Lower central series, free resolutions, and homotopy Lie algebras of arrangements

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References

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Hyperplane arrangements

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a set of hyperplanes in $\mathbb{C}^\ell$ (passing through 0).

- **Intersection lattice**:

$$\boxed{L(\mathcal{A}) = \left\{ \bigcap_{H\in\mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\}}$$

  ranked poset: $\le \longleftrightarrow \supseteq$, rank $\longleftrightarrow$ codim.

- **Complement**:

$$\boxed{M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H}$$

  a **formal** space, $\simeq$ to an ℓ-dim CW-complex.

Main topological invariants associated to $M(\mathcal{A})$:

- **Cohomology ring**: $A := H^*(M, \mathbb{Q}) = E/I$

  E = exterior algebra, I = **Orlik-Solomon** ideal

  —determined by $L(\mathcal{A})$

- **Fundamental group**: $G = \pi_1(M)$

  —not always combinatorially determined, though its LCS ranks are (since M formal).

Lower central series

Let G be a finitely-generated group. Set:

- **LCS series**: $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_k \triangleright \cdots$
  where $G_{k+1} = [G_k, G]$
- **LCS quotients**: $\operatorname{gr}_k G = G_k/G_{k+1}$
- **LCS ranks**: $\phi_k(G) = \operatorname{rank}(\operatorname{gr}_k G)$
- **Chen ranks**: $\theta_k(G) = \operatorname{rank}(\operatorname{gr}_k G/G'')$

Define **associated graded Lie algebra**:

$$L_*(G) := \operatorname{gr}_* G \otimes \mathbb{Q} = \bigoplus_{k \geq 1} G_k/G_{k+1} \otimes \mathbb{Q}$$

with Lie bracket $[\ ,\ ]\colon L_i \times L_j \to L_{i+j}$ induced by the group commutator.

Problem. Find an *explicit* combinatorial formula for the LCS ranks $\phi_k(G)$ of an arrangement group G (at least for certain classes of arrangements).

Known LCS formulas

- **Witt formula**

  $\mathcal{A} = \{n \text{ points in } \mathbb{C}\}, \quad M(\mathcal{A}) \simeq \bigvee^n S^1$

  $G = F_n$ (free group of rank n)

  $\operatorname{gr} G = \mathbb{L}_n$ (free Lie algebra of rank n)

$$\boxed{\prod_{k=1}^{\infty} (1-t^k)^{\phi_k} = 1 - nt}$$

  Hence: $\quad \phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}$

  $\phi_1 = n, \quad \phi_2 = \binom{n}{2}, \quad \phi_3 = 2\binom{n+1}{3}, \quad \phi_4 = \frac{n^2(n^2-1)}{4}.$

- **Kohno formula** [1985]

  $\mathcal{A} = \{z_i - z_j = 0\}_{1 \leq i < j \leq \ell}$ braid arrangement in $\mathbb{C}^\ell$

  $G = P_\ell$ (pure braid group on ℓ strings)

$$\boxed{\prod_{k=1}^{\infty} (1-t^k)^{\phi_k} = \prod_{j=1}^{\ell-1} (1 - jt)}$$

- **Falk-Randell LCS formula** [1985]

  $\mathcal{A}$ **fiber-type** $\left(\overset{\text{Terao}}{\Longleftrightarrow} L(\mathcal{A})\right.$ **supersolvable**$\left.\right)$,
  with exponents $d_1, \dots, d_\ell$. Then:

  $$G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$$
  $$\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i})$$

  $$\boxed{\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_M(-t)}$$

  where, for all $\mathcal{A}$:

  $$P_M(t) := \sum_{i=0}^{\ell} b_i t^i, \quad b_i = \sum_{X \in L_i(\mathcal{A})} (-1)^i \mu(X)$$

- **Shelton-Yuzvinsky** [1997], **Papadima-Yuz** [99]

  A Koszul algebra $\Longrightarrow$ LCS formula holds

Remark. (**Peeva**) There are arrangements for which

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k} \neq \mathrm{Hilb}(N, -t),$$

for any graded commutative algebra N.

LCS and free resolutions

(joint work with Hal Schenck)

We want to reduce the problem of computing $\phi_k(G)$ to that of computing the graded Betti numbers of certain free resolutions involving the OS-algebra $A = E/I$.

Starting point:

$$\prod_{k=1}^{\infty} (1 - t^k)^{-\phi_k} = \sum_{i=0}^{\infty} b_{ii}\, t^i$$

where

$$b_{ij} = \dim_{\mathbb{Q}} \operatorname{Tor}_i^A(\mathbb{Q}, \mathbb{Q})_j$$

i^{th} Betti number (in degree j) of a minimal free resolution of $\mathbb{Q}$ over A:

$$\cdots \longrightarrow \oplus_j A^{b_{2j}}(-j) \longrightarrow \oplus_j A^{b_{1j}}(-j) \longrightarrow A \longrightarrow \mathbb{Q} \to 0$$

with Betti diagram:

$$\begin{array}{lcccccl} 0: & 1 & b_1 & b_{22} & b_{33} & \ldots & \leftarrow \text{linear strand} \\ 1: & . & . & b_{23} & b_{34} & \ldots & \\ 2: & . & . & b_{24} & b_{35} & \ldots & \end{array}$$

This (known) formula follows from:

- **Sullivan** Since M formal:

$$\operatorname{gr} G \cong \mathfrak{g} := \mathbb{L}(H_1)/\operatorname{im}(\nabla\colon H_2 \to H_1 \wedge H_1)$$

Holonomy Lie algebra of $H_* = H_*(M, \mathbb{Q})$

- **Poincaré-Birkhoff-Witt**:

$$\prod_{k=1}^{\infty}(1-t^k)^{-\dim \mathfrak{g}_k} = \operatorname{Hilb}(U(\mathfrak{g}), t)$$

- **Shelton-Yuzvinsky**:

$$U(\mathfrak{g}) = \overline{A}^!$$

- **Priddy, Löfwall**:

$$\overline{A}^! \cong \bigoplus_{i \geq 0} \operatorname{Ext}^i_A(\mathbb{Q}, \mathbb{Q})_i$$

Linear strand in Yoneda Ext-algebra

Here $\overline{A} = E/\langle I_2 \rangle$ is the quadratic closure of A, and $\overline{A}^!$ is its Koszul dual.

Koszul algebras

A^* connected, graded, graded-commutative $\mathbb{Q}$-algebra.

Definition. A is a **Koszul algebra** if

$$\boxed{\mathrm{Ext}^i_A(\mathbb{Q},\mathbb{Q})_j = 0, \quad \text{for } i \neq j.}$$

i.e., Betti diagram = linear strand.

- **Necessary**: $A = T/I$, T gen in deg 1, I in deg 2.
- **Sufficient**: I has quadratic Gröbner basis.
- **Koszul duality**: $\mathrm{Hilb}(A^!, t) \cdot \mathrm{Hilb}(A, -t) = 1$
  ($\rightsquigarrow$ LCS formula of Shelton-Yuzvinsky.)
- **Arrangement interpretation (Shelton-Yuz.)**

$$\mathcal{A} \text{ fiber-type} \underset{??}{\overset{\Longrightarrow}{\Longleftarrow}} H^*(M(\mathcal{A}),\mathbb{Q}) \text{ Koszul}$$

- **Topological interpretation (Pap-Yuz.)**

X finite-type CW, $X_{\mathbb{Q}}$ Bousfield-Kan rationalization.

$$H^*(X,\mathbb{Q}) \text{ Koszul} \underset{\text{if } X \text{ formal}}{\overset{\Longrightarrow}{\Longleftarrow}} X_{\mathbb{Q}} \text{ aspherical}$$

Resolution of OS-algebra

Problem. (**Eisenbud-Popescu-Yuzvinsky** [1999])
Compute the (minimal) free resolution of A over E,

$$\cdots \longrightarrow \oplus_j E^{b'_{2j}}(-j) \longrightarrow \oplus_j E^{b'_{1j}}(-j) \longrightarrow E \longrightarrow A \to 0$$

and its Betti numbers, $b'_{ij} = \dim_{\mathbb{Q}} \operatorname{Tor}_i^E(A, \mathbb{Q})_j$.

Let $a_j = \sharp\{\text{minimal generators of } I \text{ in degree } j\}$.

Clearly, $a_2 = \binom{b_1}{2} - b_2$.

Lemma. $a_j = b'_{1j} = b_{2j}$.

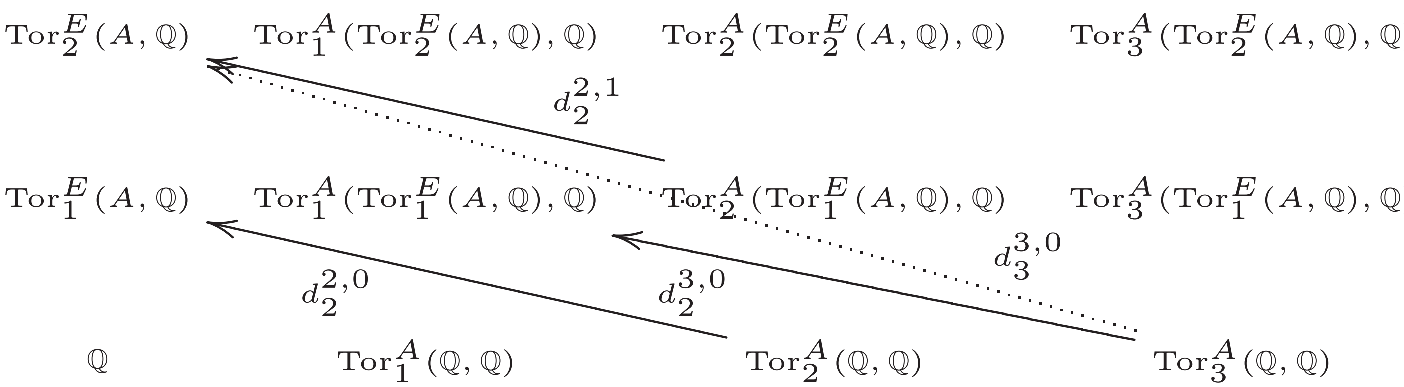
Thus, Betti diagram looks like:

$$\begin{array}{lccccc l}
0: & 1 & . & . & . & \ldots & \\
1: & . & a_2 & b'_{23} & b'_{34} & \ldots & \leftarrow \text{linear strand} \\
2: & . & a_3 & b'_{24} & b'_{35} & \ldots & \\
 & & & & & & \\
\ell-1: & . & a_\ell & b'_{2,\ell+1} & b'_{3,\ell+2} & \ldots &
\end{array}$$

Change of rings spectral sequence

Key tool (Cartan-Eilenberg):

$$\operatorname{Tor}_i^A\left(\operatorname{Tor}_j^E(A,\mathbb{Q}),\mathbb{Q}\right) \Longrightarrow \operatorname{Tor}_{i+j}^E(\mathbb{Q},\mathbb{Q})$$



The (Koszul) resolution of $\mathbb{Q}$ as a module over E is linear, with $\dim \operatorname{Tor}_i^E(\mathbb{Q},\mathbb{Q})_i = \binom{n+i-1}{i}$.

Thus, if we know $\operatorname{Tor}_i^E(A,\mathbb{Q})$, we can find $\operatorname{Tor}_i^A(\mathbb{Q},\mathbb{Q})$, provided we can compute $d_r^{p,q}$.

We carry out part of this program—mostly in low degrees. As a result, we express ϕ_k, $k \le 4$, solely in terms of the resolution of A over E.

Theorem. *For an arrangement of n hyperplanes:*

$$\phi_1 = n$$
$$\phi_2 = a_2$$
$$\phi_3 = b'_{23}$$
$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4$$

where

$$a_2 = \sharp\{\textit{generators of } I_2\} = \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X)}{2}$$
$$b'_{23} = \sharp\{\textit{linear first syzygies on } I_2\}$$
$$b'_{34} = \sharp\{\textit{linear second syzygies on } I_2\}$$
$$\delta_4 = \sharp\{\textit{minimal, quadratic, Koszul syzygies on } I_2\}$$

ϕ_1, ϕ_2: elementary

ϕ_3: recovers a formula of **Falk** [1988]

ϕ_4: new

Decomposable arrangements

Let $\mathcal{A}$ be an arrangement of n hyperplanes.

Recall:

$$\phi_1 = n, \quad \phi_2 = \sum_{X \in L_2(\mathcal{A})} \phi_2(F_{\mu(X)})$$

Falk [1989]:

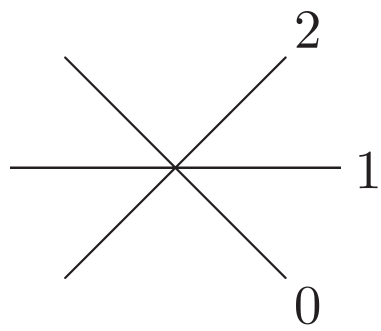
$$\boxed{\phi_k \geq \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})} \quad \text{for all } k \geq 3 \qquad (*)$$

Definition. If the lower bound is attained for $k = 3$, $\mathcal{A}$ is **decomposable** (or, *minimal linear strand*).

Conjecture (MLS LCS). If $\mathcal{A}$ is decomposable, equality holds in (*), and so

$$\boxed{\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = (1 - t)^n \prod_{X \in L_2(\mathcal{A})} \frac{1 - \mu(X)t}{1 - t}}$$

Example. $\mathcal{A} = \{H_0, H_1, H_2\}$ pencil of 3 lines in $\mathbb{C}^2$.



$E =$ exterior algebra on e_0, e_1, e_2

$I =$ ideal generated by $\partial e_{012} = (e_1 - e_2) \wedge (e_0 - e_2)$

Minimal free resolution of A over E:

$$0 \leftarrow A \longleftarrow E \xleftarrow{(\partial e_{012})} E(-1) \xleftarrow{(e_1 - e_2 \;\; e_0 - e_2)} E^2(-2)$$

$$\xleftarrow{\begin{pmatrix} e_1 - e_2 & e_0 - e_2 & 0 \\ 0 & e_1 - e_2 & e_0 - e_2 \end{pmatrix}} E^3(-3) \longleftarrow \cdots$$

Thus,

$$b'_{i,i+1} = i \qquad \text{for } i \geq 1$$

$$b'_{i,i+r} = 0 \qquad \text{for } r > 1$$

More generally, if $\mathcal{A}$ decomposable, we compute the entire linear strand of the resolution of A over E.

If moreover $\operatorname{rank}\mathcal{A} = 3$, we compute all b'_{ij} from Möbius function.

Lemma. *For any arrangement $\mathcal{A}$:*

$$b'_{i,i+1} \geq i \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X)+i-1}{i+1}$$

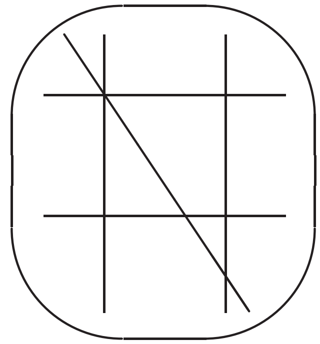
$$\delta_4 \leq \sum_{(X,Y) \in \binom{L_2(\mathcal{A})}{2}} \binom{\mu(X)}{2}\binom{\mu(Y)}{2}.$$

If $\mathcal{A}$ is decomposable, then equalities hold.

Corollary. *The MLS LCS conjecture is true for $k=4$:*

$$\phi_4 = \frac{1}{4} \sum_{X \in L_2(\mathcal{A})} \mu(X)^2(\mu(X)^2 - 1)$$

Example. X_3 arrangement



```
res of residue field over OS alg
total: 1 6 25 92 325 1138
    0: 1 6 24 80 240 672
    1: . . 1  12 84  448
    2: . . .  .   1   18
res of OS alg over exterior algebra
total: 1 4 15 42 97 195 354 595 942 1422 2065
    0: 1 .  .  .  .   .   .   .   .    .    .
    1: . 3  6  9 12  15  18  21  24   27   30
    2: . 1  9 33 85 180 336 574 918 1395 2035
```

We find: $b'_{i,i+1} = 3i, \;\; b'_{i,i+2} = \frac{1}{8}i(i+1)(i^2+5i-2)$.
Thus:

$$\phi_1 = n = 6$$

$$\phi_2 = a_2 = 3$$

$$\phi_3 = b'_{23} = 6$$

$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4 = 3 + 9 - 3 = 9$$

Conjecture says: $\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = (1-2t)^3$

i.e.: $\phi_k(G) = \phi_k(F_2^3)$, though definitely $G \not\cong F_2^3$.

Graphic arrangements

$\mathsf{G} = (\mathcal{V}, \mathcal{E})$ subgraph of complete graph K_ℓ. Set:

$$\kappa_s(\mathsf{G}) = \sharp\{\text{complete subgraphs on } s+1 \text{ vertices}\}$$

Graphic arrangement (in $\mathbb{C}^{\kappa_0}$) associated to G:

$$\mathcal{A}_\mathsf{G} = \{\ker(z_i - z_j) \mid \{i,j\} \in \mathcal{E}\}$$

Theorem. (Stanley, Fulkerson-Gross)

$\mathcal{A}_\mathsf{G}$ *is supersolvable* $\iff$ G *is chordal*

Lemma. (Cordovil-Forge [2001]**, S-S)**

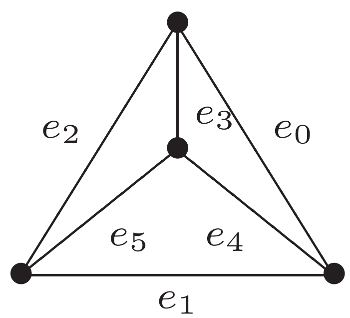
$$a_j = \sharp\{\text{chordless } j+1 \text{ cycles}\}$$

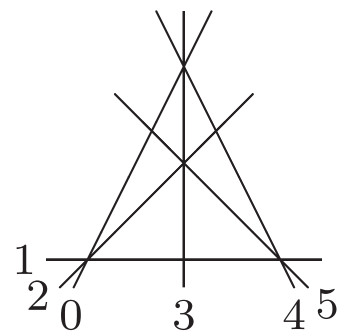
Together with a previous lemma ($a_j = b_{2j}$), we get:

Corollary. $\mathcal{A}_\mathsf{G}$ *supersolvable* $\iff$ $\mathcal{A}_\mathsf{G}$ *Koszul.*

Example. Braid arrangement $\mathcal{B}_4 = \mathcal{A}_{K_4}$

$\kappa_0 = 4,\ \kappa_1 = 6,\ \kappa_2 = 4,\ \kappa_3 = 1$





Free resolution of A over E:

$$0 \leftarrow A \longleftarrow E \xleftarrow{\partial_1} E^4(-2) \xleftarrow{\partial_2} E^{10}(-3) \longleftarrow \cdots$$

where $\partial_1 = \begin{pmatrix} \partial e_{145} & \partial e_{235} & \partial e_{034} & \partial e_{012} \end{pmatrix}$

and $\partial_2 =$

$$\begin{pmatrix} e_1 - e_4 & e_1 - e_5 & 0 & 0 & 0 & 0 & 0 & 0 & e_3 - e_0 & e_2 - e_0 \\ 0 & 0 & e_2 - e_3 & e_2 - e_5 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_4 \\ 0 & 0 & 0 & 0 & e_0 - e_3 & e_0 - e_4 & 0 & 0 & e_1 - e_5 & e_2 - e_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0 - e_1 & e_0 - e_2 & e_3 - e_5 & e_4 - e_5 \end{pmatrix}$$

Non-local component in $\mathcal{R}_1(\mathcal{B}_4) \rightsquigarrow 2$ extra syzygies

Get: $\quad b'_{i,i+1} = 5i, \quad \delta_4 = 0.$

Proposition. *For a graphic arrangement:*

$$b'_{i,i+1} = i(\kappa_2 + \kappa_3)$$

$$\delta_4 \le \binom{\kappa_2}{2} - 6(\kappa_3 + \kappa_4)$$

Corollary.

$$\phi_1 = \kappa_1$$

$$\phi_2 = \kappa_2$$

$$\phi_3 = 2(\kappa_2 + \kappa_3)$$

$$\phi_4 \ge 3(\kappa_2 + 3\kappa_3 + 2\kappa_4)$$

Moreover, if $\kappa_3 = 0$, equality holds for ϕ_4.

Conjecture (Graphic LCS).

$$\phi_k = \tfrac{1}{k} \sum_{d|k} \sum_{j=1}^{k} \sum_{s=j}^{k} (-1)^{s-j} \binom{s}{j} \kappa_s \, \mu(d) \, j^{\frac{k}{d}}$$

or:

$$\prod_{k=1}^{\infty} \left(1 - t^k\right)^{\phi_k} = \prod_{j=1}^{\kappa_0 - 1} (1 - jt)^{\sum_{s=j}^{\kappa_0 - 1} (-1)^{s-j} \binom{s}{j} \kappa_s}$$

Resonance Conjectures

$\mathcal{A} = \{H_1, \dots, H_n\}$, $A = H^*(M(\mathcal{A}), \mathbb{C})$, $G = \pi_1(M)$

$\mathcal{R}_1(\mathcal{A}) := \{\lambda \in \mathbb{C}^n \mid \dim_{\mathbb{C}} H^1(A, \cdot \sum \lambda_i e_i) > 0\}$

Then: $\mathcal{R}_1(\mathcal{A}) = \bigcup_{i=1}^{v} L_i$, where L_i linear subspaces

Set $h_r = \sharp\{L_i \mid \dim L_i = r\}$.

Chen Ranks & Linear Strand of OS-Resolution

$$\boxed{\theta_k(G) = b'_{k-1,k}} \quad \text{for } k \geq 2.$$

Resonance Formula for Chen Ranks

$$\boxed{\theta_k(G) = \sum_{r\geq 2} h_r \theta_k(F_r)} \quad \text{for } k \geq 4.$$

Resonance LCS Formula

If $\phi_4 = \theta_4 \longleftrightarrow \delta_4 = \binom{a_2}{2}$, then:

$$\boxed{\phi_k(G) = \sum_{r\geq 2} h_r \phi_k(F_r)} \quad \text{for } k \geq 4.$$

Rational homotopy groups and Koszul algebras

(joint work with Stefan Papadima)

Let $k \geq 1$ be an integer, and

- X connected space
- Y simply-connected space

(all spaces $\simeq$ to a finite-type CW-complex)

Definition. Y is a **k-rescaling** of X (over R) if:

$$\boxed{H^*(Y,R) \cong H^*(X,R)[k]} \quad \text{as graded rings}$$

i.e.: $$H^i(Y,R) \cong \begin{cases} H^j(X,R) & \text{if } i = (2k+1)j, \\ 0 & \text{otherwise,} \end{cases}$$

and isos are compatible with cup products.

(Abstracts a definition of **D.Cohen, F.Cohen, M.Xicoténcatl** [2000])

Question. When does a homological rescaling pass to a homotopical rescaling?

Examples of rescalings (over $R = \mathbb{Z}$)

- $X = S^1$ $\quad$ $Y = S^{2k+1}$
- $X = \bigvee_1^n S^1$ $\quad$ $Y = \bigvee_1^n S^{2k+1}$
- $X = \prod_1^n S^1$ $\quad$ $Y = \prod_1^n S^{2k+1}$
- $X = \#_1^g S^1 \times S^1$ $\quad$ $Y = \#_1^g S^{2k+1} \times S^{2k+1}$
- $X = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i$ $\quad$ $Y = \mathbb{C}^{(k+1)\ell} \setminus \bigcup_{i=1}^n H_i^{\times(k+1)}$

$\mathcal{A} = \{H_1, \dots, H_n\}$ hyperplane arrangement in $\mathbb{C}^\ell$

$\mathcal{A}^k := \{H_1^{\times k}, \dots, H_n^{\times k}\}$ **redundant** subspace arr.

(Cohen-Cohen-Xico.)

- $X = S^3 \setminus \bigcup_{i=1}^n K_i$ $\quad$ $Y = S^{4k+3} \setminus \bigcup_{i=1}^n K_i^{\circledast k}$

$K = (K_1, \dots, K_n)$ link of oriented circles in S^3

$K^{\circledast k}$ link of S^{2k+1}'s in S^{4k+3}: **join** of K with k copies of Hopf_n (in sense of Koschorke-Rolfsen [1989])

Rational rescalings

- Given X and k, there is a k-rescaling Y over $\mathbb{Q}$.

  Take Y to be the realization of the Sullivan minimal model for dga $\big(H^*(X,\mathbb{Q})[k], d=0\big)$. By construction, this rescaling is formal.

- In general, k-rescaling is not unique. Eg:

$$X = S^1 \vee S^1 \vee S^4$$

$$Y = S^3 \vee S^3 \vee S^{12}$$

$$Z = S^3_x \vee S^3_y \bigcup_{[x,[x,[x,[x,y]]]]} e^{12}$$

- Nevertheless, k-rescaling is unique $/\mathbb{Q}$, for k large:

  Shiga-Yagita [1982]: If $H^{>d}(X,\mathbb{Q}) = 0$, then X has a unique k-rescaling (up to $\mathbb{Q}$-equivalence), for all $k > (d-1)/2$.

- If rescaling Y is unique, it must be formal.

■ **Simple examples.**

- $X = S^1$ $\qquad Y = S^{2k+1}$
- $X = \bigvee_1^n S^1$ $\qquad Y = \bigvee_1^n S^{2k+1}$
- $X = \prod_1^n S^1$ $\qquad Y = \prod_1^n S^{2k+1}$
- $X = \#_1^g S^1 \times S^1$ $\qquad Y = \#_1^g S^{2k+1} \times S^{2k+1}$

X and Y formal, rescaling unique $/\mathbb{Q}$, for all k

■ **Arrangements.** $X = M(\mathcal{A})$, $Y = M(\mathcal{A}^{k+1})$

Arnol'd, Brieskorn: X formal

Yuzvinsky [2001]: Y formal

Y is unique k-rescaling $/\mathbb{Q}$, for $k > (\ell - 1)/2$

■ **Links.** $X = M(K)$, $Y = M(K^{\circledast k})$

X may not be formal, but Y is formal
(since $\dim X = 2$), so rescaling unique $/\mathbb{Q}$, for all k.

Rescaling Lie algebras

Lie algebras with grading

Graded $\mathbb{Q}$-vector space L_*, with $[\ ,\ ]\colon L_p \otimes L_q \to L_{p+q}$
Lie identities satisfied exactly. E.g.:

$$\boxed{\mathrm{gr}_*(G) \otimes \mathbb{Q}}$$

Graded Lie algebras

L_*, with Lie identities satisfied up to sign. E.g.:

Homotopy Lie algebra

$$\boxed{\pi_*(\Omega Y) \otimes \mathbb{Q} := \bigoplus_{r \geq 1} \pi_r(\Omega Y) \otimes \mathbb{Q}}$$

Y simply-connected space, ΩY loop space

$[\ ,\ ]$ = Samelson product

Rescaling

L_* Lie algebra with grading $\rightsquigarrow$ $L[k]$ graded Lie algebra

$L[k]_{2kq} = L_q$ and $L[k]_p = 0$ otherwise,
and Lie bracket rescaled accordingly.

The Rescaling Formula

Theorem A. *Let Y be a k-rescaling of X. If $H^*(X,\mathbb{Q})$ is a* **Koszul** *algebra, then:*

$$\pi_*(\Omega Y)\otimes\mathbb{Q}\cong \mathrm{gr}_*(\pi_1 X)\otimes\mathbb{Q}[k]$$

as graded Lie algebras.

Theorem B. *Let Y be a* **formal** *k-rescaling of X. If* $\mathrm{Hilb}(\pi_*(\Omega Y)\otimes\mathbb{Q},t)=\mathrm{Hilb}(\mathrm{gr}_*(\pi_1 X)\otimes\mathbb{Q},t^{2k})$, *then:*

1. *$H^*(X,\mathbb{Q})$ is a Koszul algebra.*
2. *Y is a* **coformal** *space (i.e., its $\mathbb{Q}$-homotopy type is determined by its homotopy Lie algebra).*

The Homotopy LCS Formula

Set $\Phi_r := \operatorname{rank} \pi_r(\Omega Y) = \operatorname{rank} \pi_{r+1}(Y)$.

Theorem C. *Let Y be a k-rescaling of X. If $H^*(X, \mathbb{Q})$ is Koszul, then:*

$$\prod_{i \geq 1} (1 - t^{(2k+1)i})^{\Phi_{2ki}} = P_X(-t^k)$$

and $\Phi_r = 0$, if $2k \nmid r$.

Hence:

$$\Omega Y \simeq_{\mathbb{Q}} \tilde{\prod}_{i \geq 1} K(\mathbb{Q}, 2ki)^{\Phi_{2ki}}$$

and so

$$P_{\Omega Y}(t) = P_X(-t^{2k})^{-1}$$

In fact, by Milnor-Moore:

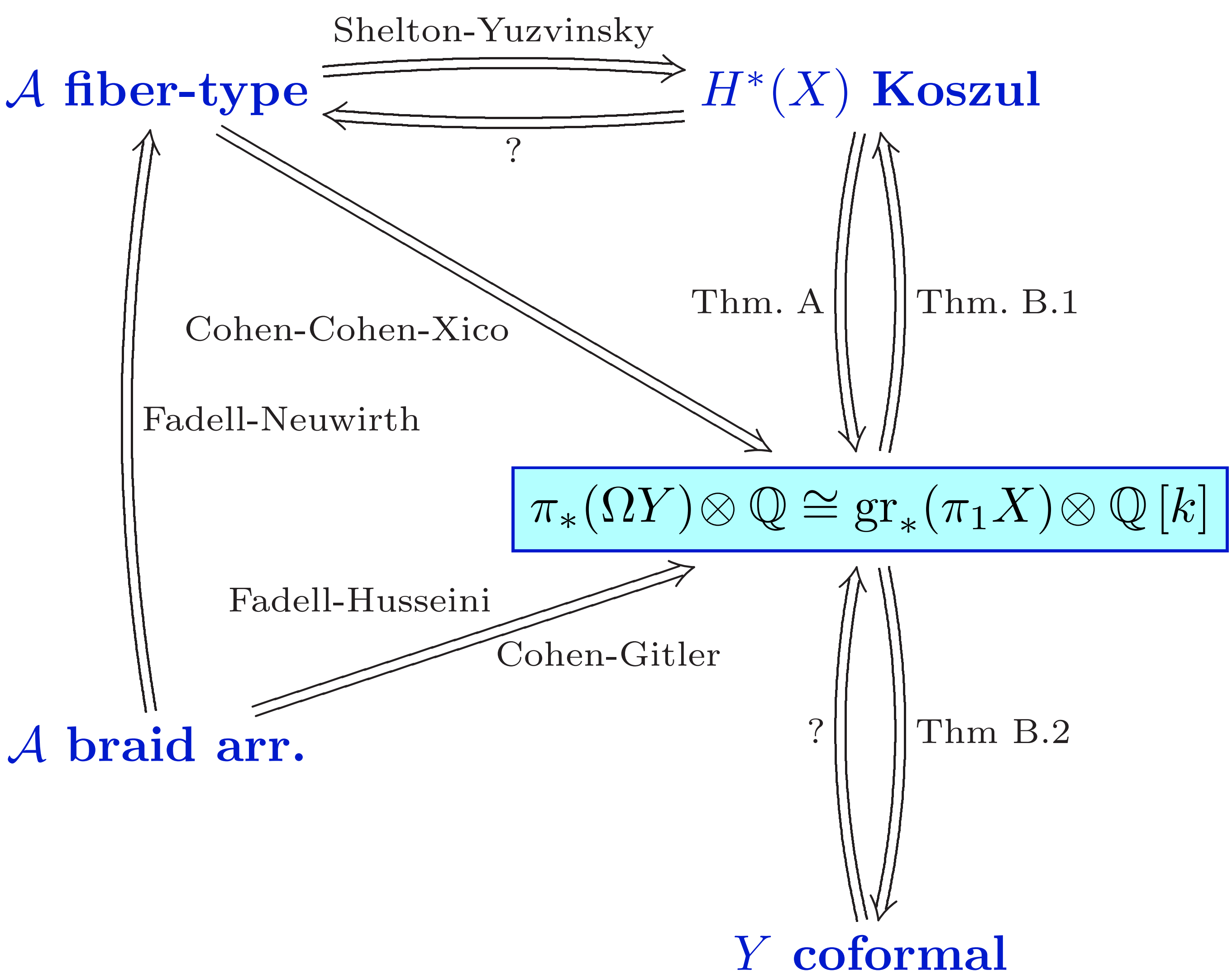
$$H_*(\Omega Y, \mathbb{Q}) \cong U\big(\operatorname{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]\big)$$

(as Hopf algebras)

Rescaling arrangements

Let $\mathcal{A}$ arrangement, $X = M(\mathcal{A})$, $Y = M(\mathcal{A}^{k+1})$.

Recall: both X, Y formal; Y rescaling of X.



Fiber-type arrangements

$\mathcal{A}$ supersolvable, $\quad \exp(\mathcal{A}) = \{d_1, \dots, d_\ell\}$

$X = M(\mathcal{A}), \quad Y = M(\mathcal{A}^{k+1})$

$\phi_r = \operatorname{rank} \operatorname{gr}_r(\pi_1 X), \quad \Phi_r = \operatorname{rank} \pi_r(\Omega Y).$

- Classical LCS formula:

$$\prod_{r=1}^{\infty} (1 - t^r)^{\phi_r} = \prod_{i=1}^{\ell} (1 - d_i t)$$

- Homotopy LCS formula:

$$\prod_{r=1}^{\infty} (1 - t^{(2k+1)r})^{\Phi_{2kr}} = \prod_{i=1}^{\ell} (1 - d_i t^{2k+1})$$

- Poincaré series of loop space:

$$P_{\Omega Y}(t) = \prod_{i=1}^{\ell} (1 - d_i t^{2ki})^{-1}$$

Generic arrangements

If $\mathcal{A}$ generic (and non-boolean), then:

(a) $H^*(X,\mathbb{Q})$ not quadratic $\implies$ not Koszul

(b) Rescaling formula fails

(c) Y not coformal

Failure of (b) and (c) is detected by higher-order Whitehead products.

Example. $\mathcal{A} = \{3 \text{ generic lines in } \mathbb{C}^2\}$

$X \simeq$ 2-skeleton of $S^1 \times S^1 \times S^1$

$Y \simeq 2(2k+1)$-skeleton of $S^{2k+1} \times S^{2k+1} \times S^{2k+1}$

Then:

$$\operatorname{gr}_*(\pi_1 X) = \mathbb{L}^{\text{ab}}(x_1, x_2, x_3)$$

$\deg x_i = 1$

$$\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^{\text{ab}}(y_1, y_2, y_3) * \mathbb{L}(w)$$

$\deg y_i = 2k$, $\deg w = 6k+1$.

The Mal'cev Formula

Theorem D. *Let Y be a k-rescaling of X. Assume $H^*(X,\mathbb{Q})$ is a Koszul algebra. Then:*

X is formal $\iff$ there are filtered group isomorphisms

$$\operatorname{Hom}^{\mathrm{coalg}}(H_*(\Omega S^{2k+1},\mathbb{Q}), H_*(\Omega Y,\mathbb{Q})) \cong [\Omega S^{2k+1}, \Omega Y]^{\wedge} \cong \pi_1 X \otimes \mathbb{Q} \quad (\ddagger)$$

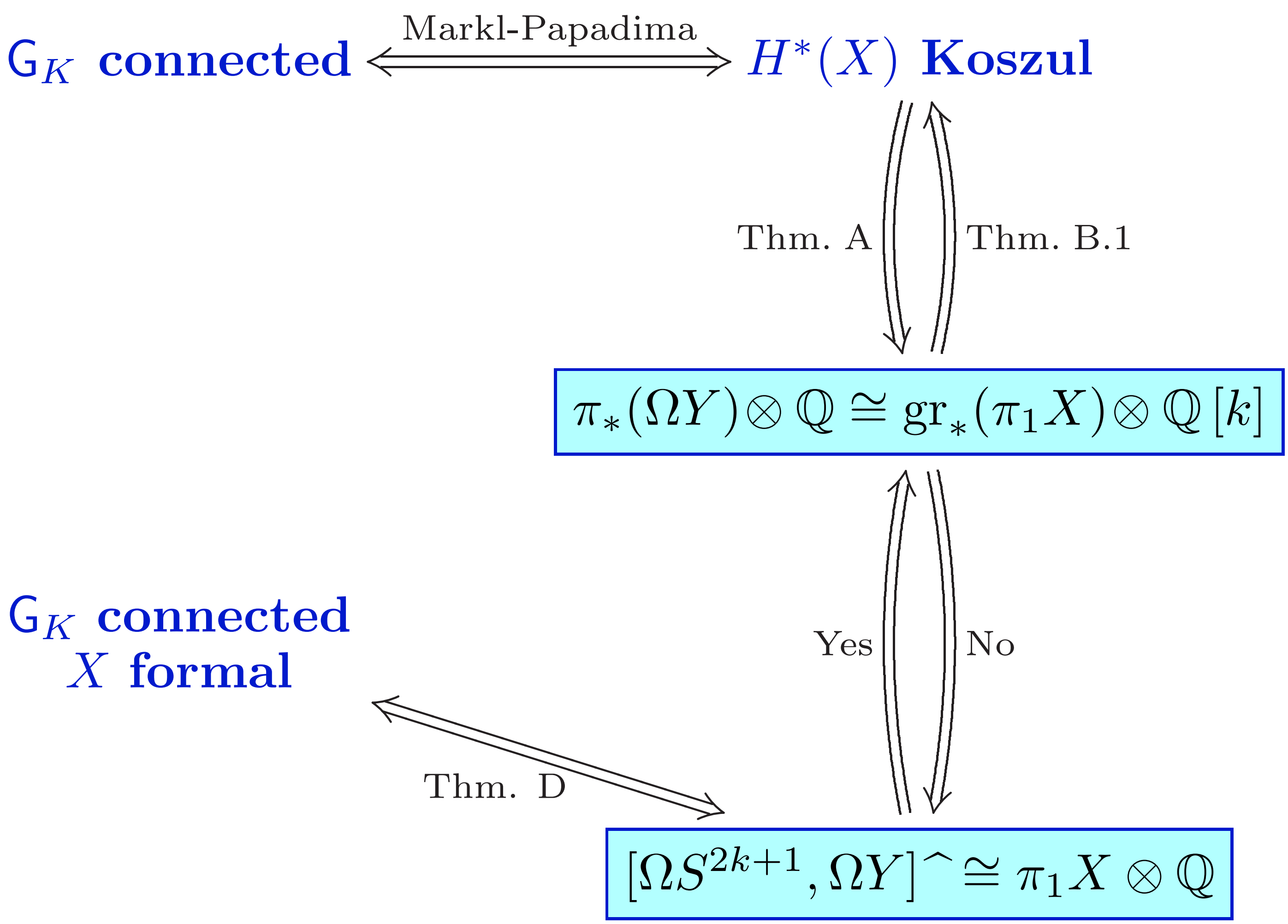
- Key ingredient in proof: A result of H. Baues [1981].
- Passing to associated graded Lie algebras: Mal'cev Formula ($\ddagger$) $\rightsquigarrow$ Rescaling Formula ($\dagger$).
- ($\ddagger$) holds for Koszul arrangements.
- For $X = \bigvee^n S^1$, $Y = \bigvee^n S^{2k+1}$: recovers a result from T. Sato's thesis [2000].
- For $\mathcal{A} =$ braid arr., $X = M(\mathcal{A})$, $Y = M(\mathcal{A}^{k+1})$: answers a question of Cohen-Gitler [2001].

Rescaling links

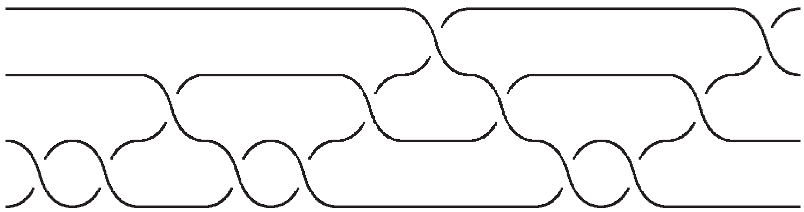
Let K link, $X = M(K)$, $Y = M(K^{\circledast k})$

G_K **linking graph** of K

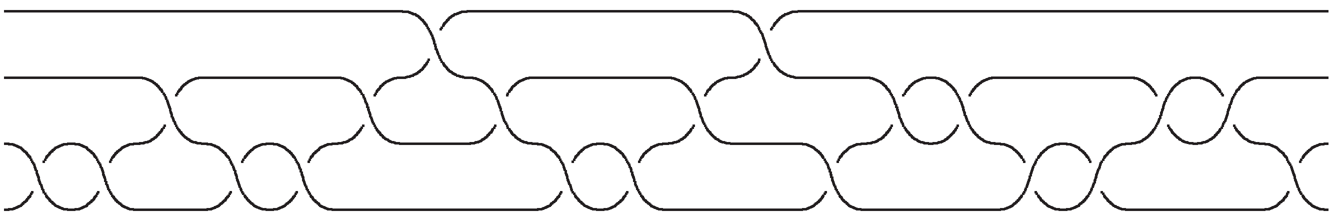
—vertices K_i, edges (K_i, K_j) if $\mathrm{lk}(K_i, K_j) \neq 0$



K_0



K



$K_0 = \widehat{\Delta_n^2} = \mathrm{Hopf}_n, \quad n \geq 4$

$K = \widehat{\Delta_n^2 \beta}, \quad \text{with } \beta \in (P_{n-1})_{r-1} \setminus (P_{n-1})_r, \quad r \geq 3$

K_0 and K have:

- $\mathsf{G} = K_n$, $\mathrm{lk}_{ij} = 1$. Thus, $H^*(X_0, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$.
- Same Milnor $\overline{\mu}$-invariants.
- Same Vassiliev invariants, up to order $r - 2$.
- $\mathrm{gr}^*(\pi_1 X_0) \cong \mathrm{gr}^*(\pi_1 X)$.

But X_0 **formal**, X **not formal**. So:

- Rescaling Formula holds for both K_0 and K.
- Mal'cev Formula **holds** for K_0, **fails** for K.