Which 3-manifold groups are Kähler groups?

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Realizing finitely presented groups

- Every finitely presented group G can be realized as
$$G = \pi_1(M),$$
for some smooth, compact, connected, orientable manifold M^n of dimension $n \geq 4$.
- The manifold M^n (n even) can be chosen to be symplectic (Gompf 1995).
- The manifold M^n (n even, $n \geq 6$) can be chosen to be complex (Taubes 1992).

If M is a compact Kähler manifold, $G = \pi_1(M)$ is called a ***Kähler group*** (or, ***projective group***, if M is actually a smooth projective variety). This puts strong restrictions on G, e.g.:

- $b_1(G)$ is even (Hodge theory).
- G is ***1-formal***, i.e., its Malcev Lie algebra is quadratic (Deligne–Griffiths–Morgan–Sullivan 1975).
- G cannot split non-trivially as a free product (Gromov 1989).

Example. Every finite group is a projective group (Serre 1958).

Remark. If G is a Kähler group, and $H < G$ is a finite-index subgroup, then H is also a Kähler group.

Requiring M to be a (compact, connected, orientable) 3-manifold also puts severe restrictions on $G = \pi_1(M)$. For example, if G is abelian, then G is either $\mathbb{Z}/n\mathbb{Z}$, or $\mathbb{Z}$, or $\mathbb{Z}^3$.

Question (Donaldson–Goldman 1989, Reznikov 1993)**.** What are the 3-manifold groups which are Kähler groups?

Partial answer:

Theorem (Reznikov 2002)**.** *Let M be an irreducible, atoroidal 3-manifold. Suppose there is a homomorphism $\rho\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ with Zariski dense image. Then $G = \pi_1(M)$ is not a Kähler group.*

We answer the question for all 3-manifold groups:

Theorem. *Let G be a 3-manifold group. If G is a Kähler group, then G is finite.*

By the Thurston Geometrization Conjecture (Perelman 2003), a closed, orientable 3-manifold M has finite fundamental group iff it admits a metric of constant positive curvature. Thus, $M = S^3/G$, where G is a finite subgroup of SO(4), acting freely on S^3. By (Milnor 1957), the list of such finite groups is:

$$1,\ D^*_{4n},\ O^*,\ I^*,\ D_{2^k(2n+1)},\ P'_{8\cdot 3^k},$$

and products of one of these with a cyclic group of relatively prime order.

Remark. The Theorem holds for fundamental groups of non-orientable (closed) 3-manifolds, as well: use the orientation double cover, and previous Remark.

Characteristic varieties

Let X be a connected, finite-type CW-complex, $G = \pi_1(X)$, and $\operatorname{Hom}(G, \mathbb{C}^*)$ the character torus ($\cong (\mathbb{C}^*)^n$, $n = b_1(G)$).

Every $\rho \in \operatorname{Hom}(G, \mathbb{C}^*)$ determines a rank 1 local system, $\mathbb{C}_\rho$, on X. The ***characteristic varieties*** of X are the jumping loci for cohomology with coefficients in such local systems:

$$V^i_d(X) = \{\rho \in \operatorname{Hom}(G, \mathbb{C}^*) \mid \dim H^i(X, \mathbb{C}_\rho) \geq d\}.$$

Note. $V_d(X) = V^1_d(X)$ depends only on $G = \pi_1(X)$, so we may write it as $V_d(G)$.

Theorem (Beauville, Green–Lazarsfeld, Simpson, Campana). *If $G = \pi_1(M)$ is a Kähler group, then $V_d(G)$ is a union of (possibly translated) subtori:*

$$V_d(G) = \bigcup_\alpha \rho_\alpha \cdot f^*_\alpha \operatorname{Hom}(\pi_1(C_\alpha), \mathbb{C}^*),$$

where each $f_\alpha \colon M \to C_\alpha$ is a surjective, holomorphic map to a compact, complex curve of positive genus.

Resonance varieties

Consider now the cohomology algebra $H^*(X, \mathbb{C})$.

Left-multiplication by $x \in H = H^1(X, \mathbb{C})$ yields a cochain complex $(H^*(X, \mathbb{C}), x)$:

$$H^0(X, \mathbb{C}) \xrightarrow{x\cdot} H^1(X, \mathbb{C}) \xrightarrow{x\cdot} H^2(X, \mathbb{C}) \longrightarrow \cdots$$

The ***resonance varieties*** of X are the jumping loci for the homology of this complex:

$$R^i_d(X) = \{x \in H \mid \dim H^i(H^*(X, \mathbb{C}), x) \geq d\}.$$

Note. $x \in H$ belongs to $R^1_d(X) \iff \exists$ subspace $W \subset H$ of dim $d+1$ such that $x \cup y = 0$, $\forall y \in W$.

Note. $R_d(X) = R^1_d(X)$ depends only on $G = \pi_1(X)$, so write it as $R_d(G)$.

Set $n = b_1(X)$, $m = b_2(X)$. Fix bases $\{e_1, \ldots, e_n\}$ for $H = H^1(X, \mathbb{C})$ and $\{f_1, \ldots, f_m\}$ for $H^2(X, \mathbb{C})$, and write

$$e_i \cup e_j = \sum_{k=1}^{m} \mu_{i,j,k} f_k.$$

Define an $m \times n$ matrix Δ of linear forms in variables $x_1, \ldots, x_n$, with entries

$$\Delta_{k,j} = \sum_{i=1}^{n} \mu_{i,j,k} x_i.$$

Then:

$$R^1_d(X) = V(E_d(\Delta)),$$

where

$$E_d = \text{ideal of } (n-d) \times (n-d) \text{ minors}$$

Note. $x \cup x = 0$ $(\forall x \in H)$ implies $\Delta \cdot \vec{x} = 0$, where $\vec{x}$ is the column vector $(x_1, \ldots, x_n)$.

Remark. When G is a commutator-relators group, $\Delta = A^{\text{lin}}$, the ***linearized Alexander matrix***, from Cohen-S. [1999, 2006], Matei-S. [2000].

The tangent cone theorem

Let $H^1(X, \mathbb{C}) = \mathrm{Hom}(G, \mathbb{C})$ be the Lie algebra of the character group $\mathrm{Hom}(G, \mathbb{C}^*)$, and consider the exponential map,

$$\begin{array}{ccc} \mathrm{Hom}(G,\mathbb{C}) & \xrightarrow{\exp} & \mathrm{Hom}(G,\mathbb{C}^*) \\ \uparrow & & \uparrow \\ R^i_d(X) & & V^i_d(X) \end{array}$$

The tangent cone to $V^i_d(X)$ at 1 is contained in $R^i_d(X)$ (Libgober 2002).

In general, the inclusion is strict (Matei–S. 2002).

Theorem (Dimca–Papadima–S. 2005). *Let G be a 1-formal group (e.g., a Kähler group). Then, $\forall d \geq 1$,*

$$\exp\colon (R_d(G), 0) \xrightarrow{\simeq} (V_d(G), 1)$$

is an iso of complex analytic germs. Consequently,

$$\mathrm{TC}_1(V_d(G)) = R_d(G).$$

Resonance varieties of Kähler groups

The description of the irreducible components of $V_1(M)$ in terms of pullbacks of tori $H^1(C, \mathbb{C}^*)$ along holomorphic maps $f\colon M \to C$, together with the Tangent Cone Theorem yield:

Theorem (Dimca–Papadima–S. 2005). *Let G be a Kähler group. Then every positive-dimensional component of $R_1(G)$ is an 1-isotropic linear subspace of $H^1(G, \mathbb{C})$, of dimension at least 4.*

Here, a subspace $W \subseteq H^1(G, \mathbb{C})$ is 1-*isotropic* with respect to the cup-product map

$$\cup_G\colon H^1(G, \mathbb{C}) \wedge H^1(G, \mathbb{C}) \to H^2(G, \mathbb{C})$$

if the restriction of $\cup_G$ to $W \wedge W$ has rank 1.

Corollary. *Let G be a Kähler group. Suppose $R_1(G) = H^1(G, \mathbb{C})$, and $H^1(G, \mathbb{C})$ is not 1-isotropic. Then $b_1(G) = 0$.*

Resonance varieties of 3-manifold groups

Let M be a compact, connected, orientable 3-manifold. Fix an orientation $[M] \in H^3(M, \mathbb{Z}) \cong \mathbb{Z}$.

With this choice, the cup product on M determines an alternating 3-form μ_M on $H^1(M, \mathbb{Z})$:

$$\mu_M(x, y, z) = \langle x \cup y \cup z, [M] \rangle,$$

where $\langle \, , \rangle$ is the Kronecker pairing.

In turn, $\cup_M \colon H^1(M, \mathbb{Z}) \wedge H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is determined by μ_M, via $\langle x \cup y, \gamma \rangle = \mu_M(x, y, z)$, where $z = \mathrm{PD}(\gamma)$ is the Poincaré dual of $\gamma \in H_2(M, \mathbb{Z})$.

Now fix a basis $\{e_1, \dots, e_n\}$ for $H^1(M, \mathbb{C})$, and choose as basis for $H^2(X, \mathbb{C})$ the set $\{e_1^\vee, \dots, e_n^\vee\}$, where $e_i^\vee$ is the Kronecker dual of the Poincaré dual of e_i. Then

$$\mu(e_i, e_j, e_k) = \langle \sum_{1 \le m \le n} \mu_{i,j,m} e_m^\vee, \mathrm{PD}(e_k) \rangle = \mu_{i,j,k}.$$

Recall the $n \times n$ matrix Δ, with $\Delta_{k,j} = \sum_{i=1}^n \mu_{i,j,k} x_i$. Since μ is an alternating form, Δ is skew-symmetric.

Proposition. *Let M be a closed, orientable 3-manifold. Then:*

1. $H^1(M, \mathbb{C})$ *is not* 1*-isotropic.*
2. *If $b_1(M)$ is even, then $R_1(M) = H^1(M, \mathbb{C})$.*

Proof. To prove (1), suppose $\dim \operatorname{im}(\cup_M) = 1$. This means there is a hyperplane $E \subset H := H^1(M, \mathbb{C})$ such that $x \cup y \cup z = 0$, for all $x, y \in H$ and $z \in E$. Hence, the skew 3-form $\mu\colon \bigwedge^3 H \to \mathbb{C}$ factors through a skew 3-form $\bar{\mu}\colon \bigwedge^3 (H/E) \to \mathbb{C}$. But $\dim H/E = 1$ forces $\bar{\mu} = 0$, and so $\mu = 0$, a contradiction.

To prove (2), recall $R_1(M) = V(E_1(\Delta))$. Since Δ is a skew-symmetric matrix of even size, it follows from (Buchsbaum–Eisenbud 1977) that

$$V(E_1(\Delta)) = V(E_0(\Delta)).$$

But $\Delta \vec{x} = 0 \Rightarrow \det \Delta = 0$; hence, $V(E_0(\Delta)) = H$. $\square$

Kazhdan's property T

Definition. A discrete group G satisfies ***Kazhdan's property*** T if

$$H^1(G, \mathbb{C}^k_\rho) = 0,$$

for all representations $\rho\colon G \to \mathrm{U}(k)$.

In particular, $b_1(G) \neq 0 \implies G$ not Kazhdan.

Theorem (Reznikov 2002). *Let G be a Kähler group. If G is not Kazhdan, then $b_2(G) \neq 0$.*

Theorem (Fujiwara 1999). *Let G be a 3-manifold group. If G is Kazhdan, then G is finite.*

Remark. The last theorem holds for any subgroup G of $\pi_1(M)$, where M is a compact (not necessarily boundaryless), orientable 3-manifold. Fujiwara assumes that each piece of the JSJ decomposition of M admits one of the 8 geometric structures in the sense of Thurston, but this is now guaranteed by the work of Perelman.

Proof of Main Theorem

Let G be the fundamental group of a compact, connected, orientable 3-manifold M.

Suppose G is a Kähler group, and G is not finite.

Step 1. Claim: M is irreducible.

Otherwise, M splits as a connected sum $M_1 \# M_2$, with $M_i \not\cong S^3$. Thus, by van Kampen's Theorem,

$$G = G_1 * G_2.$$

Each group $G_i = \pi_1(M_i)$ is non-trivial, by the Poincaré conjecture (proved by Perelman).

But G is a Kähler group, so it does not admit such a non-trivial splitting, by Gromov's Theorem.

Step 2. Since M is irreducible and $G = \pi_1(M)$ is infinite, the Sphere Theorem of Papakyriakopoulos implies

$$M = K(G, 1).$$

Thus, by Poincaré duality,

$$b_1(G) = b_2(G).$$

Step 3. Since G is an infinite 3-manifold group, G is not Kazhdan, by Fujiwara's Theorem.

Since G is Kähler and not Kazhdan, $b_2(G) \neq 0$, by Reznikov's Theorem.

Thus, by Step 2,

$$b_1(G) \neq 0.$$

Step 4. Since G is Kähler, $b_1(G)$ must be even.

Since M is a closed 3-manifold with $G = \pi_1(M)$, the Proposition gives

$$R_1(G) = H^1(G, \mathbb{C})$$

and $H^1(G, \mathbb{C})$ is not 1-isotropic.

Since, on the other hand, G is Kähler, the Corollary gives

$$b_1(G) = 0.$$

Our assumptions have led to a contradiction. Thus, the Theorem is proved.

Quasi-Kähler groups

A group G is ***quasi-Kähler*** (***quasi-projective***) if $G = \pi_1(M \setminus D)$, where M is a Kähler (projective) manifold and D is a divisor with normal crossings. E.g., arrangement groups are quasi-projective.

Question. Which 3-manifold groups are quasi-Kähler?

We have partial results, including a complete answer in the case of *boundary manifolds* of line arrangements.

Theorem (Cohen–S. 2008, Dimca–Papadima–S. 2008)**.** *Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be a line arrangement in $\mathbb{CP}^2$. Let M be the boundary of a regular neighborhood of $\mathcal{A}$, and $G = \pi_1(M)$. The following are equivalent:*

1. *G is 1-formal.*
2. *$\mathrm{TC}_1(V_d(G)) = R_d(G)$.*
3. *G is quasi-Kähler.*
4. *G is quasi-projective.*
5. *$\mathcal{A}$ is either a pencil or a near-pencil.*

In this case, $M = \sharp^n S^1 \times S^2$ or $M = S^1 \times \Sigma_{n-1}$.