6 a different proof of Theorem 6.1, which does not rely on the (forward) implications from (42), yet reaches a stronger conclusion.

Now suppose the space X is *formal*, in the sense of rational homotopy theory. Then, by the Tangent Cone Theorem from [19, 21],

$$(43) \quad \tau_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X)$$

for all $i \leq q$, where $\mathcal{R}^i(X) \subseteq H^1(X, \mathbb{C})$ are the *resonance varieties* associated to the cohomology algebra $H^*(X, \mathbb{C})$; see [59, 60] for more on this. Letting $\mathcal{R}^i(X, \mathbb{R}) = \mathcal{R}^i(X) \cap H^1(X, \mathbb{R})$ be the real resonance varieties, this allows us to replace $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X))$ by $\mathcal{R}^i(X, \mathbb{R})$ in Proposition 4.4; moreover, the following inclusion holds in the formal setting,

$$(44) \quad \Sigma^q(X, \mathbb{Z}) \subseteq S(\mathcal{R}^{\leq q}(X, \mathbb{R}))^c.$$

In some instances, which are treated in detail in [48], the inclusions (41) or (44) hold as equalities. We briefly recall those examples.

Example 6.2. Let X be a nilmanifold. Then $\Sigma^i(X, \mathbb{Z}) = S(X)$, while $\mathcal{V}^i(X) = \{1\}$ and so $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) = \{0\}$, for all i . Thus, $\Sigma^q(X, \mathbb{Z}) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)))^c$, for all q .

Example 6.3. Associated to every finite simple graph Γ there is a right-angled Artin group, $G = G_\Gamma$. Such a group admits as classifying space a finite CW-complex which is formal. As shown in [48] (based on computations from [42] and [47]), the equality $\Sigma^q(G, \mathbb{R}) = S(\mathcal{R}^{\leq q}(G, \mathbb{R}))^c$ holds for all q . Moreover, $\Sigma^q(G, \mathbb{Z}) = S(\mathcal{R}^{\leq q}(G, \mathbb{R}))^c$, provided the homology groups of certain subcomplexes in the flag complex of Γ are torsion-free. This condition is always satisfied in degree $q = 1$, giving $\Sigma^1(G) = S(\mathcal{R}^1(G, \mathbb{R}))^c$.

In general, though—as we shall see in a number of examples in the last few sections—the inclusions (41) and (44) are strict, even when $q = 1$.

6.2. Characters and valuations. In order to establish our main result, we will rely instead on [48, Thm. 10.1]. For completeness, we outline the proof of this theorem, with some additional details and explanations provided.

Theorem 6.4 ([48]). *Let X be a space as above, and let \mathbb{k} be an arbitrary field. Suppose $\rho: \pi_1(X) \rightarrow \mathbb{k}^\times$ is a multiplicative character such that $\rho \in \mathcal{V}^{\leq q}(X, \mathbb{k})$. Let $\nu: \mathbb{k}^\times \rightarrow \mathbb{R}$ be the homomorphism defined by a valuation on \mathbb{k} , and write $\chi = \nu \circ \rho$. If the additive character $\chi: \pi_1(X) \rightarrow \mathbb{R}$ is nonzero, then $\chi \notin \Sigma^q(X, \mathbb{Z})$.*

Proof. Let $\hat{\mathbb{k}}$ be the topological completion of \mathbb{k} with respect to the absolute value defined by the valuation ν . Then $\hat{\mathbb{k}}$ is a field, and the map to the completion, $\iota: \mathbb{k} \hookrightarrow \hat{\mathbb{k}}$, is a field extension.

Now let $G = \pi_1(X)$. Every character $\rho: G \rightarrow \mathbb{k}^\times$ extends linearly to a ring map, $\bar{\rho}: \mathbb{Z}G \rightarrow \mathbb{k}$. Since $\chi = \nu \circ \rho$, formula (39) allows us to extend $\bar{\rho}$ to a morphism of topological rings, $\hat{\rho}: \widehat{\mathbb{Z}G}_{-\chi} \rightarrow \hat{\mathbb{k}}$. This makes $\hat{\mathbb{k}}$ into a $\widehat{\mathbb{Z}G}_{-\chi}$ -module, denoted $\hat{\mathbb{k}}_{\hat{\rho}}$; restricting scalars via the inclusion $\mathbb{Z}G \hookrightarrow \widehat{\mathbb{Z}G}_{-\chi}$ yields the $\mathbb{Z}G$ -module $\hat{\mathbb{k}}_{\iota \circ \rho}$, defined by the character $\iota \circ \rho: G \rightarrow \hat{\mathbb{k}}^\times$.

For a ring R , a bounded below chain complex of flat right R -modules K_* , and a left R -module M , there is a (right half-plane, boundedly converging) Künneth spectral sequence,

$$(45) \quad E_{ij}^2 = \text{Tor}_i^R(H_j(K), M) \Rightarrow H_{i+j}(K \otimes_R M),$$

see [67, Thm. 5.6.4]. We will apply this spectral sequence to the ring $R = \widehat{\mathbb{Z}G}_{-\chi}$, the chain complex of free R -modules $K_* = C_*(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}_{-\chi}$, and the R -module $M = \hat{\mathbb{k}}_{\hat{\rho}}$.

Let $\rho \in \mathcal{V}^{\leq q}(X, \mathbb{k})$, and suppose that $\chi = \nu \circ \rho$ belongs to $\Sigma^q(X, \mathbb{Z})$. By Theorem 5.6, this condition is equivalent to the vanishing of $H_j(X, \widehat{\mathbb{Z}G}_{-\chi})$ for all $j \leq q$; that is, $H_j(K) = 0$ for $j \leq q$. Therefore, $E_{ij}^2 = 0$ for $j \leq q$. Noting that

$$(46) \quad K \otimes_R M = C_*(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}_{-\chi} \otimes_{\widehat{\mathbb{Z}G}_{-\chi}} \hat{\mathbb{k}}_{\hat{\rho}} = C_*(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}G} \hat{\mathbb{k}}_{\iota \circ \rho},$$

we infer from (45) that $H_{i+j}(X, \hat{\mathbb{k}}_{\iota \circ \rho}) = 0$ for $j \leq q$, and so $H_j(X, \hat{\mathbb{k}}_{\iota \circ \rho}) = 0$ for $j \leq q$. From the definition of the characteristic varieties, this is equivalent to $\iota \circ \rho \notin \mathcal{V}^{\leq q}(X, \hat{\mathbb{k}})$. Hence, by (23), $\rho \notin \mathcal{V}^{\leq q}(X, \mathbb{k})$, contradicting our hypothesis on ρ . Therefore, $\chi \notin \Sigma^q(X, \mathbb{Z})$, and we are done. \square

The above theorem builds on an idea that goes back to Bieri and Groves [5]. A particular case of Theorem 6.4—for $X = K(G, 1)$, $q = 1$, and ν a discrete valuation—was previously proved by Delzant in [14, Prop. 1], using the interpretation of $\Sigma^1(G)$ in terms of G -actions on trees given by Brown in [9].

6.3. Main result. We are now in a position to state and prove our main result. Once again, let X be a connected CW-complex with finite q -skeleton, for some $q \geq 1$. We place ourselves in the framework from Sections 2 and 4, and work with the base field $\mathbb{k} = \mathbb{C}$ and its extension to the field of Puiseux series, $\mathbb{K} = \mathbb{C}\{\{t\}\}$, endowed with its usual valuation map, $\text{val}: \mathbb{K}^\times \rightarrow \mathbb{Q}$.

Theorem 6.5. *Let $\mathcal{V}^{\leq q}(X) \subset H^1(X, \mathbb{K}^\times)$ be the union of the characteristic varieties of X in degrees up to q , and let $\text{Trop}(\mathcal{V}^{\leq q}(X)) \subset H^1(X, \mathbb{R})$ be its tropicalization. Then*

$$(47) \quad \Sigma^q(X, \mathbb{Z}) \subseteq S(\text{Trop}(\mathcal{V}^{\leq q}(X)))^c.$$

Proof. Let $\rho \in H^1(X, \mathbb{K}^\times)$ be a \mathbb{K} -valued, multiplicative character. Composing the valuation map $\text{val}: \mathbb{K}^\times \rightarrow \mathbb{Q}$ with the homomorphism $\rho: \pi_1(X) \rightarrow \mathbb{K}^\times$, we obtain an additive character, $\chi := \text{val} \circ \rho: \pi_1(X) \rightarrow \mathbb{Q}$, which may be viewed as an element of $H^1(X, \mathbb{Q})$, that is, a rational point on $H^1(X, \mathbb{R})$. Moreover, if χ is nonzero, then χ determines a rational point on $S(X)$.

Now suppose ρ belongs to the characteristic variety $\mathcal{V}^{\leq q}(X, \mathbb{K}) = \mathcal{V}^{\leq q}(X) \times_{\mathbb{C}} \mathbb{K}$. By (35), the set of rational points on the tropicalization of $\mathcal{V}^{\leq q}(X)$ is $\nu_X(\mathcal{V}^{\leq q}(X, \mathbb{K}))$. Thus, $\chi = \text{val} \circ \rho = \nu_X(\rho)$ is a rational point on $\text{Trop}(\mathcal{V}^{\leq q}(X)) = \overline{\nu_X(\mathcal{V}^{\leq q}(X, \mathbb{K}))}$, and conversely, all rational points on $\text{Trop}(\mathcal{V}^{\leq q}(X))$ are of the form $\nu_X(\rho)$, for some $\rho \in \mathcal{V}^{\leq q}(X, \mathbb{K})$.

Finally, assume that χ is also nonzero, so that it represents an (arbitrary) rational point in $S(\text{Trop}(\mathcal{V}^{\leq q}(X)))$. Then, by Theorem 6.4, the additive character χ belongs to $\Sigma^q(X, \mathbb{Z})^c$. By the above, though, the set of rational points is dense in $S(\text{Trop}(\mathcal{V}^{\leq q}(X)))$. Moreover, we also know from §5.2 that $\Sigma^q(X, \mathbb{Z})^c$ is a closed subset of $S(X)$. Thus, $S(\text{Trop}(\mathcal{V}^{\leq q}(X))) \subseteq \Sigma^q(X, \mathbb{Z})^c$. \square

Since we are working over the field $\mathbb{k} = \mathbb{C}$, we also have at our disposal the exponential map, $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$, and the resulting exponential tangent cone construction, $\tau_1(W)$, for subvarieties $W \subset (\mathbb{C}^\times)^n$. The next result recovers (in a much stronger form) Theorem 6.1.

Corollary 6.6. *With notation as above,*

$$(48) \quad \Sigma^q(X, \mathbb{Z}) \subseteq S(\text{Trop}(\mathcal{V}^{\leq q}(X)))^c \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)))^c.$$

Proof. The first inclusion comes from Theorem 6.5, while the second inclusion follows from Proposition 2.7. \square

In a very special situation (which we will encounter several times later on), the above theorem allows us to precisely identify the sets $\Sigma^q(X, \mathbb{Z})$, even in cases when Theorem 6.1 does not.

Corollary 6.7. *If $\mathcal{V}^{\leq q}(X)$ contains a connected component of $H^1(X, \mathbb{C}^\times)$, then $\Sigma^q(X, \mathbb{Z}) = \emptyset$.*

Proof. By Lemma 4.2, $\text{Trop}(\mathcal{V}^{\leq q}(X)) = H^1(X, \mathbb{R})$. The claim now follows from Theorem 6.5. \square

7. UPPER BOUNDS FOR THE BNS INVARIANT

In most of the applications and examples described in the following sections, we will only deal with the original BNS sets $\Sigma^1(X) = \Sigma^1(\pi_1(X))$, which recall are equal to $-\Sigma^1(X, \mathbb{Z})$. For completeness, we restate our results from §6.3 in this setting.

Corollary 7.1. *Let X be a connected CW-complex with finite 1-skeleton. Then*

$$(49) \quad \Sigma^1(X) \subseteq -S(\text{Trop}(\mathcal{V}^1(X)))^c.$$

Corollary 7.2. *Let G be a finitely generated group. Then*

$$(50) \quad \Sigma^1(G) \subseteq -S(\text{Trop}(\mathcal{V}^1(G)))^c \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))^c.$$

Corollary 7.3. *If $\mathcal{V}^1(G)$ contains a connected component of \mathbb{T}_G , then $\Sigma^1(G) = \emptyset$.*

We shall see applications of the last corollary in Sections 10–11 on orbifold Riemann surfaces and Seifert manifolds.

Under some more restrictive hypothesis, it is possible to recast these tropical upper bounds in terms of simpler, more computable data. Given a finitely generated group G with $b_1(G) > 0$, we prove two such results: one in which the upper bound for $\Sigma^1(G)$ is given in terms of the

tropicalization of the Alexander polynomial $\Delta_G \in \mathbb{Z}H$, where $H = G_{\text{ab}} / \text{Tors}(G_{\text{ab}})$, and the other in which the upper bound is given in purely homological terms, using a family of projections onto suitably chosen quotients of G .

Theorem 7.4. *Suppose Δ_G is symmetric and $I_H^p \cdot (\Delta_G) = E_1(A_G)$, for some $p \geq 0$. Then*

$$(51) \quad \Sigma^1(G) \subseteq \bigcup_F S(F),$$

where F runs through the open facets of B_A .

Proof. By Corollary 7.2, $\Sigma^1(G)$ is contained in $-S(\text{Trop}(\mathcal{V}^1(G)))^c$; in turn, this set is contained in $-S(\text{Trop}(\mathcal{V}^1(G) \cap \mathbb{T}_G^0))^c$.

By Proposition 4.5, the set $\text{Trop}(\mathcal{V}^1(G) \cap \mathbb{T}_G^0)$ is symmetric and coincides with the positive-codimension skeleton of $\mathcal{F}(B_A)$, the face fan of the unit ball in the Alexander norm. Thus, its complement in $H^1(G, \mathbb{R})$ is the union of the open cones on the facets of B_A . The claim follows. \square

We shall give applications of this theorem in Sections 8–9 on 1-relator groups and 3-manifold groups.

Theorem 7.5. *Let G be a finitely generated group, and let $f_\alpha: G \rightarrow G_\alpha$ be a finite collection of epimorphisms. Suppose each characteristic variety $\mathcal{V}^1(G_\alpha)$ contains a component of \mathbb{T}_{G_α} . Then*

$$(52) \quad \Sigma^1(G) \subseteq \left(\bigcup_\alpha S(f_\alpha^*(H^1(G_\alpha, \mathbb{R}))) \right)^c.$$

Proof. Fix an index α , and consider the induced homomorphism $f_\alpha^*: H^1(G_\alpha, \mathbb{R}) \rightarrow H^1(G, \mathbb{R})$. Since f_α is assumed to be surjective, f_α^* is an \mathbb{R} -linear injection, and, by Proposition 4.3, the map f_α^* takes $\text{Trop}(\mathcal{V}^1(G_\alpha))$ to $\text{Trop}(\mathcal{V}^1(G))$. Therefore,

$$(53) \quad \bigcup_\alpha f_\alpha^*(\text{Trop}(\mathcal{V}^1(G_\alpha))) \subseteq \text{Trop}(\mathcal{V}^1(G)).$$

By our other assumption, we have that $\mathcal{V}^1(G_\alpha) \supseteq \rho \mathbb{T}_{G_\alpha}^0$, for some (torsion) character $\rho \in \mathbb{T}_{G_\alpha}$. Thus, by Lemma 4.2, $\text{Trop}(\mathcal{V}^1(G_\alpha)) = H^1(G_\alpha, \mathbb{R})$. Therefore,

$$(54) \quad \begin{aligned} \Sigma^1(G) &\subseteq -S(\text{Trop}(\mathcal{V}^1(G)))^c && \text{by Corollary 7.2} \\ &\subseteq -\left(\bigcup_\alpha S(f_\alpha^*(\text{Trop}(\mathcal{V}^1(G_\alpha))) \right)^c && \text{by (53)} \\ &= \left(\bigcup_\alpha S(f_\alpha^*(H^1(G_\alpha, \mathbb{R}))) \right)^c, \end{aligned}$$

where at the last step we used the aforementioned equality, together with the homogeneity of the linear subspace $f_\alpha^*(H^1(G_\alpha, \mathbb{R}))$ to dispense with the sign. \square

Remark 7.6. It is possible to give an alternative argument, relying on Proposition 5.5 instead of Proposition 4.3. In this approach, one starts by noting that each map $f_\alpha^*: S(G_\alpha) \hookrightarrow S(G)$ is an embedding taking $\Sigma^1(G_\alpha)^c$ to $\Sigma^1(G)^c$, and then proceeds in like manner.

We shall provide applications of this theorem in Sections 12–13 on Kähler manifolds and hyperplane arrangements.

8. ONE-RELATOR GROUPS

In [9], Brown gave an algorithm for computing the BNS invariant of a 1-relator group G (see [7, 30] for other approaches). If G has at least 3 generators, then $\Sigma^1(G) = \emptyset$, so the interesting case is that of a 2-generator, 1-relator group, $G = \langle x_1, x_2 \mid r \rangle$.

First suppose $b_1(G) = 1$. Let $\chi: G \rightarrow \mathbb{R}$ be a nonzero homomorphism, and assume without loss of generality that $\chi(x_1) > 0$. Then one determines whether $[\chi]$ belongs to $\Sigma^1(G)$ according to whether the “leading term” of $\partial_2(r)$ in the χ -direction is of the form $\pm g \in \mathbb{Z}G$.

Example 8.1. Let $G = \mathbb{Z} * \mathbb{Z}_2 = \langle x_1, x_2 \mid x_2^2 \rangle$. Then $S(G) = \{\pm\chi\}$, where $\chi(x_1) = 1$ and $\chi(x_2) = 0$. The Fox derivative $\partial_2(r) = 1 + x_2$ has leading term $1 + x_2$ in both directions; thus, $\Sigma^1(G) = \emptyset$. On the other hand, $\mathbb{T}_G = \mathbb{C}^\times \times \{\pm 1\}$ and $\mathcal{V}^1(G) = \{1\} \cup \mathbb{C}^\times \times \{-1\}$. Hence, $\Sigma^1(G) = S(\text{Trop}(\mathcal{V}^1(G)))^c$, although $S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))^c = \{\pm\chi\}$.

Example 8.2. Let $G = \langle x_1, x_2 \mid x_1 x_2 x_1^{-1} x_2^{-2} \rangle$ be the Baumslag–Solitar group $\text{BS}_{1,2}$. Then $S(G) = \{\pm\chi\}$, where $\chi(x_1) = 1$ and $\chi(x_2) = 0$. The Fox derivative $\partial_2(r) = x_1 - x_2 - 1$ has leading term x_1 in the χ -direction, and $-(1 + x_2)$ in the $-\chi$ -direction; thus, $\Sigma^1(G) = \{\chi\}$. On the other hand, $\mathbb{T}_G = \mathbb{C}^\times$ and $\mathcal{V}^1(G) = \{1, 2\}$; thus, $\text{Trop}(\mathcal{V}^1(G)) = \tau_1^{\mathbb{R}}(\mathcal{V}^1(G)) = \{0\}$, and so inclusion (55) is strict in this case.

Now suppose $b_1(G) = 2$. Then, in fact, $G_{\text{ab}} \cong \mathbb{Z}^2$, and without loss of essential generality, we may assume that r is a nontrivial, cyclically reduced word in the commutator subgroup of the free group on generators x_1 and x_2 . Let P be the boundary of the convex hull of the walk traced out by the relator r in the x_1 - x_2 plane \mathbb{R}^2 ; vertices of the polygon P traversed only once are called simple, while horizontal or vertical edges are called special if they contain precisely two simple vertices. Then $\Sigma^1(G)$ is a finite union of open arcs on the unit circle S^1 : for each simple vertex v or special edge e of P , the corresponding arc runs from the normal vector to the edge preceding v or e to the one following it (when proceeding in a counterclockwise direction).

On the other hand, the Alexander module A_G is presented by the matrix $(\partial_1(r)^{\text{ab}} \ \partial_2(r)^{\text{ab}})$, while the Alexander polynomial Δ_G is the gcd of the entries of this matrix; hence, $I_{G_{\text{ab}}} \cdot (\Delta_G) = E_1(A_G)$. By Proposition 3.2, we have $\mathcal{V}^1(G) = V(\Delta_G) \cup \{1\}$, and so the tropical upper bound from (50) takes the form

$$(55) \quad \Sigma^1(G) \subseteq -S(\text{Trop}(V(\Delta_G)))^c,$$

where $\text{Trop}(V(\Delta_G))$ is the union of the rays in $H^1(G, \mathbb{R}) = \mathbb{R}^2$ starting at 0 and inner normal to the edges of the Newton polygon of Δ_G . On the other hand, $\tau_1^{\mathbb{R}}(\mathcal{V}^1(G))$ is a union of lines through 0 with rational slopes, and equals $\{0\}$ if $\Delta_G(1) \neq 0$. We illustrate the way this works in three examples (corresponding to the first three pictures from Figure 1).

Example 8.3. Let $G = \langle x_1, x_2 \mid x_1 x_2^2 x_1^{-1} x_2^{-2} \rangle$. Then $\Delta_G = t_2 + 1$ and $\Sigma^1(G) = S^1 \setminus \{(\pm 1, 0)\}$.

Example 8.4. Let $G = \langle x_1, x_2 \mid x_2^2(x_1 x_2^{-1})^2 x_1^{-2} \rangle$. Then $\Delta_G = t_1 + t_2 + 1$ and $\Sigma^1(G) = S^1 \setminus \{(-1, 0), (0, -1), (1/\sqrt{2}, 1/\sqrt{2})\}$.

In both of the above examples, $\Sigma^1(G)$ coincides with $-S(\text{Trop}(\mathcal{V}^1(G)))^c$, but is properly contained in $S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))^c = S^1$.

Example 8.5. Let $G = \langle x_1, x_2 \mid x_1^{-1} x_2^{-1} x_1 x_2^2 x_1^{-1} x_2^{-1} x_1^2 x_2^{-1} x_1^{-1} x_2 x_1^{-1} x_2 x_1 x_2^{-1} \rangle$. As an application of his algorithm, Brown showed that $\Sigma^1(G)$ consists of two open arcs on the unit circle, joining the

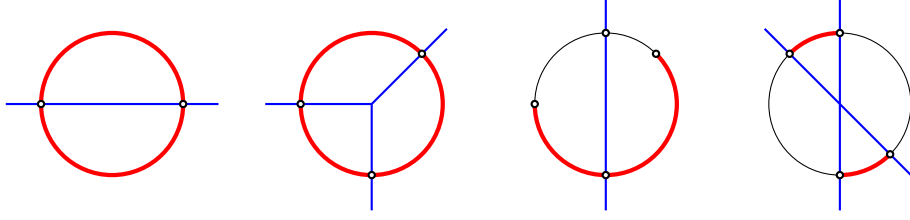


FIGURE 1. BNS invariants $\Sigma^1(G)$ (in red) and tropical curves $-\text{Trop}(\mathcal{V}^1(G))$ (in blue) for the one-relator groups G from Examples 8.3, 8.4, 8.5, and 9.6

points $(-1, 0)$ to $(0, -1)$ and $(0, -1)$ to $(1/\sqrt{2}, 1/\sqrt{2})$, respectively. On the other hand, $\Delta_G = t_1 - 1$ and so $S(\text{Trop}(\mathcal{V}^1(G))) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))$ consists of the whole circle with the points $(0, \pm 1)$ removed. Thus, inclusion (55) is strict in this case.

9. COMPACT 3-MANIFOLDS

In this section, M will denote a compact, connected 3-manifold with $b_1(M) > 0$. We say that a nontrivial cohomology class $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$ is a fibered class if there exists a fibration $p: M \rightarrow S^1$ such that the induced map $p_*: \pi_1(M) \rightarrow \pi_1(S^1) = \mathbb{Z}$ coincides with ϕ .

The Thurston norm $\|\phi\|_T$ of a class $\phi \in H^1(M; \mathbb{Z})$ is defined as the infimum of $-\chi(S_0)$, where S runs through all the properly embedded, oriented surfaces in M dual to ϕ , and S_0 denotes the result of discarding all components of S which are disks or spheres. In [63], Thurston proved that $\|\cdot\|_T$ defines a seminorm on $H^1(M; \mathbb{Z})$, which can be extended to a continuous seminorm on $H^1(M; \mathbb{R})$.

The unit norm ball, $B_T = \{\phi \in H^1(M; \mathbb{R}) \mid \|\phi\|_T \leq 1\}$, is a rational polyhedron with finitely many sides and which is symmetric in the origin. Moreover, there are facets of B_T , called the *fibered faces* (coming in antipodal pairs), so that a class $\phi \in H^1(M; \mathbb{Z})$ fibers if and only if it lies in the cone over the interior of a fibered face. In [6, Thm. E], Bieri, Neumann, and Strebel showed that the BNS invariant of $G = \pi_1(M)$ is the projection onto $S(G)$ of the open fibered faces of the Thurston norm ball B_T ; in particular, $\Sigma^1(G) = -\Sigma^1(G)$.

Under some mild assumptions, McMullen [41] established an inequality between the Alexander and Thurston norms of a 3-manifold M . When combined with the above theorem from [6], this inequality leads to an upper bound for the BNS invariant of G in terms of the unit ball in the Alexander norm, see [22]. We show next how to recover this upper bound from our tropical bound; for the sake of brevity, we restrict to the case $b_1(M) \geq 2$. For more on the characteristic varieties of 3-manifolds, as well as their (exponential) tangent cones, we refer to [60].

Proposition 9.1. *Let M be a compact, connected, orientable, 3-manifold with empty or toroidal boundary. Set $G = \pi_1(M)$ and assume $b_1(M) \geq 2$. Then*

- (1) $\text{Trop}(\mathcal{V}^1(G) \cap \mathbb{T}_G^0)$ coincides with the positive-codimension skeleton of $\mathcal{F}(B_A)$, the face fan of the unit ball in the Alexander norm.
- (2) $\Sigma^1(G)$ is contained in the union of the open cones on the facets of B_A .

Proof. Work of Milnor [43] and Turaev [64, 65] shows that the Alexander polynomial of a 3-manifold is symmetric (up to units). Moreover, as shown by McMullen [41] (see also [25, 65]), we have that $I_H^p \cdot (\Delta_G) = E_1(A_G)$, where $H = G_{\text{ab}}/\text{Tors}$, and $p = 1$ if $\partial M = \emptyset$ and $p = 2$, otherwise. Thus, claim (1) follows from Proposition 4.5, while claim (2) follows from Theorem 7.4. \square

Remark 9.2. Here is a sketch of the original proof of claim (2). As shown by McMullen in [41], $\|\phi\|_A \leq \|\phi\|_T$ for all $\phi \in H^1(M, \mathbb{R})$, with equality if $\phi \in H^1(M, \mathbb{Z})$ is a fibered class; thus $B_T \subseteq B_A$, and each fibered face of B_T is included in a face of B_A . Since $\Sigma^1(G)$ is the projection onto $S(G)$ of the fibered faces of B_T , the claim follows.

Remark 9.3. Let G be a 3-dimensional Poincaré duality group such that G admits a finite $K(G, 1)$ and $\mathbb{Z}G$ embeds in a division algebra. Under these hypotheses, Kielak shows in [36, Thm. 5.32] that $\Sigma^1(G)$ is the projection onto $S(G)$ of the open cones on the marked faces of a certain polytope in $H^1(G, \mathbb{R})$. It would be interesting to know whether an analogue of Proposition 9.1 still holds, with B_T replaced by Kielak's marked polytope.

The next example shows that the tropical bound from Corollary 7.1 (or Proposition 9.1) gives more information than the exponential tangent cone bound from Theorem 6.1.

Example 9.4. Let M be a closed, orientable 3-manifold with $H_1(M, \mathbb{Z}) = \mathbb{Z}^2$ and $\Delta_M = (t_1 + t_2)(t_1 t_2 + 1) - 4t_1 t_2$ (such a manifold exists by [65, VII.5.3]). It is readily seen that $\tau_1(\mathcal{V}_1^1(M)) = \{0\}$, yet $\text{Trop}(V(\Delta_M))$ consists of the two diagonals in the plane. Thus, $\Sigma^1(M)$ is contained in the complement in S^1 of the four points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ and $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

We conclude this section with a couple of examples from link theory.

Example 9.5. Let L be the n -component Hopf link, consisting of $n \geq 3$ fibers of the Hopf fibration $S^3 \rightarrow S^2$, and let G be the link group. Then $\Delta_G = (t_1 \cdots t_n - 1)^{n-2}$, and a quick computation shows that $S(\text{Trop}(\mathcal{V}^1(G))) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))$ is the great sphere cut out by the hyperplane $\sum x_i = 0$, while $\Sigma^1(G) = S^{n-1} \setminus S^{n-2}$ is the complement of this great sphere.

As shown by Dunfield [22], though, the fibered faces of B_T may be strictly included in the corresponding faces of B_A .

Example 9.6. Let L be the 2-component studied in [22, §6]. The link group G is a 2-generator, 1-relator group with $\Delta_G = (t_1 - 1)(t_1 t_2 - 1)$. Thus, $S(\text{Trop}(\mathcal{V}^1(G))) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))$ consists of the four points $(\pm 1, 0)$ and $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$. On the other hand, $\Sigma^1(G)$ consists of two open arcs joining those two pairs of points (see the last picture in Figure 1).

10. ORBIFOLD RIEMANN SURFACES

Let Σ_g be a Riemann surface of genus $g \geq 1$ and let $k \geq 0$ be an integer. If $k > 0$, fix points q_1, \dots, q_k in Σ_g , assign to these points an integer weight vector $\mathbf{m} = (m_1, \dots, m_k)$ with $m_i \geq 2$, and set $|\mathbf{m}| = k$. The orbifold Euler characteristic of the surface with marked points is defined as $\chi^{\text{orb}}(\Sigma_g, \mathbf{m}) = 2 - 2g - \sum_{i=1}^k (1 - 1/m_i)$, while the orbifold fundamental group $\Gamma = \pi_1^{\text{orb}}(\Sigma_g, \mathbf{m})$ has presentation

$$(56) \quad \Gamma = \left\langle \begin{array}{c} x_1, \dots, x_g, y_1, \dots, y_g, \\ z_1, \dots, z_k \end{array} \left| \begin{array}{l} [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_k = 1, \\ z_1^{m_1} = \cdots = z_k^{m_k} = 1 \end{array} \right. \right\rangle.$$

There is an obvious epimorphism $\Gamma \twoheadrightarrow \pi_1(\Sigma_g)$, obtained by sending each z_i to 1. Upon abelianizing, we obtain an isomorphism $\Gamma_{\text{ab}} \cong \pi_1(\Sigma_g)_{\text{ab}} \oplus \Theta$, where the torsion subgroup,

$$(57) \quad \Theta := \text{Tors}(\Gamma_{\text{ab}}) = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k} / (1, \dots, 1),$$

has order $|\Theta| = \theta(\mathbf{m})$, where

$$(58) \quad \theta(\mathbf{m}) := m_1 \cdots m_k / \text{lcm}(m_1, \dots, m_k).$$

Let us identify the character group $\mathbb{T}_\Gamma = \text{Hom}(\Gamma, \mathbb{C}^\times)$ with $\mathbb{T}_\Gamma^0 \times \mathbb{T}_\Theta$, where $\mathbb{T}_\Gamma^0 \cong (\mathbb{C}^\times)^{2g}$ and $\mathbb{T}_\Theta \cong \Theta$. As noted in [2] (see also [58]), we then have that

$$(59) \quad \mathcal{V}^1(\Gamma) = \begin{cases} \mathbb{T}_\Gamma & \text{if } g > 1, \\ (\mathbb{T}_\Gamma \setminus \mathbb{T}_\Gamma^0) \cup \{1\} & \text{if } g = 1 \text{ and } \theta(\mathbf{m}) > 1, \\ \{1\} & \text{otherwise.} \end{cases}$$

Let us identify $\text{Hom}(\Gamma, \mathbb{R})$ with \mathbb{R}^{2g} . In the first case, $\text{Trop}(\mathcal{V}^1(\Gamma)) = \tau_1(\mathcal{V}^1(\Gamma)) = \mathbb{R}^{2g}$, while in the third case $\text{Trop}(\mathcal{V}^1(\Gamma)) = \tau_1(\mathcal{V}^1(\Gamma)) = \{0\}$. On the other hand, in the second case $\mathbb{T}_\Gamma \setminus \mathbb{T}_\Gamma^0 \cong \mathbb{T}_\Gamma^0 \times (\Theta \setminus \{1\})$, where the second factor is nonempty; thus, $\text{Trop}(\mathcal{V}^1(\Gamma)) = \mathbb{R}^{2g}$ yet $\tau_1(\mathcal{V}^1(\Gamma)) = \{0\}$. Using Corollary 7.3, we obtain the following result.

Proposition 10.1. *Let (Σ_g, \mathbf{m}) be a compact orbifold surface such that either $g \geq 2$ or $g = 1$ and $\theta(\mathbf{m}) > 1$. Then $\Sigma^1(\pi_1^{\text{orb}}(\Sigma_g, \mathbf{m})) = \emptyset$.*

Now let $\Sigma_{g,r} = \Sigma_g \setminus \{p_1, \dots, p_r\}$ be a Riemann surface of genus $g \geq 0$ with r points removed ($r \geq 1$). Then $\pi_1(\Sigma_{g,r}) = F_n$, where $n = b_1(\Sigma_{g,r}) = 2g + r - 1$. To avoid trivialities, we will assume $n > 0$; note that $n = 1$ if and only if $g = 0$ and $r = 2$, i.e., $\Sigma_{g,r} = \mathbb{C}^\times$.

Next, consider a non-compact 2-orbifold, i.e., a surface $(\Sigma_{g,r}, \mathbf{m})$ with marked points q_1, \dots, q_k , weight vector $\mathbf{m} = (m_1, \dots, m_k)$, and r points removed. The orbifold Euler characteristic of the surface is defined as $\chi^{\text{orb}}(\Sigma_{g,r}, \mathbf{m}) = 1 - n - \sum_{i=1}^k (1 - 1/m_i)$, while the orbifold fundamental group $\Gamma = \pi_1^{\text{orb}}(\Sigma_{g,r}, \mathbf{m})$ associated to these data is the free product

$$(60) \quad \Gamma = F_n * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_k}.$$

(The case $n = k = 1$ and $m_1 = 2$ is the one analyzed in Example 8.1.) Thus, $\Gamma_{\text{ab}} = \mathbb{Z}^n \oplus \Theta$, where now $\Theta = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$. A similar computation as above shows that

$$(61) \quad \mathcal{V}^1(\Gamma) = \begin{cases} \mathbb{T}_\Gamma & \text{if } n > 1, \\ (\mathbb{T}_\Gamma \setminus \mathbb{T}_\Gamma^0) \cup \{1\} & \text{if } n = 1 \text{ and } |\mathbf{m}| > 0, \\ \{1\} & \text{if } n = 1 \text{ and } |\mathbf{m}| = 0. \end{cases}$$

Identifying $\text{Hom}(\Gamma, \mathbb{R}) = \mathbb{R}^n$, we find that $\text{Trop}(\mathcal{V}^1(\Gamma)) = \tau_1(\mathcal{V}^1(\Gamma)) = \mathbb{R}^n$ in the first case and $\text{Trop}(\mathcal{V}^1(\Gamma)) = \tau_1(\mathcal{V}^1(\Gamma)) = \{0\}$ in the last one. On the other hand, in the second case $\text{Trop}(\mathcal{V}^1(\Gamma)) = \mathbb{R}^n$, yet $\tau_1(\mathcal{V}^1(\Gamma)) = \{0\}$. Applying Corollary 7.3 once again, we obtain the following result.

Proposition 10.2. *Let $(\Sigma_{g,r}, \mathbf{m})$ be a punctured orbifold surface with negative orbifold Euler characteristic. Then $\Sigma^1(\pi_1^{\text{orb}}(\Sigma_{g,r}, \mathbf{m})) = \emptyset$.*

11. SEIFERT MANIFOLDS

A compact 3-manifold is a Seifert fibered space if it is foliated by circles. One can think of such a manifold M as a bundle in the category of orbifolds, in which the circles of the foliation are the fibers, and the base space of the orbifold bundle is the quotient space of M obtained by identifying each circle to a point. We point to [35, 44, 53] as general references for the subject.

For our purposes here, we will only consider compact, connected, orientable Seifert manifolds with orientable base. Every such manifold M admits an effective circle action, with orbit space a Riemann surface Σ_g , and finitely many exceptional orbits, encoded by pairs of coprime integers $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ with $\alpha_i > 0$. The fundamental group $G = \pi_1(M)$ admits a presentation of the form

$$(62) \quad G = \left\langle \begin{array}{l} x_1, \dots, x_g, y_1, \dots, y_g, \\ z_1, \dots, z_r, h \end{array} \left| \begin{array}{l} [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_r = 1, \\ z_1^{\alpha_1} h^{\beta_1} = \cdots = z_r^{\alpha_r} h^{\beta_r} = 1, h \text{ central} \end{array} \right. \right\rangle,$$

where h is the homotopy class of a regular orbit. We will denote by $e = -\sum_{i=1}^r \beta_i / \alpha_i$ the Euler number of the orbifold bundle. If $e = 0$, then $G_{\text{ab}} = \mathbb{Z}^{2g+1}$, and if $e \neq 0$, then $G_{\text{ab}} = \mathbb{Z}^{2g} \oplus \text{Tors}(G_{\text{ab}})$, where $|\text{Tors}(G_{\text{ab}})| = \alpha_1 \cdots \alpha_r |e|$. Write $\alpha = (\alpha_1, \dots, \alpha_r)$.

Proposition 11.1. *Let M be Seifert manifold with genus g base surface, r exceptional fibers, and orbifold Euler number e . Suppose $e \neq 0$ and either $g > 1$, or $g = 1$ and $\theta(\alpha) > 1$. Then $\Sigma^1(M) = \emptyset$.*

Proof. The orbit map $p: M \rightarrow \Sigma_g$ induces an epimorphism $\pi_1(M) \twoheadrightarrow \pi_1(\Sigma_g)$ sending $h \mapsto 1$. This map factors through an epimorphism $G \twoheadrightarrow \Gamma$, where $\Gamma = \pi_1^{\text{orb}}(\Sigma_g, \alpha)$. Passing to abelianizations, we obtain an epimorphism $p_*: G_{\text{ab}} \twoheadrightarrow \Gamma_{\text{ab}}$, which in turn induces a monomorphism $p^*: \mathbb{T}_{\Gamma} \hookrightarrow \mathbb{T}_G$.

Since $e \neq 0$, the groups G_{ab} and Γ_{ab} have the same rank (equal to $2g$); thus, the map p^* restricts to an isomorphism $p^*: \mathbb{T}_{\Gamma}^0 \xrightarrow{\cong} \mathbb{T}_G^0$. On the other hand, we also know from Proposition 3.1 that the map $p^*: \mathbb{T}_{\Gamma} \hookrightarrow \mathbb{T}_G$ sends $\mathcal{V}^1(\Gamma)$ to $\mathcal{V}^1(G)$. There are two cases to consider.

- If $g > 1$, then by (59), $\mathcal{V}^1(\Gamma) = \mathbb{T}_{\Gamma}$. Therefore $\mathcal{V}^1(G)$ contains $p^*(\mathbb{T}_{\Gamma}) = p^*(\mathcal{V}^1(\Gamma))$, and thus it must also contain $p^*(\mathbb{T}_{\Gamma}^0) = \mathbb{T}_G^0$.
- If $g = 1$ and $\theta(\alpha) > 1$, then again by (59), $\mathcal{V}^1(\Gamma) = \mathbb{T}_{\Gamma} \setminus \mathbb{T}_{\Gamma}^0$, which is a non-empty union of torsion-translated copies of \mathbb{T}_{Γ}^0 , since $\text{Tors}(\Gamma_{\text{ab}}) \neq 0$. Arguing as above, we infer that $\mathcal{V}^1(G)$ must contain a translated copy of \mathbb{T}_G^0 .

In either case, the variety $\mathcal{V}^1(G)$ contains a connected component of \mathbb{T}_G . Hence, by Corollary 7.3, $\Sigma^1(M) = \emptyset$. \square

A well-known class of Seifert manifolds arises in singularity theory. Let (a_1, \dots, a_n) be an n -tuple of integers, with $a_j \geq 2$. Consider the variety X in \mathbb{C}^n defined by the equations $c_{j1}x_1^{a_1} + \cdots + c_{jn}x_n^{a_n} = 0$, for $1 \leq j \leq n-2$. Assuming all maximal minors of the matrix (c_{jk}) are non-zero, X is a quasi-homogeneous surface, with an isolated singularity at 0. The space X admits a good \mathbb{C}^{\times} -action. Set $X^* = X \setminus \{0\}$, and let $p: X^* \rightarrow \Sigma_g$ be the corresponding projection onto a smooth projective curve.

By definition, the Brieskorn manifold $M = \Sigma(a_1, \dots, a_n)$ is the link of the quasi-homogeneous singularity $(X, 0)$. As such, M is a closed, smooth, oriented 3-manifold which is homotopy equivalent to X^* . In fact, M is a Seifert manifold, with S^1 -action obtained by restricting the \mathbb{C}^{\times} -action on X . Put $\ell = \text{lcm}(a_1, \dots, a_n)$, $\ell_i = \text{lcm}(a_1, \dots, \widehat{a_i}, \dots, a_n)$, and $a = a_1 \cdots a_n$, and note that $s_i := a / (a_i \ell_i) \in \mathbb{Z}$. Set $\alpha_i = \ell / \ell_i$ and define integers $0 < \beta_i < \alpha_i$ by $\beta_i(\ell / a_i) \equiv 1 \pmod{\alpha_i}$. The S^1 -equivariant homeomorphism type of M is then determined by the following Seifert invariants associated to the projection $p|_M: M \rightarrow \Sigma_g$:

- The exceptional orbit data, $(s_1(\alpha_1, \beta_1), \dots, s_n(\alpha_n, \beta_n))$, where $s_i(\alpha_i, \beta_i)$ signifies (α_i, β_i) repeated s_i times, unless $\alpha_i = 1$, in which case $s_i(\alpha_i, \beta_i)$ is to be removed from the list.
- The genus of the base curve, $g = \frac{1}{2} \left(2 + \frac{(n-2)a}{\ell} - \sum_{i=1}^n s_i \right)$.
- The orbifold Euler number, $e = -a / \ell^2 < 0$.

The group $H_1(M, \mathbb{Z})$ has rank $2g$ and torsion subgroup of order $\alpha_1^{s_1} \cdots \alpha_n^{s_n} \cdot |e|$. Set $\alpha = \alpha_1^{s_1} \cdots \alpha_n^{s_n} / \text{lcm}(\alpha_1, \dots, \alpha_n)$. Applying Proposition 11.1, we obtain the following corollary.

Corollary 11.2. *Let M be a Brieskorn manifold as above. If either $g > 1$, or $g = 1$ and $\alpha > 1$, then $\Sigma^1(M) = \emptyset$.*

Here is a concrete example, which builds on computations from [20, 62].

Example 11.3. The Brieskorn manifold $M = \Sigma(2, 4, 8)$ fibers over the torus with two exceptional fibers, each of multiplicity 2, and with orbifold Euler number $e = -1$; thus, $g = 1$ and $\alpha = 2$. By Corollary 11.2, we have that $\Sigma^1(M) = \emptyset$. Alternatively, it is readily seen that $H_1(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}_4$, and hence $\mathbb{T}_M = (\mathbb{C}^\times)^2 \times \{\pm 1, \pm i\}$. A short computation shows that $\mathcal{V}^1(M) = \{1\} \cup (\mathbb{C}^\times)^2 \times \{-1\}$. Thus, $\tau_1(\mathcal{V}^1(M)) = \{0\}$, and so the bound from Theorem 6.1 does not say anything in this case. On the other hand, $\text{Trop}_1(\mathcal{V}^1(M)) = H^1(M, \mathbb{R})$, and so the bound from Corollary 7.1 gives the precise answer for $\Sigma^1(M)$.

12. KÄHLER MANIFOLDS

Let M be a compact, connected, Kähler manifold. The structure of the characteristic varieties of such manifolds was determined by Green and Lazarsfeld in [32, 33], building on work of Castelnuovo and de Franchis, Beauville [3], and Catanese [11]. For this reason, the varieties $\mathcal{V}^i(M)$ are also known in this context as the *Green–Lazarsfeld sets* of M . The theory was further amplified by Simpson [55], Ein and Lazarsfeld [23], and Arapura [1], developed in [18, 14, 21, 2, 10], and completed by Wang in [66]. The cornerstone of this theory is that all the Green–Lazarsfeld sets of M are finite unions of torsion translates of algebraic subtori of $H^1(M, \mathbb{C}^\times)$.

In degree $i = 1$, the structure of the Green–Lazarsfeld set $\mathcal{V}^1(M)$ can be made more precise. As before, let Σ_g be a Riemann surface of genus $g \geq 1$, with marked points q_1, \dots, q_k , and weight vector $\mathbf{m} = (m_1, \dots, m_k)$ with $m_i \geq 2$, and set $|\mathbf{m}| := k$. A surjective map $f: M \rightarrow (\Sigma_g, \mathbf{m})$ is called an *orbifold fibration* if f is holomorphic, the fiber over any non-marked point is connected, and, for every marked point q_i the multiplicity of the fiber $f^{-1}(q_i)$ equals m_i . Such a map induces an epimorphism $f_\# : \pi_1(M) \twoheadrightarrow \Gamma$, where $\Gamma = \pi_1^{\text{orb}}(\Sigma_g, \mathbf{m})$. The induced morphism of character groups, $f_\#^* : \mathbb{T}_\Gamma \hookrightarrow \mathbb{T}_{\pi_1(M)}$, sends $\mathcal{V}^1(\Gamma)$ to a union of (possibly torsion-translated) subtori inside $\mathcal{V}^1(M)$.

Two orbifold fibrations, $f: M \rightarrow (\Sigma_g, \mathbf{m})$ and $f': M \rightarrow (\Sigma_{g'}, \mathbf{m}')$, are equivalent if there is a biholomorphic map, $h: \Sigma_g \rightarrow \Sigma_{g'}$, which sends marked points to marked points, while preserving multiplicities. As shown by Delzant [14, Thm. 2], a Kähler manifold M admits only finitely many equivalence classes of orbifold fibrations for which the orbifold Euler characteristic of the base is negative. The next theorem, which summarizes several results from [1, 18, 14, 2], shows that *all* positive-dimensional components in the first characteristic variety of M arise by pullback along this finite set of orbifold fibrations.

Theorem 12.1. *Let M be a compact Kähler manifold. Then*

$$(63) \quad \mathcal{V}^1(M) = \bigcup_{\alpha \in \text{Fib}(M)} (f_\alpha)_\#^*(\mathcal{V}^1(\pi_1^{\text{orb}}(\Sigma_{g_\alpha}, \mathbf{m}_\alpha))) \cup Z,$$

where Z is a finite set of torsion characters, and the union runs over a (finite) set of equivalence classes of orbifold fibrations $f_\alpha: M \rightarrow (\Sigma_{g_\alpha}, \mathbf{m}_\alpha)$ with either $g_\alpha \geq 2$, or $g_\alpha = 1$ and $|\mathbf{m}_\alpha| \geq 2$.

It now follows from Theorem 7.5 and formula (59) that

$$(64) \quad \Sigma^1(M) \subseteq \left(\bigcup_{\alpha \in \text{Fib}(M)} S(f_\alpha^*(H^1(C_\alpha, \mathbb{R}))) \right)^c.$$

Remarkably, a result of Delzant ([15, Thm. 1.1]) shows that the above inclusion holds as an equality. In view of Corollary 2.4, we may recast this result in the tropical setting, as follows.

Theorem 12.2 ([15]). *Let M be a compact Kähler manifold. Then*

$$(65) \quad \Sigma^1(M) = S(\text{Trop}(\mathcal{V}^1(M)))^c.$$

Remark 12.3. As shown by Friedl and Vidussi in [31, Lemma 2.3], the set $\Sigma^1(M)$ is non-empty if and only if $\pi_1(M)$ virtually algebraically fibers (i.e., it contains a finite-index subgroup which maps onto \mathbb{Z} with finitely generated kernel). By Theorem 12.2, this condition is equivalent to $\text{Trop}(\mathcal{V}^1(M)) \not\subseteq H^1(M, \mathbb{R})$.

Example 12.4. Let C_1 be a smooth, complex curve of genus 2 with an elliptic involution σ_1 , and let C_2 be a curve of genus 3 with a free involution σ_2 . Then $\Sigma_1 = C_1/\sigma_1$ is a curve of genus one, $\Sigma_2 = C_2/\sigma_2$ is a curve of genus two, and $M = (C_1 \times C_2)/\sigma_1 \times \sigma_2$ is the smooth, complex projective surface constructed in [12]. Projection onto the first coordinate yields an orbifold fibration $f_1: M \rightarrow \Sigma_1$ with two multiple fibers, each of multiplicity 2, while projection onto the second coordinate defines a smooth fibration $f_2: M \rightarrow \Sigma_2$. It is readily seen that $H_1(M, \mathbb{Z}) = \mathbb{Z}^6$, and so $H^1(M, \mathbb{C}^\times) = (\mathbb{C}^\times)^6$. In [58], we showed that $\mathcal{V}^1(M) = \{t_4 = t_5 = t_6 = 1, t_3 = -1\} \cup \{t_1 = t_2 = 1\}$, with the two components obtained by pullback along the maps f_1 and f_2 . Thus, $\Sigma^1(M)$ is the complement in S^5 of the two subspheres cut out by the subspaces $\{x_3 = \dots = x_6 = 0\}$ and $\{x_1 = x_2 = 0\}$, respectively.

13. HYPERPLANE ARRANGEMENTS

13.1. Arrangement complements. Let \mathcal{A} be an arrangement of hyperplanes in the complex affine space \mathbb{C}^ℓ . We denote by $L(\mathcal{A})$ the intersection poset of \mathcal{A} , and by $L_k(\mathcal{A})$ those subspaces in $L(\mathcal{A})$ of codimension k . We also let $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ be the complement of the arrangement. Without much loss of generality, we may assume \mathcal{A} is central and essential, that is, $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

The complement $M = M(\mathcal{A})$ has the homotopy type of a finite, connected CW-complex of dimension ℓ . The homology groups $H_i(M, \mathbb{Z})$ are all torsion-free, and $b_1(M)$ is equal to $|\mathcal{A}|$, the number of hyperplanes in \mathcal{A} . The cohomology ring of M was computed by Brieskorn in the early 1970s, building on work of Arnol'd; a description of this ring in terms of $L(\mathcal{A})$ was given by Orlik and Solomon in 1980. Brieskorn's work implies that the space M is formal; thus, by (43), we have that $\tau_1(\mathcal{V}^q(M)) = \mathcal{R}^q(M)$, for all q .

13.2. BNSR invariants of arrangements. We now turn to the Σ -invariants of arrangement complements. Although we do not know whether these sets are symmetric in general, the next result (which generalizes an observation from [48]) singles out a situation where they are.

Proposition 13.1. *Let \mathcal{A} be the complexification of an arrangement $\mathcal{A}_{\mathbb{R}}$ of real hyperplanes in \mathbb{R}^ℓ , and let M be the complement of \mathcal{A} in \mathbb{C}^ℓ . Then $\Sigma^q(M, \mathbb{Z}) = -\Sigma^q(M, \mathbb{Z})$, for all q .*

Proof. The group $H_1(M, \mathbb{Z})$ is freely generated by the meridians of the hyperplanes in \mathcal{A} . If \mathcal{A} is defined by real equations, complex conjugation in \mathbb{C}^ℓ restricts to a diffeomorphism $f: M \rightarrow M$

sending each meridian S^1 to itself by a map of degree -1 . Thus, $f_* = -\text{id}: H_1(M, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$, and the claim follows from Proposition 5.4. \square

At a 2007 Oberwolfach Mini-Workshop [29], the question was raised whether $\Sigma^1(M)$ equals $S(\mathcal{R}^1(M, \mathbb{R}))^c$ for $M = M(\mathcal{A})$. In an important particular case, the answer turned out to be yes.

Example 13.2. Let \mathcal{A} be the braid arrangement in \mathbb{C}^n , consisting of all the hyperplanes $z_i - z_j = 0$ with $1 \leq i < j \leq n$. The complement M is the configuration space of n ordered points in \mathbb{C} , and is thus a classifying space for the pure braid group P_n . As shown in [37], $\Sigma^1(M)$ is equal to $S(\mathcal{R}^1(M, \mathbb{R}))^c$, which is the complement of a collection of $\binom{n+1}{4}$ great circles in $S^{(n)}^{-1}$.

In general, though, the answer to the aforementioned question is no: as we showed in [57], there is an arrangement complement M for which $\Sigma^1(M) \subsetneq S(\mathcal{R}^1(M, \mathbb{R}))^c$. We will revisit this example in Proposition 13.6, and explain it from a tropical point of view.

13.3. A tropical upper bound. For a central, essential arrangement \mathcal{A} in \mathbb{C}^ℓ with complement $M = M(\mathcal{A})$, setting $n = |\mathcal{A}|$ and ordering the hyperplanes as H_1, \dots, H_n allows us to identify $H^1(M, \mathbb{R}) = \mathbb{R}^n$ and $S(M) = S^{n-1}$. Since arrangement complements are formal, the ‘‘resonance upper bound’’ from (44) holds. The next theorem provides a much improved ‘‘tropical upper bound’’ for the BNSR invariants of arrangements.

Theorem 13.3. *Let M be the complement of an arrangement of n hyperplanes in \mathbb{C}^ℓ . Then, for each $1 \leq q \leq \ell - 1$, the following hold.*

- (1) $\text{Trop}(\mathcal{V}^q(M))$ is the union of a subspace arrangement in \mathbb{R}^n .
- (2) $\Sigma^q(M, \mathbb{Z}) \subseteq S(\text{Trop}(\mathcal{V}^q(M)))^c$.

Consequently, each BNSR invariant of M is contained in the complement of an arrangement of great subspheres in S^{n-1} .

Proof. A theorem of Arapura [1] implies that all the irreducible components of $\mathcal{V}^q(M)$ are translated subtori of $(\mathbb{C}^\times)^n$. The first claim then follows from formulas (13) and (15).

As shown in [17], M is an abelian duality space, and consequently, its characteristic varieties propagate, that is, $\mathcal{V}^1(M) \subseteq \dots \subseteq \mathcal{V}^{\ell-1}(M)$. Therefore, $\mathcal{V}^{\leq q}(M) = \mathcal{V}^q(M)$. The second claim now follows from Theorem 6.5. \square

13.4. Lower bounds for the Σ -invariants. Let $G = \pi_1(M)$. In [38], Kohno and Pajitnov showed that the homology groups of M with coefficients in the Novikov–Sikorav completion $\widehat{\mathbb{Z}G}_\chi$ vanish in degrees $i < \ell$, provided χ belongs to the positive orthant of S^{n-1} . In view of Theorem 5.6, this implies that the negative orthant, S_-^{n-1} , is contained in $\Sigma^q(M, \mathbb{Z})$, for all $q < \ell$. In particular, if $\ell > 1$, then $S_-^{n-1} \subseteq \Sigma^1(M)$. In fact, as shown in [57],

$$(66) \quad \Sigma^1(M) = S^{n-1} \setminus (S^{n-2} \setminus \Sigma^1(U)),$$

where S^{n-2} is the great sphere cut out by the hyperplane $\sum \chi_j = 0$, and U is the complement of the projectivized arrangement. In particular, $S^{n-1} \setminus S^{n-2} \subseteq \Sigma^1(M)$. In the case of an arrangement of $n \geq 3$ lines through the origin of \mathbb{C}^2 , the complement deform-retracts onto the complement of the n -component Hopf link from Example 9.5. This lower bound coincides with the upper bound from Theorem 13.3, and so $\Sigma^1(M) = S^{n-1} \setminus S^{n-2}$. In general, though, the inclusion is strict.

13.5. BNS invariants of arrangement groups. To see why this is the case, we need a more detailed description of the first characteristic variety of a hyperplane arrangement. The notion of orbifold fibration is defined for arrangement complements (and, indeed, for all smooth, quasi-projective varieties) in exact analogy with Kähler manifolds (see §12). The next theorem summarizes several results from [1, 18, 26, 50, 68, 2] in this direction.

Theorem 13.4. *The decomposition into irreducible components of the first characteristic variety of an arrangement complement M is of the form*

$$(67) \quad \mathcal{V}^1(M) = \bigcup_{\alpha \in \text{Fib}(M)} (f_\alpha)_\#^*(\mathcal{V}^1(\pi_1^{\text{orb}}(\Sigma_{0,s_\alpha}, \mathbf{m}_\alpha))) \cup Z,$$

where Z is a finite set of torsion characters, and the union runs over a finite set of equivalence classes of orbifold fibrations $f_\alpha: M \rightarrow (\Sigma_{0,s_\alpha}, \mathbf{m}_\alpha)$ for which either $s_\alpha = 3$ or 4, or $s_\alpha = 2$ and $|\mathbf{m}_\alpha| > 0$.

By formula (61), each positive-dimensional component T_α in (67) is a torus of dimension $s_\alpha - 1$, translated by a non-trivial torsion character if $|\mathbf{m}_\alpha| > 0$. As shown by Falk and Yuzvinsky in [26], each component T_α passing through the origin arises from a k -multinet on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ with $k = s_\alpha$. Such a multinet consists of a partition $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_k)$, together with multiplicities m_H attached to each hyperplane H and a subset $X \subset L_2(\mathcal{A})$ such that several conditions are satisfied; most importantly, for each $X \in \mathcal{X}$, the sum $n_X := \sum_{H \in \mathcal{B}_i: H \supset X} m_H$ is independent of i . For instance, each flat $X \in L_2(\mathcal{A})$ of size $k \geq 3$ gives rise to a k -multinet on the corresponding “local” sub-arrangement, and thus, to a component $T_X \subset \mathcal{V}^1(M)$ of dimension $k - 1$. On the other hand, the non-local components of $\mathcal{V}^1(M)$ passing through 1 all have dimension 2 or 3, see [50, 68].

As a consequence, we have the following description of $\Sigma^1(M)$ in terms of the orbifold fibrations supported by M , with negative orbifold Euler characteristic of the base.

Theorem 13.5. *Let $M = M(\mathcal{A})$ be an arrangement complement. Then,*

$$(68) \quad \Sigma^1(M) \subseteq \left(\bigcup_{\alpha \in \text{Fib}(M)} S(f_\alpha^*(H^1(\Sigma_{0,s_\alpha}, \mathbb{R}))) \right)^c.$$

Proof. By Theorem 13.4, there is a finite set, $\text{Fib}(M)$, indexing equivalence classes of orbifold fibrations, $f_\alpha: M \rightarrow (\Sigma_{0,s_\alpha}, \mathbf{m}_\alpha)$, for which the induced homomorphism from $G = \pi_1(M)$ to $G_\alpha = \pi_1^{\text{orb}}(\Sigma_{0,s_\alpha}, \mathbf{m}_\alpha)$ is surjective. From the definition of this set, we also know that $\chi^{\text{orb}}(\Sigma_{0,s_\alpha}, \mathbf{m}_\alpha) < 0$, for each α . Thus, by formula (61), $\mathcal{V}^1(T_{G_\alpha})$ is equal to either \mathbb{T}_{G_α} or $\mathbb{T}_{G_\alpha} \setminus \mathbb{T}_{G_\alpha}^0$. In either case, $\mathcal{V}^1(T_{G_\alpha})$ contains a connected component of \mathbb{T}_{G_α} . Hence, Theorem 7.5 applies, and gives the desired conclusion. \square

13.6. Pointed multinets. Following [16], we describe a combinatorial construction which produces translated subtori in the first characteristic variety of a certain class of arrangements. Fix a hyperplane $H \in \mathcal{A}$, and let $\mathcal{A}' = \mathcal{A} \setminus \{H\}$. A *pointed multinet* on \mathcal{A} consists of a multinet on \mathcal{A} , together with a distinguished hyperplane $H \in \mathcal{A}$ such that $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$. Given these data, let M' be the complement of \mathcal{A}' ; then $\mathcal{V}^1(M')$ has a component which is a 1-dimensional subtorus of $\mathbb{T}_{M'}$, translated by a character of order m_H .

Applying Proposition 4.4 to this setup, we obtain the following result.

Proposition 13.6. *Let \mathcal{A} be an arrangement which admits a pointed multinet, and let \mathcal{A}' be the arrangement obtained from \mathcal{A} by deleting the distinguished hyperplane H . Then*

- (1) *The resonance variety $\mathcal{R}^1(M', \mathbb{R})$ is properly contained in $\text{Trop}(\mathcal{V}^1(M'))$.*
- (2) *The BNS invariant $\Sigma^1(M')$ is properly contained in $S(\mathcal{R}^1(M', \mathbb{R}))^c$.*

Taking \mathcal{A} to be the reflection arrangement of type B_3 , and \mathcal{A}' to be the deleted B_3 arrangement from [56] recovers the result that $\Sigma^1(M') \subsetneq S(\mathcal{R}^1(M', \mathbb{R}))^c$, proved in a different way in [57, Example 11.8]. In view of Proposition 13.6, a more refined question to ask, then, is the following.

Question 13.7. For a complex hyperplane arrangement \mathcal{A} with complement $M = M(\mathcal{A})$, is the BNS invariant $\Sigma^1(M)$ equal to $S(\text{Trop}(\mathcal{V}^1(M)))^c$?

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