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ALEXANDER INVARIANTS OF COMPLEX HYPERPLANE ARRANGEMENTS

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ABSTRACT. Let $\mathcal{A}$ be an arrangement of n complex hyperplanes. The fundamental group of the complement of $\mathcal{A}$ is determined by a braid monodromy homomorphism, $\alpha : F_s \to P_n$. Using the Gassner representation of the pure braid group, we find an explicit presentation for the Alexander invariant of $\mathcal{A}$. From this presentation, we obtain combinatorial lower bounds for the ranks of the Chen groups of $\mathcal{A}$. We also provide a combinatorial criterion for when these lower bounds are attained.

INTRODUCTION

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^d$, with complement $M = \mathbb{C}^d \setminus \cup_{i=1}^n H_i$, and group $G = \pi_1(M)$. Let M' be the maximal abelian cover, corresponding to the abelianization $\mathrm{ab} : G \to \mathbb{Z}^n$. The action of $\mathbb{Z}^n$ on M' puts on $H_*(M')$ the structure of a module over the group ring $\mathbb{Z}\mathbb{Z}^n$. This ring can be identified with the ring of Laurent polynomials $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, with t_i corresponding to a standardly oriented meridional loop around H_i. The object of our study is the Alexander invariant, $B(\mathcal{A}) = H_1(M')$, viewed as a module over the ring Λ.

Let $L(\mathcal{A})$ denote the intersection lattice of $\mathcal{A}$, with rank function given by codimension (see Orlik and Terao [27] as a general reference for arrangements). Let s denote the cardinality of $L_2(\mathcal{A})$, the set of rank two elements in $L(\mathcal{A})$. From the defining polynomial of $\mathcal{A}$, one can compute the Moishezon-Libgober braid monodromy homomorphism, $\alpha : F_s \to P_n$, see [7]. This homomorphism determines a finite presentation for the group of the arrangement: $G = \langle t_1, \dots, t_n \mid \alpha_k(t_i) = t_i \rangle$, where $\alpha_1, \dots, \alpha_s$ generate the image of α. The braid monodromy may also be used to obtain a finite presentation for the Alexander invariant $B(\mathcal{A})$. We accomplish this here, by means of the Gassner representation, $\Theta : P_n \to \mathrm{GL}(n, \Lambda)$, the Fox calculus, and homological algebra.

Surprisingly, the size of the presentation depends only on the first two betti numbers of the complement: there are $\binom{n}{2}$ generators and $\binom{n}{3} + b_2(M)$ relations. When $\mathcal{A}$ is the complexification of a real arrangement, the presentation of $B(\mathcal{A})$ can

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be simplified to $\binom{n}{2} - b_2(M)$ generators and $\binom{n}{3}$ relations. More generally, if G is the group of a collection of s basis-conjugating automorphisms of a finitely generated free group F_n, our methods yield a presentation of $B(G)$ with $\binom{n}{2}$ generators and $\binom{n}{3} + ns$ relations. In particular, the Alexander invariant of any pure link has such a presentation (with $s = 1$). This should be compared with the general situation for links in S^3, where there is no upper bound on the number of relations, see Traldi [31].

Note that the Alexander invariant is isomorphic to G'/G'' (with the usual G/G' action), and so depends only on the isomorphism type of G. Consequently, we may obtain invariants of an arrangement $\mathcal{A}$ from the module $B = B(\mathcal{A})$ and its presentation. For instance, if Δ is a presentation matrix for B, the elementary ideal $E_k(B)$ is defined to be the ideal generated by the codimension k minors of Δ. It is well-known that these ideals depend only on the module B. These ideals, and the closely related characteristic varieties, arise in the study of plane algebraic curves; see for instance the recent works of Hironaka [20] and Libgober [22]. The structure of the elementary ideals and characteristic varieties of the Alexander invariant of an arrangement will be the subject of a future work.

In this paper, we focus on Chen groups. The Chen groups of G are the lower central series quotients of the maximal metabelian quotient G/G''. Using an observation of Massey [23] relating the Chen groups and the Alexander invariant, together with Mora's tangent cone algorithm, we obtain an algorithm for computing the Chen groups of an arrangement from the presentation of the Alexander invariant B. The ranks of the Chen groups often serve to distinguish the groups of combinatorially "similar" arrangements. This is particularly useful for fiber-type arrangements, where the ranks of the lower central series quotients of G itself are determined by the exponents of the arrangement. On the other hand, we know of no combinatorially equivalent arrangements whose Chen groups differ.

The precise relation between the Chen groups and the intersection lattice of a central arrangement $\mathcal{A}$ is not known. We obtain partial results toward this end here. To each element $V \in L_2(\mathcal{A})$, we associate a "local" Alexander invariant B_V. Algebraic considerations yield a surjective homomorphism $B \to B^{\mathrm{cc}}$, where $B^{\mathrm{cc}} = \oplus_V B_V$ is the "coarse combinatorial Alexander invariant" of $\mathcal{A}$, determined by (only) the multiplicities of the elements of $L_2(\mathcal{A})$. From this map, we obtain combinatorial lower bounds on the ranks of the Chen groups of $\mathcal{A}$. These ranks are determined by the I-adic completion, $\widehat{B}$, of the Alexander invariant B, where I is the augmentation ideal of Λ. We find a combinatorial criterion for when the completion of the Alexander invariant of $\mathcal{A}$ decomposes as a direct sum, i.e., $\widehat{B} \xrightarrow{\sim} \widehat{B}^{\mathrm{cc}}$. We also obtain a combinatorial formula for the rank of the third Chen group of any arrangement.

The above results may be viewed as evidence that the ranks, θ_k, of the Chen groups of $\mathcal{A}$ are combinatorially determined. In [5], we conjectured an explicit combinatorial formula for θ_k, for sufficiently large k. This formula involved the number β of subarrangements of $\mathcal{A}$ lattice-isomorphic to the braid arrangement $\mathcal{A}_4 \subset \mathbb{C}^4$. In the present context, we show by example that if $\beta > 1$, then the map $\widehat{B} \to \widehat{B}^{\mathrm{cc}}$ is not an isomorphism. Other examples exhibit combinatorially different ways this map can fail to be an isomorphism. These provide counterexamples to the aforementioned formula, and illustrate the subtlety of the relationship between the Chen groups and the lattice of an arrangement.

Our results on Chen groups parallel a portion of Falk's work on the LCS quotients of an arrangement group. The combinatorial lower bounds we obtain for the ranks of the Chen groups are analogous to those for the ranks, ϕ_k, of the LCS quotients found in [13]. Moreover, the formula we obtain for $\theta_3 = \phi_3$ may be viewed as dual to the description of ϕ_3 found in [12], [13]. The precise relationship between the Chen groups and LCS quotients of an arrangement will be explored elsewhere.

The structure of the paper is as follows:

- In section 1, we review Alexander invariants and Chen groups, and present a Groebner basis algorithm for determining the latter. The section concludes with an analysis of the Alexander invariant and Chen groups of a product of spaces.
- In section 2, we introduce our basic computational tools: the Fox free differential calculus and the Magnus representations.
- In section 3, we study the Alexander invariant of the group of a free automorphism. An explicit presentation is given when the automorphism is basis-conjugating.
- In section 4, we find presentations for the local Alexander invariants of an arrangement.
- In section 5, the presentation for the Alexander invariant of an arrangement is obtained.
- In section 6, the homomorphism $B \to B^{\mathrm{cc}}$ is defined, and its completion proven to be an isomorphism when a certain criterion is satisfied.
- In section 7, the aforementioned criterion is shown to be combinatorial, and lower bounds on the ranks of the Chen groups of an arrangement are obtained.
- In section 8, we illustrate our results by means of several explicit examples.

Conventions. Given a group G, we will denote by $\mathrm{Aut}(G)$ the group of *right* automorphisms of G, with multiplication $\alpha \cdot \beta = \beta \circ \alpha$. We will regard all modules over the group ring $\mathbb{Z}G$ as *left* modules. Elements of the free module $(\mathbb{Z}G)^n$ are viewed as *row* vectors, and $\mathbb{Z}G$-linear maps $(\mathbb{Z}G)^n \to (\mathbb{Z}G)^m$ are viewed as $n \times m$ matrices which act on the *right* (so that the matrix of $B \circ A$ is $A \cdot B$). We will write $A^\top$ for the transpose of A, and $(A_1 \ \cdots \ A_s)^\top$ for $\left(\begin{smallmatrix} A_1 \\ \vdots \\ A_s \end{smallmatrix}\right)$. If $\phi : G \to H$ is a homomorphism, $\tilde{\phi} : \mathbb{Z}G \to \mathbb{Z}H$ denotes its $\mathbb{Z}$-linear extension to group rings. We will abuse notation and also write $\tilde{\phi} : (\mathbb{Z}G)^n \to (\mathbb{Z}H)^n$ for the map $\oplus_1^n \tilde{\phi}$.

1. Alexander Invariants, Chen Groups, and Products

We start by reviewing the definition of the Alexander invariant of a finite complex. We then present an algorithm for computing the ranks of the Chen groups of a group, based on a presentation of this module. Finally, we determine the structure of the Alexander invariant of a product of spaces in terms of those of the factors.

1.1. **Alexander Invariants.** Let M be a path-connected space that has the homotopy type of a finite CW-complex. Let $G = \pi_1(M, *)$ be the fundamental group, and $K = H_1(M)$ its abelianization. Let $p : M' \to M$ be the maximal abelian cover. The action of K on M' passes to an action of K on the homology groups $H_*(M')$. This defines on $H_*(M')$ the structure of a module over the group ring $\mathbb{Z}K$. The $\mathbb{Z}K$-module $B = H_1(M')$ is called the (first) *Alexander invariant* of M. Closely related to it is the (first) *Alexander module*, $A = H_1(M', p^{-1}(*))$. These

two modules, together with the augmentation ideal $I = IK = \ker(\epsilon : \mathbb{Z}K \to \mathbb{Z})$, comprise the Crowell exact sequence, $0 \to B \to A \to I \to 0$, of [10].

The two Alexander modules depend only on the group G. Indeed, $A = \mathbb{Z}K \otimes_{\mathbb{Z}G} IG$, with $K = G/G'$ acting by multiplication on the left factor, and $B = G'/G''$, with the action of K defined by the extension $1 \to G'/G'' \to G/G'' \to G/G' \to 1$. Since M is by assumption a finite complex, G is a finitely presented group. Hence, the $\mathbb{Z}K$-module A is finitely presented; Fox's free differential calculus provides an explicit presentation (see [11], and also sections 2 and 3). Less evident, but still true, is the fact that B also admits a finite presentation as a $\mathbb{Z}K$-module (see [10], [23], and also section 3).

1.2. **Chen Groups.** Let $\Gamma_k(G)$ denote the k^{th} lower central series subgroup of G, defined inductively by $\Gamma_1(G) = G$ and $\Gamma_{k+1}(G) = [\Gamma_k(G), G]$ for $k \geq 1$. The projection of G onto its maximal metabelian quotient G/G'' induces an epimorphism

$$\frac{\Gamma_k(G)}{\Gamma_{k+1}(G)} \twoheadrightarrow \frac{\Gamma_k(G/G'')}{\Gamma_{k+1}(G/G'')}$$

from the k^{th} lower central series quotient of G to the k^{th} Chen group of G. Since G is finitely presented, these quotients are finitely generated abelian groups, whose ranks we will denote by ϕ_k, respectively θ_k. It is readily seen that $\phi_k = \theta_k$ for $k \leq 3$, and $\phi_k \geq \theta_k$ for $k > 3$.

The Chen groups of G can be determined from the Alexander invariant of G. Indeed, Massey [23] noted the following isomorphism, for $k \geq 2$:

$$\frac{\Gamma_k(G/G'')}{\Gamma_{k+1}(G/G'')} = \frac{I^{k-2}B}{I^{k-1}B}.$$

Thus, the Chen groups are determined by $\operatorname{gr} B = \bigoplus_{k\geq 0} I^k B/I^{k+1}B$, viewed as a graded module over the graded ring $\operatorname{gr} \mathbb{Z}K = \bigoplus_{k\geq 0} I^k/I^{k+1}$.

Now assume K is free abelian, and fix a system of generators, $t_1, \dots, t_n$. The group ring $\mathbb{Z}K$ can be identified with the ring of Laurent polynomials in n variables, $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. The ring Λ can be viewed as a subring of the formal power series ring $P = \mathbb{Z}[[x_1, \dots, x_n]]$ via the "Magnus embedding," given by $t_i \mapsto 1 - x_i$ and $t_i^{-1} \mapsto \sum_{k=0}^{\infty} x_i^k$. Let $\widehat{\Lambda} = \varprojlim \Lambda/I^k$ be the completion of Λ relative to the I-adic topology. Then, the Magnus embedding extends to a ring isomorphism $\widehat{\Lambda} \xrightarrow{\sim} P$.

Consider the $\mathfrak{m}$-adic filtration on P, where $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$, and its associated graded ring, $\operatorname{gr} P = \bigoplus_{k\geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$. As is well-known, this ring is isomorphic to the polynomial ring $R = \mathbb{Z}[x_1, \dots, x_n]$. Moreover, the Magnus embedding induces a graded ring isomorphism $\operatorname{gr} \Lambda \xrightarrow{\sim} \operatorname{gr} P = R$. Let $\widehat{B}$ be the I-adic completion of B, and $\operatorname{gr} \widehat{B} = \bigoplus_{k\geq 0} \mathfrak{m}^k \widehat{B}/\mathfrak{m}^{k+1}\widehat{B}$ the associated graded module. Then, the canonical map $B \to \widehat{B}$ induces an isomorphism $\operatorname{gr} B \xrightarrow{\sim} \operatorname{gr} \widehat{B}$ of graded modules over the ring R.

Combining these facts, we can restate Massey's result as follows:

Theorem 1.3 ([23])**.** *The generating series for the ranks of the Chen groups of G, $\sum_{k=0}^{\infty} \theta_{k+2} t^k$, is equal to the Hilbert series of the graded module associated to the I-adic completion of $B(G)$, $\sum_{k=0}^{\infty} \operatorname{rank}(\mathfrak{m}^k \widehat{B}/\mathfrak{m}^{k+1}\widehat{B}) t^k$.*

An immediate consequence of this theorem is that, for k sufficiently large, θ_k is given by a polynomial in k. Indeed, this is just the Hilbert-Serre polynomial of $\operatorname{gr} \widehat{B}$, see [32].

1.4. **Groebner Bases.** Let $\Lambda^a \xrightarrow{\Delta} \Lambda^b \to B \to 0$ be a (finite) presentation of the Alexander invariant. Note that, by replacing the generators of the free module Λ^a by suitable multiples if necessary, we may assume that the entries of the matrix of Δ are polynomials in the variables t_i. Let $J = \operatorname{im}\Delta$. A presentation for the I-adic completion of B is given by $\widehat{\Lambda}^a \xrightarrow{\widehat{\Delta}} \widehat{\Lambda}^b \to \widehat{B} \to 0$, where $\widehat{\Delta}$ is obtained from Δ via the Magnus embedding. Clearly, $\operatorname{im}\widehat{\Delta} = \widehat{J}$. Since all the entries of the matrix for $\widehat{\Delta}$ belong to the subring $R \subset P$, we may restrict $\widehat{\Delta}$ to a map $\mathbf{\Delta} : R^a \to R^b$, whose image, $\widehat{J} \cap R^b$, we denote by $\mathbf{J}$.

We must find a presentation for the associated graded module $\operatorname{gr}\widehat{B} = \operatorname{gr}(P^b/\widehat{J})$. This module is isomorphic to $R^b/\textsc{lt}(\mathbf{J})$, where $\textsc{lt}(\mathbf{J})$ is the submodule of R^b consisting of lowest degree homogeneous forms of elements in $\mathbf{J}$, see [32]. We are left with finding a finite generating set for $\textsc{lt}(\mathbf{J})$. Such a set is provided by Mora's algorithm for obtaining the tangent cone of an affine variety at the origin, see [9], [2]. Essentially, we must determine a (minimal) *Groebner basis* $\mathcal{G} = \{g_1, \dots, g_c\}$ for the module $\mathbf{J}$, with respect to a suitable monomial ordering. Then, $\textsc{lt}(\mathbf{J})$ has Groebner basis $\textsc{lt}(\mathcal{G}) = \{\textsc{lt}g_1, \dots, \textsc{lt}g_c\}$, from which we can extract a minimal Groebner basis $\mathcal{H} = \{h_1, \dots, h_d\}$. Putting all these facts together, we obtain the following.

Theorem 1.5. *The module* $\operatorname{gr}\widehat{B}$ *has presentation* $R^d \xrightarrow{\operatorname{gr}\Delta} R^b \to \operatorname{gr}\widehat{B} \to 0$*, where the rows of* $\operatorname{gr}\widehat{\Delta}$ *constitute a minimal Groebner basis for the module generated by the rows of the matrix* $\widehat{\Delta}$*, obtained from a presentation matrix* Δ *for* B *by replacing* t_i *by* $1 - x_i$*.*

Example 1.6. Let $G = F_n$ be a free group of rank n. A presentation for the Alexander invariant B of G is given by $\Lambda^a \xrightarrow{d_3} \Lambda^b \to B \to 0$, where $a = \binom{n}{3}$, $b = \binom{n}{2}$, and d_3 is the differential in the standard Λ-resolution of $\mathbb{Z}$. In this instance, it is readily checked that the rows of the matrix $\hat{d}_3$ form a Groebner basis for the module $\mathbf{J}$. A standard argument then yields the ranks of the Chen groups of F_n: $\theta_1 = n$ and $\theta_k = (k-1)\binom{k+n-2}{k}$ for $k \geq 2$, a calculation originally due to Murasugi [26] (see also [24]).

1.7. **Products.** Let M_1 and M_2 be two path connected finite CW-complexes, with $K_i = H_1(M_i)$ free abelian, and let M_i' be the corresponding maximal abelian covers. Then $M = M_1 \times M_2$ has maximal abelian cover $M' = M_1' \times M_2'$, corresponding to $K = H_1(M) = K_1 \times K_2$.

Proposition 1.8. *There is an isomorphism of* $\mathbb{Z}K$*-modules,*

$$H_1(M') \cong \big((H_1(M_1') \otimes_{\mathbb{Z}K_1} \mathbb{Z}K) \otimes_{\mathbb{Z}K_2} \mathbb{Z}\big) \oplus \big((H_1(M_2') \otimes_{\mathbb{Z}K_2} \mathbb{Z}K) \otimes_{\mathbb{Z}K_1} \mathbb{Z}\big).$$

Proof. By the (homological) Künneth formula, the group $H_1(M')$ is isomorphic to $H_1(M_1') \otimes H_0(M_2') \oplus H_0(M_1') \otimes H_1(M_2')$. When viewed as a $\mathbb{Z}K$-module, the first summand is isomorphic to

$$\begin{aligned}
&((H_1(M_1') \otimes_{\mathbb{Z}K_1} \mathbb{Z}K) \otimes_{\mathbb{Z}K} \mathbb{Z}K_1) \otimes ((\mathbb{Z} \otimes_{\mathbb{Z}K_2} \mathbb{Z}K) \otimes_{\mathbb{Z}K} \mathbb{Z}K_2) \\
&\quad = (H_1(M_1') \otimes_{\mathbb{Z}K_1} \mathbb{Z}K) \otimes_{\mathbb{Z}K} (\mathbb{Z}K_1 \otimes \mathbb{Z}K_2) \otimes_{\mathbb{Z}K} (\mathbb{Z} \otimes_{\mathbb{Z}K_2} \mathbb{Z}K) \\
&\quad = (H_1(M_1') \otimes_{\mathbb{Z}K_1} \mathbb{Z}K) \otimes_{\mathbb{Z}K_2} \mathbb{Z},
\end{aligned}$$

where we made use of the obvious isomorphism $(\mathbb{Z}K_1 \otimes \mathbb{Z}K_2) \cong \mathbb{Z}(K_1 \times K_2)$, and viewed the induced module $H_1(M_1') \otimes_{\mathbb{Z}K_1} \mathbb{Z}K$ as a $\mathbb{Z}K_2$-module by restriction of scalars. The second summand is treated exactly the same way. □

We want to find now a presentation for the Alexander invariant $B(M) = H_1(M')$, given presentations for the Alexander invariants $B(M_i) = H_1(M_i')$. Fix generators $t_1^{(i)}, \dots, t_{n_i}^{(i)}$ for K_i, and use them to identify $\mathbb{Z}K_i$ with Λ_i.

Theorem 1.9. *If the Alexander invariants of M_1 and M_2 have presentations*

$$\Lambda_i^{a_i} \xrightarrow{\Delta_i} \Lambda_i^{b_i} \to B(M_i) \to 0,$$

then the Alexander invariant of $M = M_1 \times M_2$ has presentation

$$\Lambda^a \xrightarrow{\begin{pmatrix} \Delta_1 \\ D_2^{b_1} \end{pmatrix} \oplus \begin{pmatrix} \Delta_2 \\ D_1^{b_2} \end{pmatrix}} \Lambda^b \to B(M) \to 0,$$

where $a = a_1 + n_2 b_1 + a_2 + n_1 b_2$, $b = b_1 + b_2$, and $D_i = (t_1^{(i)} - 1, \dots, t_{n_i}^{(i)} - 1)^\top$.

Proof. Let us look at the first summand in the direct sum decomposition of $H_1(M')$ from Proposition 1.8. It is the tensor product over $\mathbb{Z}K$ of two induced modules. The first one is the $\mathbb{Z}K$-module induced from the $\mathbb{Z}K_1$-module $H_1(M_1')$, and has presentation

$$(\mathbb{Z}K)^{a_1} \xrightarrow{\Delta_1} (\mathbb{Z}K)^{b_1} \to H_1(X_1) \otimes_{\mathbb{Z}K_1} \mathbb{Z}K \to 0. \tag{1.1}$$

The second one is the $\mathbb{Z}K$-module induced from the trivial $\mathbb{Z}K_2$-module $\mathbb{Z}$, and has presentation

$$(\mathbb{Z}K)^{n_2} \xrightarrow{D_2} \mathbb{Z}K \to \mathbb{Z} \otimes_{\mathbb{Z}K_2} \mathbb{Z}K \to 0. \tag{1.2}$$

Taking the tensor product (over $\mathbb{Z}K$) of the complexes (1.1) and (1.2) and truncating yields the following presentation for the first summand of $H_1(M')$:

$$(\mathbb{Z}K)^{a_1 + n_2 b_1} \xrightarrow{\begin{pmatrix} \Delta_1 \\ D_2^{b_1} \end{pmatrix}} (\mathbb{Z}K)^{b_1} \to (H_1(X_1) \otimes_{\mathbb{Z}K_1} \mathbb{Z}K) \otimes_{\mathbb{Z}K_2} \mathbb{Z} \to 0.$$

The second summand is handled the same way, and that finishes the proof. □

Corollary 1.10. *The ranks of the Chen groups of $G = \pi_1(M_1 \times M_2)$ are given by*

$$\theta_k(G) = \theta_k(G_1) + \theta_k(G_2),$$

where $G_i = \pi_1(M_i)$.

Example 1.11. Let $G = F_{d_1} \times \cdots \times F_{d_\ell}$ be a direct product of finitely generated free groups. Using the above result, and the calculation in Example 1.6, one can easily recover the ranks of the Chen groups of G announced in [5]: $\theta_1 = \sum_{i=1}^{\ell} d_i$ and $\theta_k = (k-1) \cdot \sum_{i=1}^{\ell} \binom{k+d_i-2}{k}$ for $k \geq 2$.

Example 1.12. Let $\mathcal{A}$ be an affine arrangement of n hyperplanes in $\mathbb{C}^d$, and let $\overline{\mathcal{A}}$ be the *cone* of $\mathcal{A}$, a central arrangement of $n+1$ hyperplanes in $\mathbb{C}^{d+1}$ (see [27]). It is well-known that the complement $\overline{M}$ of $\overline{\mathcal{A}}$ is homeomorphic to the product of the complement M of $\mathcal{A}$ and $\mathbb{C}^*$, $\overline{M} = M \times \mathbb{C}^*$.

Fix a generator x for $\pi_1(\mathbb{C}^*) = \mathbb{Z}$. Let $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, and suppose that the Alexander invariant B of $\mathcal{A}$ has presentation $\Lambda^a \xrightarrow{\Delta} \Lambda^b \to B \to 0$ (see section 5). Using Theorem 1.9, we obtain a presentation $\overline{\Lambda}^{a+b} \xrightarrow{\overline{\Delta}} \overline{\Lambda}^b \to \overline{B} \to 0$ for the Alexander invariant of $\overline{\mathcal{A}}$, where $\overline{\Lambda} = \Lambda[x^{\pm 1}]$ and $\overline{\Delta} = \begin{pmatrix} \Delta & (x-1) \cdot \mathrm{id} \end{pmatrix}^\top$. Thus

the ranks of the Chen groups of $G(\overline{\mathcal{A}})$ coincide with those of $G(\mathcal{A})$ for $k \geq 2$. Note that $\theta_1(\overline{\mathcal{A}}) = \theta_1(\mathcal{A}) + 1 = n + 1$.

2. A Quick Trip through Fox Calculus

In this section we review the basics of Fox's free differential calculus, as introduced in [17], and developed in [3], and derive some consequences.

2.1. **Fox Gradient.** Let F_n be the free group on generators $t_1, \dots, t_n$, and $\mathbb{Z}F_n$ its group ring. Let $W_n = \bigvee_1^n S^1$ be a wedge of n circles, with basepoint $*$ at the wedge point. Let $\widetilde{W}_n$ be the universal cover, and let $\widetilde{C}_\bullet(\widetilde{W}_n)$ be the augmented, equivariant chain complex of $\widetilde{W}_n$. Identifying $C_0(\widetilde{W}_n)$ with $\mathbb{Z}F_n$, and $C_1(\widetilde{W}_n)$ with the free $\mathbb{Z}F_n$-module of rank n (with basis $e_1, \dots, e_n$ given by the lifts of the 1-cells at the basepoint $\widetilde{*}$), we obtain the standard free $\mathbb{Z}F_n$-resolution of $\mathbb{Z}$,

$$0 \to (\mathbb{Z}F_n)^n \xrightarrow{\partial_1} \mathbb{Z}F_n \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $\partial_1(e_i) = t_i - 1$ and $\epsilon(t_i) = 1$. The Fox Calculus is based on the observation that the augmentation ideal, $IF_n = \ker \epsilon$, is a free $\mathbb{Z}F_n$-module of rank n, generated by the entries of the matrix of ∂_1. This can be rephrased as follows: Given any $w \in \mathbb{Z}F_n$, there exist unique elements $\frac{\partial w}{\partial t_i} \in \mathbb{Z}F_n$ (called the Fox derivatives of w) such that the following "fundamental formula of Fox Calculus" holds:

$$w - \epsilon(w) = \sum_{i=1}^{n} \frac{\partial w}{\partial t_i}(t_i - 1). \tag{2.1}$$

Define the *Fox gradient* to be the $\mathbb{Z}F_n$-linear homomorphism $\nabla : \mathbb{Z}F_n \to (\mathbb{Z}F_n)^n$ given by

$$\nabla(w) = \sum_{i=1}^{n} \frac{\partial w}{\partial t_i}\, e_i.$$

Then, formula (2.1) takes the form $\partial_1(\nabla(w)) = w - \epsilon(w)$. From this can be deduced the following "product rule" for the Fox gradient: $\nabla(uv) = \nabla(u) \cdot \epsilon(v) + u \cdot \nabla(v)$. In particular, $\nabla(z^{-1}) = -z^{-1}\nabla(z)$, for $z \in F_n$.

Now consider an endomorphism $\alpha : F_n \to F_n$. This defines a map $\alpha : W_n \to W_n$ (unique up to homotopy). The induced chain map $\alpha_\bullet : C_\bullet(\widetilde{W}_n) \to C_\bullet(\widetilde{W}_n)$ can be written as

$$\begin{array}{ccc} C_1(\widetilde{W}_n) & \xrightarrow{\partial_1} & C_0(\widetilde{W}_n) \\ \big\downarrow{\scriptstyle J(\alpha)\circ\tilde{\alpha}} & & \big\downarrow{\scriptstyle \tilde{\alpha}} \\ C_1(\widetilde{W}_n) & \xrightarrow{\partial_1} & C_0(\widetilde{W}_n) \end{array} \tag{2.2}$$

where $J(\alpha) : (\mathbb{Z}F_n)^n \to (\mathbb{Z}F_n)^n$ is the *Fox Jacobian* of α; namely, the $\mathbb{Z}F_n$-linear homomorphism given by $J(\alpha)(e_i) = \nabla(\alpha(t_i))$. If $\beta : F_n \to F_n$ is another endomorphism, the fact that $(\beta \circ \alpha)_\bullet = \beta_\bullet \circ \alpha_\bullet$ may be rephrased as the "chain rule of Fox Calculus:" $J(\alpha \cdot \beta) = \tilde{\beta}(J(\alpha)) \cdot J(\beta)$. In particular, $J(\alpha)^{-1} = \tilde{\alpha} \circ J(\alpha^{-1}) \circ \tilde{\alpha}^{-1}$.

2.2. **Abelianized Fox Jacobian.** Let $\mathbb{Z}^n$ be the free abelian group on generators $t_1, \dots, t_n$, and identify the group ring $\mathbb{Z}\mathbb{Z}^n$ with $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Let $T^n = \times_1^n S^1$ be the n-torus. The augmented, equivariant chain complex, $\widetilde{C}_\bullet = \widetilde{C}_\bullet(\widetilde{T}^n)$, of the universal (abelian) cover can be written as

$$0 \to C_n \xrightarrow{d_n} \cdots \to C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0. \tag{2.3}$$

Identifying C_0 with Λ, C_1 with Λ^n, and C_k with $\bigwedge^k C_1 = \Lambda^{\binom{n}{k}}$, we obtain the standard free Λ-resolution of $\mathbb{Z}$. The differentials of this resolution are given by $d_k(e_J) = \sum_{r=1}^k (-1)^{k+r}(t_{j_r} - 1) \cdot e_{J \setminus \{j_r\}}$, where $e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}$ if $J = \{j_1, \dots, j_k\}$.

Let $\mathrm{ab} : F_n \to \mathbb{Z}^n$, $x \mapsto x^{\mathrm{ab}}$, be the abelianization homomorphism. For an element $w \in \mathbb{Z}F_n$, let $\nabla^{\mathrm{ab}}(w) := \widetilde{\mathrm{ab}}(\nabla(w)) \in \Lambda^n$ be its abelianized Fox gradient. This defines a Λ-linear homomorphism

$$\nabla^{\mathrm{ab}}(w) : C_0 \to C_1, \qquad u \mapsto \nabla^{\mathrm{ab}}(w) \cdot u.$$

For an endomorphism α of F_n, let $\Theta(\alpha) := \widetilde{\mathrm{ab}}(J(\alpha)) : C_1 \to C_1$ be its abelianized Fox Jacobian. This is a Λ-linear map, whose matrix has rows $\Theta(\alpha)(e_i) = \nabla^{\mathrm{ab}}(\alpha(t_i))$. Abelianizing diagram (2.2) yields the chain map:

$$\begin{array}{ccc} C_1 & \xrightarrow{d_1} & C_0 \\ \big\downarrow{\scriptstyle \Theta(\alpha)\circ\tilde{\alpha}} & & \big\downarrow{\scriptstyle \tilde{\alpha}} \\ C_1 & \xrightarrow{d_1} & C_0 \end{array} \tag{2.4}$$

Set $\Theta_k(\alpha) = \bigwedge^k \Theta(\alpha) : C_k \to C_k$ (in particular, $\Theta_0 = \mathrm{id}$). A computation in the exterior algebra $C_\bullet = \bigwedge^\bullet C_1$ shows that $\Theta_{k-1}(\alpha) \circ \tilde{\alpha} \circ d_k = d_k \circ \Theta_k(\alpha) \circ \tilde{\alpha}$ for each k, $1 \le k \le n$. Thus, (2.4) extends to a chain map

$$\Theta_\bullet(\alpha) \circ \tilde{\alpha} : C_\bullet \to C_\bullet \tag{2.5}$$

This chain map is the composite of two chain maps. The first is the (non-Λ-linear) map $\tilde{\alpha} : (C_\bullet, d_\bullet) \to (C_\bullet, \tilde{\alpha} \circ d_\bullet \circ \tilde{\alpha}^{-1})$. The second is the ($\Lambda$-linear) map $\Theta_\bullet(\alpha) : (C_\bullet, \tilde{\alpha} \circ d_\bullet \circ \tilde{\alpha}^{-1}) \to (C_\bullet, d_\bullet)$.

2.3. **Magnus Representations.** An automorphism $\alpha \in \mathrm{Aut}(F_n)$ is called an *IA-automorphism* if its abelianization, $\mathrm{ab}(\alpha) : \mathbb{Z}^n \to \mathbb{Z}^n$, is the identity map. In this case, $\tilde{\alpha} = \mathrm{id}$, and so $\Theta_\bullet(\alpha) : C_\bullet \to C_\bullet$ is a chain map.

The set of IA-automorphisms forms a subgroup of $\mathrm{Aut}(F_n)$, denoted by $\mathrm{IA}(F_n)$. By the chain rule, $\Theta(\alpha \cdot \beta) = \Theta(\alpha) \cdot \Theta(\beta)$, for $\alpha, \beta \in \mathrm{IA}(F_n)$. Thus $\Theta : \mathrm{IA}(F_n) \to \mathrm{Aut}_\Lambda(C_1) \cong \mathrm{GL}(n, \Lambda)$ is a linear representation of $\mathrm{IA}(F_n)$, called the *Magnus representation*, see [3]. From the above discussion, we see that this representation generalizes to $\Theta_k : \mathrm{IA}(F_n) \to \mathrm{Aut}_\Lambda(C_k) \cong \mathrm{GL}(\binom{n}{k}, \Lambda)$.

Remark 2.4. For $\alpha \in \mathrm{IA}(F_n)$, the chain automorphism $\Theta_\bullet(\alpha) : C_\bullet \to C_\bullet$ admits the following topological interpretation. The map $\alpha : W_n \to W_n$ lifts to a map of the maximal abelian covers, $\alpha' : W_n' \to W_n'$. View W_n' as the 1-skeleton of $\widetilde{T}^n$. The map α' extends to a $\mathbb{Z}^n$-equivariant map $\bar{\alpha} : \widetilde{T}^n \to \widetilde{T}^n$. The induced chain map, $\bar{\alpha}_\bullet : C_\bullet(\widetilde{T}^n) \to C_\bullet(\widetilde{T}^n)$, is chain-equivalent to $\Theta_\bullet(\alpha)$.

3. The Alexander Invariant of a Free Automorphism

In this section, we find presentations for the Alexander module and the Alexander invariant of the group of an IA-automorphism. A more explicit presentation for the latter is given in case the automorphism is basis-conjugating.

3.1. **The Group of a Free Automorphism.** Associated to an automorphism α of the free group $F_n = \langle t_1, \dots, t_n \rangle$ is the group

$$G(\alpha) = \langle t_1, \dots, t_n \mid t_1 = \alpha(t_1), \dots, t_n = \alpha(t_n) \rangle.$$

Notice that α induces the identity automorphism on $G(\alpha)$. In fact, $G(\alpha)$ is the maximal quotient of F_n with this property. Also, note that $G(\alpha)$ is independent of the choice of free generators for F_n: If $x_1, \dots, x_n$ is another such choice, then $\langle t_1, \dots, t_n \mid t_i = \alpha(t_i) \rangle \cong \langle x_1, \dots, x_n \mid x_i = \alpha(x_i) \rangle$. Finally, notice that the group of a free automorphism depends only on the conjugacy class of that automorphism: If $\beta \in \mathrm{Aut}(F_n)$, then

$$\begin{aligned} G(\beta^{-1} \circ \alpha \circ \beta) &= \langle t_i \mid t_i = \beta^{-1} \circ \alpha \circ \beta(t_i) \rangle \\ &= \langle t_i \mid \beta(t_i) = \alpha(\beta(t_i)) \rangle \\ &= \langle x_i \mid x_i = \alpha(x_i) \rangle \cong G(\alpha). \end{aligned}$$

See [25] for details.

Topologically, the group $G(\alpha)$ can be interpreted as follows. Recall that W_n denotes a wedge of n circles, and that $\alpha : W_n \to W_n$ also denotes a basepoint preserving homotopy equivalence that induces $\alpha : F_n \to F_n$ on fundamental groups. Let $Y(\alpha) = W_n \times_\alpha S^1$ be the mapping torus of α; its fundamental group is the semidirect product $F_n \rtimes_\alpha \mathbb{Z} = \langle t_1, \dots, t_n, x \mid x^{-1} t_i x = \alpha(t_i) \rangle$. Let $X(\alpha) = W_n \times_\alpha S^1 \bigcup_{* \times S^1} * \times D^2$. Then $\pi_1(X(\alpha)) = G(\alpha)$, and, in fact, $X(\alpha)$ is homotopy equivalent to the 2-complex associated to the above presentation of $G(\alpha)$.

3.2. **Alexander Invariants.** Let α be an IA-automorphism of F_n, and $G = G(\alpha)$ the associated group. Then $H_1(G) = \mathbb{Z}^n$, the free abelian group generated by $t_1, \dots, t_n$. Let $p : X' \to X$ be the corresponding (maximal abelian) cover of $X = X(\alpha)$. We call the Λ-modules $A(\alpha) = H_1(X', p^{-1}(*))$, resp. $B(\alpha) = H_1(X')$ the *Alexander module*, resp. *Alexander invariant* of $G(\alpha)$. We wish to find presentations for these modules.

First consider $Y = W_n \times_\alpha S^1$. The chain complex of its maximal abelian cover is obtained using the Fox calculus as in [6]:

$$C_\bullet(Y') : \qquad C_2(Y') \xrightarrow{\begin{pmatrix} \mathrm{id} - x \cdot \Theta(\alpha) & d_1 \end{pmatrix}} C_1(Y') \xrightarrow{\begin{pmatrix} d_1 \\ x - 1 \end{pmatrix}} C_0(Y') \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where the chain groups are the modules over $\overline{\Lambda} = \Lambda[x^{\pm 1}]$ given by $C_2(Y') = C_1 \otimes_\Lambda \overline{\Lambda}$, $C_1(Y') = (C_1 \oplus C_0) \otimes_\Lambda \overline{\Lambda}$, and $C_0(Y') = C_0 \otimes_\Lambda \overline{\Lambda}$. It follows that the chain complex of the maximal abelian cover of $X = X(\alpha)$ is

$$C_\bullet(X') : \quad C_1 \xrightarrow{\mathrm{id} - \Theta(\alpha)} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0. \tag{3.1}$$

Hence, $A(\alpha) = \mathrm{coker}(\mathrm{id} - \Theta(\alpha))$ and $B(\alpha) = \ker(d_1)/\operatorname{im}(\mathrm{id} - \Theta(\alpha))$. By homological algebra, there exists a chain map from the chain complex (3.1) to the free

Λ-resolution (2.3), extending the identity map of $\mathbb{Z}$:

$$\begin{array}{ccccccccccc}
 & & C_1 & \xrightarrow{\mathrm{id}-\Theta(\alpha)} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi(\alpha)} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \\
C_3 & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

A diagram chase shows that $B(\alpha) = \operatorname{coker}\left(\begin{smallmatrix}\Phi(\alpha)\\ d_3\end{smallmatrix}\right)$. To summarize, we have:

Proposition 3.3. *If $\alpha \in \mathrm{IA}(F_n)$, the Alexander module and the Alexander invariant of $G(\alpha)$ have presentations*

$$C_1 \xrightarrow{\mathrm{id}-\Theta(\alpha)} C_1 \to A(\alpha) \to 0,$$
$$C_1 \oplus C_3 \xrightarrow{(\,\Phi(\alpha)\quad d_3\,)^{\top}} C_2 \to B(\alpha) \to 0.$$

Remark 3.4. The map $\Phi(\alpha)$ is not unique, but rather, it is unique up to chain homotopy: Given two choices, $\Phi_1(\alpha)$ and $\Phi_2(\alpha)$, there is a homomorphism $D : C_1 \to C_3$ such that $\Phi_1(\alpha) - \Phi_2(\alpha) = d_3 \circ D$. We abbreviate this by saying that $\Phi_1(\alpha) = \Phi_2(\alpha) \mod d_3$. Of course, any two choices yield equivalent presentations for $B(\alpha)$.

As noted previously, the group $G(\beta \circ \alpha \circ \beta^{-1})$ is isomorphic to $G(\alpha)$. The relationship between the corresponding chain maps is as follows:

Proposition 3.5. *For $\alpha, \beta \in \mathrm{IA}(F_n)$, we have*

$$\Phi(\beta \circ \alpha \circ \beta^{-1}) = \Theta_2(\beta) \circ \Phi(\alpha) \circ \Theta(\beta^{-1}) \mod d_3.$$

Proof. By Remark 3.4, it is enough to show that

$$d_2 \circ \Theta_2(\beta) \circ \Phi(\alpha) \circ \Theta(\beta^{-1}) = \mathrm{id} - \Theta(\beta \circ \alpha \circ \beta^{-1}).$$

Since the right-hand side equals $\Theta(\beta) \circ (\mathrm{id} - \Theta(\alpha)) \circ \Theta(\beta^{-1})$, the claim follows from the equalities $d_2 \circ \Theta_2(\beta) = \Theta(\beta) \circ d_2$ and $d_2 \circ \Phi(\alpha) = \mathrm{id} - \Theta(\alpha)$. □

3.6. **Basis-Conjugating Automorphisms.** Let $F_n = \langle t_1, \dots, t_n \rangle$. An automorphism α of F_n is called a *basis-conjugating automorphism* if there exists an n-tuple $\mathbf{z} = (z_1, \dots, z_n)$, with $z_i \in F_n$, such that $\alpha = \gamma_{\mathbf{z}}$, where $\gamma_{\mathbf{z}}(t_i) = z_i t_i z_i^{-1}$. The basis-conjugating automorphisms of F_n form a subgroup, $\mathrm{CA}(F_n)$, of $\mathrm{Aut}(F_n)$. For $\alpha \in \mathrm{CA}(F_n)$, the following definition/proposition gives an explicit formula for $\Phi(\alpha)$.

Proposition 3.7. *For $\gamma_{\mathbf{z}} \in \mathrm{CA}(F_n)$, define the Λ-homomorphism $\Phi(\gamma_{\mathbf{z}}) : C_1 \to C_2$ by*

$$\Phi(\gamma_{\mathbf{z}})(e_i) = \nabla^{\mathrm{ab}}(z_i) \wedge e_i. \tag{3.2}$$

Then $\mathrm{id} - \Theta(\gamma_{\mathbf{z}}) = d_2 \circ \Phi(\gamma_{\mathbf{z}})$.

Proof. First, note that the Magnus representation of $\gamma_{\mathbf{z}}$ is given by:

$$\Theta(\gamma_{\mathbf{z}})(e_i) = (1 - t_i) \cdot \nabla^{\mathrm{ab}}(z_i) + z_i^{\mathrm{ab}} \cdot e_i.$$

Hence:

$$\begin{aligned}
(\mathrm{id} - \Theta(\gamma_{\mathbf{z}}))(e_i) &= (t_i - 1) \cdot \nabla^{\mathrm{ab}}(z_i) + (1 - z_i^{\mathrm{ab}}) \cdot e_i \\
&= d_1(e_i) \cdot \nabla^{\mathrm{ab}}(z_i) - d_1(\nabla^{\mathrm{ab}}(z_i)) \cdot e_i \\
&= d_2(\Phi(\gamma_{\mathbf{z}})(e_i)).
\end{aligned}$$

□

Remark 3.8. As mentioned before, an explicit formula for the Alexander invariant $B(L)$ of an arbitrary link $L \subset S^3$ is lacking. If L is a pure link of n components, though, Propositions 3.3 and 3.7 provide a presentation for $B(L)$, with $\binom{n}{2}$ generators and $\binom{n}{3} + n$ relations. Indeed, as shown by Artin, the braid group B_n admits a faithful representation $B_n \hookrightarrow \mathrm{Aut}(F_n)$, which restricts to $P_n \hookrightarrow \mathrm{CA}(F_n)$. Moreover, any link L is the closure, $\hat{\alpha}$, of a braid $\alpha \in B_n$, and $\pi_1(S^3 \setminus L) = G(\alpha)$. Now assume L is a pure link, i.e, $L = \hat{\alpha}$, for some $\alpha \in P_n$. Then $\alpha = \gamma_{\mathbf{z}}$, where z_i is the longitude corresponding to the meridian t_i, and we get $B(L) = \mathrm{coker}\left(\begin{smallmatrix} \Phi(\gamma_{\mathbf{z}}) \\ d_3 \end{smallmatrix}\right)$, with $\Phi(\gamma_{\mathbf{z}})$ given by (3.2).

3.9. **Alexander Invariant of Several Automorphisms.** The above notions generalize in a straightforward manner, from a single automorphism α to several automorphisms $\alpha_1, \dots, \alpha_s$ of F_n. Namely, let

$$G(\alpha_1, \dots, \alpha_s) = \langle t_1, \dots, t_n \mid t_i = \alpha_k(t_i),\ 1 \le k \le s \rangle$$

be the maximal quotient of F_n on which all α_k act trivially. This group can also be characterized as the quotient of the semidirect product $F_n \rtimes F_s = \langle t_1, \dots, t_n, x_1, \dots, x_s \mid x_k^{-1} t_i x_k = \alpha_k(t_i) \rangle$ by the normal closure of $F_s = \langle x_1, \dots, x_s \rangle$.

Assume $\alpha_k \in \mathrm{IA}(F_n)$, for $1 \le k \le s$. Let Y be the presentation 2-complex for $F_n \rtimes F_s$, and X that of $G = G(\alpha_1, \dots, \alpha_s)$. The chain complex of the maximal abelian cover of Y has the following form:

$$C_2(Y') \xrightarrow{\begin{pmatrix} \mathrm{id} - x_1 \cdot \Theta(\alpha_1) & d_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \mathrm{id} - x_s \cdot \Theta(\alpha_s) & 0 & \cdots & d_1 \end{pmatrix}} C_1(Y') \xrightarrow{\begin{pmatrix} d_1 \\ \overline{d}_1 \end{pmatrix}} C_0(Y') \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where the chain groups are the modules over $\overline{\Lambda} = \Lambda[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$ given by $C_2(Y') = \oplus_1^s C_1 \otimes_\Lambda \overline{\Lambda}$, $C_1(Y') = (C_1 \oplus \oplus_1^s C_0) \otimes_\Lambda \overline{\Lambda}$, and $C_0(Y') = C_0 \otimes_\Lambda \overline{\Lambda}$, and where $\overline{d}_1 = (x_1 - 1 \ \cdots \ x_s - 1)^\top$. The chain complex of the maximal abelian cover of X is then

$$C_1^s \xrightarrow{\begin{pmatrix} \mathrm{id} - \Theta(\alpha_1) \\ \cdots \\ \mathrm{id} - \Theta(\alpha_s) \end{pmatrix}} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

This chain complex provides a presentation matrix—the so-called *Alexander matrix*, $(\mathrm{id} - \Theta(\alpha_1) \ \cdots \ \mathrm{id} - \Theta(\alpha_s))^\top$—for the Alexander module $A = A(\alpha_1, \dots, \alpha_s)$. Furthermore, if $\Phi(\alpha_k) : C_1 \to C_2$ satisfy $d_2 \circ \Phi(\alpha_k) = \mathrm{id} - \Theta(\alpha_k)$, then the Alexander invariant $B = B(\alpha_1, \dots, \alpha_s)$ has presentation matrix $(\Phi(\alpha_1) \ \cdots \ \Phi(\alpha_s) \ d_3)^\top$. When $\alpha_k \in \mathrm{CA}(F_n)$, we obtain an explicit presentation for B.

Theorem 3.10. *Let $\gamma_{\mathbf{z}^1}, \dots, \gamma_{\mathbf{z}^s}$ be a collection of basis-conjugating automorphisms of F_n. Let $\Phi(\gamma_{\mathbf{z}^k}) : C_1 \to C_2$ be the homomorphisms defined by* (3.2)*. Then the Alexander invariant of $G(\gamma_{\mathbf{z}^1}, \dots, \gamma_{\mathbf{z}^s})$ has presentation*

$$C_1^s \oplus C_3 \xrightarrow{(\Phi(\gamma_{\mathbf{z}^1}) \ \cdots \ \Phi(\gamma_{\mathbf{z}^s}) \ d_3)^\top} C_2 \to B \to 0.$$

4. Local Alexander Invariants

We now find presentations for the Alexander invariant of the group of a full-twist braid automorphism, and that of a related "vertex" group. These presentations are given in terms of the Gassner representation, $\Theta : P_n \to \mathrm{GL}(n,\Lambda)$, which is simply the restriction of the Magnus representation, $\Theta : \mathrm{IA}(F_n) \to \mathrm{GL}(n,\Lambda)$, to the pure braid group P_n.

4.1. **Alexander Invariant of a Twist Automorphism.** Let $V = \{i_1, \dots, i_r\}$ be an increasingly ordered subset of $[n] = \{1, \dots, n\}$. Let A_V the pure braid in P_n which performs a full twist on the strands corresponding to V, leaving the other strands fixed. Let σ_i $(1 \le i < n)$ be the standard generators of B_n, and $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ $(1 \le i < j \le n)$ the standard generators of P_n, see [3]. The twist on V is given by

$$A_V = (A_{i_1,i_2})(A_{i_1,i_3}A_{i_2,i_3})(A_{i_1,i_4}A_{i_2,i_4}A_{i_3,i_4}) \cdots (A_{i_1,i_r} \cdots A_{i_{r-1},i_r}). \tag{4.1}$$

A computation with the Artin representation reveals that $A_V = \gamma_{\mathbf{w}}$, where $\mathbf{w} = (w_1, \dots, w_n)$ is defined as follows:

$$w_i = \begin{cases} t_V & \text{if } i \in V, \\ [t_{V^i}, t_{{}^iV}] & \text{if } i \in \overline{V} \setminus V, \\ 1 & \text{otherwise,} \end{cases} \tag{4.2}$$

where $\overline{V} = \{i \in [n] \mid i_1 \le i \le i_r\}$, $V^i = \{j \in V \mid j < i\}$, ${}^iV = \{j \in V \mid i < j\}$, and $t_V = \prod_{j \in V} t_j = t_{i_1} \cdots t_{i_r}$.

Let $G(A_V) = \langle t_1, \dots, t_n \mid A_V(t_i) = t_i \rangle$ be the group associated to $A_V \in \mathrm{Aut}(F_n)$. A computation with (4.2) shows that

$$G(A_V) = \langle t_1, \dots, t_n \mid t_V t_i t_V^{-1} = t_i \quad , i \in V \rangle,$$

and so $G(A_V) \cong (F_{r-1} \times F_1) * F_{n-r}$. By Proposition 3.3, the Alexander invariant of $G(A_V)$ has presentation

$$C_1 \oplus C_3 \xrightarrow{(\,\Phi(A_V) \quad d_3\,)^{\top}} C_2 \to B(A_V) \to 0. \tag{4.3}$$

By Proposition 3.7, and a Fox calculus computation, the map $\Phi_V := \Phi(A_V) : C_1 \to C_2$ is given by

$$\Phi_V(e_i) = \begin{cases} \nabla_V \wedge e_i & \text{if } i \in V \\ (1 - t_i)\nabla_V \wedge \nabla_{{}^iV} & \text{if } i \in \overline{V} \setminus V \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

where $\nabla_V := \nabla^{\mathrm{ab}}(t_V) = \sum_{i \in V} t_{V^i} e_i$.

4.2. **Simplified Presentation for $B(A_V)$.** Set $C_k(V) = \mathrm{span}\{e_J \mid J \subset V\}$, and let $\iota_V : C_k(V) \to C_k$ be the inclusion, and $\pi_V : C_k \to C_k(V)$ the natural projection. Write $V' := V \setminus \{\min V\} = \{i_2, \dots, i_r\}$. From (4.4) it is apparent that $\Phi_V(C_1) = \Phi_V(C_1(V)) \subset C_2(V)$. Since $0 = \nabla_V \wedge \nabla_V = \nabla_V \wedge e_{i_1} + \nabla_V \wedge \sum_{i \in V'} t_{V^i} e_i$, we see that $\Phi_V(e_{i_1}) \in \Phi_V(C_1(V'))$. Thus,

$$\Phi_V(C_1) = \Phi_V(C_1(V')) \subset C_2(V). \tag{4.5}$$

Define an automorphism $\mu_V : F_n \to F_n$ by:

$$\mu_V(t_i) = \begin{cases} t_V & \text{if } i = \min V, \\ t_i & \text{otherwise.} \end{cases}$$

Note that $\Theta(\mu_V)(e_{i_1}) = \nabla_V$, and $\Theta_2(\mu_V)(e_{i_1} \wedge e_i) = \Phi_V(e_i)$, for $i \in V'$. Thus,

$$\Theta_2(\mu_V)^{-1} \circ \Phi_V(e_i) = e_{i_1} \wedge e_i \quad \text{for } i \in V'. \tag{4.6}$$

Let $C_2'(V)$ be the direct summand of C_2 spanned by $\{e_{i_1} \wedge e_i \mid i \in V'\}$, let $C_2^{\perp}(V)$ be the complementary summand, and let $\pi^V : C_2 \to C_2^{\perp}(V)$ be the canonical projection. Putting together (4.3), (4.5), and (4.6), we obtain:

Proposition 4.3. *The Alexander invariant of $G(A_V)$ has presentation*

$$C_3 \xrightarrow{\Delta(V)} C_2^{\perp}(V) \to B(A_V) \to 0, \tag{4.7}$$

where $\Delta(V) = \pi^V \circ \Theta_2(\mu_V)^{-1} \circ d_3$.

4.4. **Alexander Invariant of a Vertex Group.** To the twist automorphism A_V, we also associate a "vertex group," $G_V := G(\{A_V, A_{i,j} \mid |\{i,j\} \cap V| \leq 1\})$. Using (4.2), we obtain the following presentation:

$$G_V = \langle t_1, \dots, t_n \mid t_V t_i t_V^{-1} = t_i \text{ if } i \in V, \;\; t_j t_i t_j^{-1} = t_i \text{ if } \{i,j\} \not\subset V \rangle. \tag{4.8}$$

Remark 4.5. The group G_V has the following geometric interpretation. Let $\mathcal{A}_V = \{H_1, \dots, H_n\}$ be an arrangement of n distinct lines in $\mathbb{C}^2$ such that H_i passes through 0 for $i \in V$, and H_j is in general position with H_k, for $j \notin V$, and $k \neq j$. Then G_V is isomorphic to the group of the arrangement $\mathcal{A}_V$: $G_V \cong \pi_1\left(\mathbb{C}^2 \setminus \bigcup_{i=1}^n H_i\right)$.

A (minimal) presentation for the Alexander invariant $B_V = B(G_V)$ may be obtained from (4.7) by restricting the map $\Delta(V)$ to a map $C_2(V') \wedge C_1 \to C_2(V')$, and some further matrix operations. Alternatively, it may be obtained by applying Theorem 1.9 to the direct product decomposition $G_V \cong F_{r-1} \times \mathbb{Z}^{n-r+1}$, apparent from (4.8). The result is as follows:

Proposition 4.6. *The Alexander invariant of G_V has presentation*

$$C_2(V') \wedge C_1 \xrightarrow{\Delta_V} C_2(V') \to B_V \to 0,$$

where $\Delta_V = \pi_{V'} \circ \tilde{\mu}_V \circ d_3 \circ \tilde{\mu}_V^{-1} \circ (\iota_{V'} \wedge \mathrm{id})$.

(This will be useful only when $|V| \geq 3$; if $|V| = 2$, then $G_V = \mathbb{Z}^n$, $B_V = 0$, and $C_2(V') = 0$.)

The above presentation may be extended to a free resolution,

$$\cdots \to C_2(V') \wedge C_2 \xrightarrow{\Delta_V^2} C_2(V') \wedge C_1 \xrightarrow{\Delta_V} C_2(V') \to B_V \to 0, \tag{4.9}$$

with boundary maps $\Delta_V^\bullet$ given by

$$\Delta_V^k = (\pi_{V'} \wedge \mathrm{id}) \circ \tilde{\mu}_V \circ d_{k+2} \circ \tilde{\mu}_V^{-1} \circ (\iota_{V'} \wedge \mathrm{id}) : C_2(V') \wedge C_k \to C_2(V') \wedge C_{k-1}.$$

Furthermore, by the discussion following (2.5), there exists a naturally defined chain map $\Psi_{V,\bullet} : (C_\bullet, d_\bullet) \to (C_2(V') \wedge C_{\bullet-2}, \Delta_V^{\bullet-2})$, given by

$$\Psi_{V,k} = (\pi_{V'} \wedge \mathrm{id}) \circ \Theta_k(\mu_V)^{-1} : C_k \to C_2(V') \wedge C_{k-2}, \quad \text{for } k \geq 2. \tag{4.10}$$

5. The Alexander Invariant of an Arrangement

In this section, we use the results of the previous sections to obtain a presentation for the Alexander invariant of the group of a hyperplane arrangement.

5.1. **Braid Monodromy.** The fundamental group of the complement of an arrangement of complex hyperplanes is, by a Lefschetz-type theorem of Zariski ([19], [18]), isomorphic to that of a generic two-dimensional section. So, for the purpose of computing the Alexander invariant, it is enough to consider affine line arrangements in $\mathbb{C}^2$. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be such an arrangement, with vertices $\mathcal{V} = \{v_1, \dots, v_s\}$. If $v_k = H_{i_1} \cap \cdots \cap H_{i_r}$, let $V_k = \{i_1, \dots, i_r\}$ denote the corresponding "vertex set." We identify the set $L_2(\mathcal{A})$ of rank two elements in the lattice of $\mathcal{A}$ and the collection $\{V_1, \dots, V_s\}$ of vertex sets of $\mathcal{A}$.

The braid monodromy of $\mathcal{A}$ is determined as follows (see [7] for details). Choose coordinates (x, z) in $\mathbb{C}^2$ so that the projection $\mathrm{pr}_1 : \mathbb{C}^2 \to \mathbb{C}$ is generic with respect to $\mathcal{A}$. Let $f(x, z) = \prod_{i=1}^{n} (z - a_i(x))$ be a defining polynomial for $\mathcal{A}$. The root map $a = (a_1, \dots, a_n) : \mathbb{C} \to \mathbb{C}^n$ restricts to a map from the complement of $\mathcal{Y} = \mathrm{pr}_1(\mathcal{V})$ to the complement of the braid arrangement $\mathcal{A}_n = \{\ker(y_i - y_j)\}_{1 \le i < j \le n}$. Identify $\pi_1(\mathbb{C} \backslash \mathcal{Y})$ with the free group $F_s = \langle x_1, \dots, x_s \rangle$, and $\pi_1(\mathbb{C}^n \backslash \bigcup_{H \in \mathcal{A}_n} H)$ with the pure braid group P_n. Then, the *braid monodromy* of $\mathcal{A}$ is the induced homomorphism on fundamental groups, $\alpha : F_s \to P_n$.

The braid monodromy generators $\alpha_k = \alpha(x_k)$ can be written explicitly using a *braided wiring diagram* $\mathcal{W}$ associated to $\mathcal{A}$. Such a diagram, determined by the choices made above, may be (abstractly) specified by a sequence of vertex sets and braids, $\mathcal{W} = \mathcal{W}_s = \{V_1, \beta_1, V_2, \beta_2, \dots, \beta_{s-1}, V_s\}$. The braid monodromy generators are given by $\alpha_k = A_{V_k}^{\delta_k}$, where A_{V_k} is the twist braid defined in (4.1) and δ_k is a pure braid determined by the subdiagram $\mathcal{W}_k$.

5.2. **The Presentation for $B(\mathcal{A})$.** Let $M = M(\mathcal{A})$ be the complement of $\mathcal{A}$. Let $G = G(\alpha_1, \dots, \alpha_s)$ be the fundamental group of M, with Alexander invariant $B = B(\mathcal{A})$. Theorem 3.10 provides the following presentation for B:

$$C_1^s \oplus C_3 \xrightarrow{\begin{pmatrix} \Phi_{V_1}^{\delta_1} & \cdots & \Phi_{V_s}^{\delta_s} & d_3 \end{pmatrix}^{\top}} C_2 \to B \to 0,$$

where $\Phi_V^{\delta} := \Phi(A_V^{\delta}) = \Theta_2(\delta) \circ \Phi_V \circ \Theta(\delta^{-1}) : C_1 \to C_2$, and Φ_V is given by (4.4).

This presentation can be simplified, based on the following elementary observation: If R is a ring, and B is an R-module, with presentation $R^p \xrightarrow{\Delta} R^q \to B \to 0$, where $\Delta = \Upsilon \circ \Xi$, or $\Delta = \Xi \circ \Upsilon$, with Ξ invertible, then B can also be presented as $R^p \xrightarrow{\Upsilon} R^q \to B \to 0$. Since the maps $\Theta(\delta_k^{-1})$ are invertible we may replace $\Phi_{V_k}^{\delta_k}$ by $\Theta_2(\delta_k) \circ \Phi_{V_k}$. Furthermore, by (4.5), we may subsequently restrict each of the maps Φ_{V_k} to $\Phi_k : C_1(V_k') \to C_2$. Thus, we obtain the following:

Theorem 5.3. *The Alexander invariant of an arrangement $\mathcal{A}$, with braid monodromy generators $A_{V_1}^{\delta_1}, \dots, A_{V_s}^{\delta_s}$, has presentation*

$$K_1 \xrightarrow{\Delta} K_0 \to B(\mathcal{A}) \to 0,$$

where $K_1 = \bigoplus_{k=1}^{s} C_1(V_k') \oplus C_3$, $K_0 = C_2$, *and* $\Delta = \begin{pmatrix} \Phi & d_3 \end{pmatrix}^{\top}$, *with* $\Phi|_{C_1(V_k')} = \Theta_2(\delta_k) \circ \Phi_k$.

Note that this presentation has $\binom{n}{2}$ generators and $\sum_{k=1}^{s}(|V_k|-1)+\binom{n}{3}$ relations, and that $\sum_{k=1}^{s}(|V_k|-1) = b_2(M)$.

5.4. **Real Arrangements.** The presentation can be simplified in the case where $\mathcal{A}$ is the complexification of a line arrangement $\mathcal{A}_{\mathbb{R}}$ in $\mathbb{R}^2$. In this instance, the wiring diagram $\mathcal{W}$ can be chosen so that it contains no intermediary braids, and each "conjugating braid," δ_k, is a subword of the full twist, $A_{[n]}$, on n strands. Let U_k denote the set of indices of wires of $\mathcal{W}$ which lie above the vertex v_k in $\mathrm{pr}_1^{-1}(y_k)$, and let $J_k = (\overline{V}_k \setminus V_k) \cap U_k$. Then the conjugating braids may be written as $\delta_k = \prod_{j<i} A_{j,i}$, where the product is over all $i \in V_k$ and $j \in J_k$, see [8], [7].

Define a homomorphism $\mathbf{\Theta}_2(\mu) : C_2 \to C_2$ by

$$\mathbf{\Theta}_2(\mu)(e_i \wedge e_j) = \begin{cases} \Theta_2(\mu_{V_k})(e_i \wedge e_j) & \text{if } \{i,j\} \subset V_k \\ e_i \wedge e_j & \text{otherwise.} \end{cases}$$

It is readily seen that $\mathbf{\Theta}_2(\mu)$ is invertible. Similarly, define $\mathbf{\Theta}_2(\delta) : C_2 \to C_2$. A computation shows that $\delta_k(t_i) = t_{J_k^i} \cdot t_i \cdot t_{J_k^i}^{-1}$, and that $\Theta(\delta_k)(e_i) = (1-t_i)\cdot \nabla_{J_k^i} + t_{J_k^i} \cdot e_i$, for $i \in V_k$. Thus, $\mathbf{\Theta}_2(\delta)$ is also invertible. Proceeding as in 4.2, we obtain the following.

Theorem 5.5. *The Alexander invariant of a complexified real arrangement $\mathcal{A}$ has presentation*

$$C_3 \xrightarrow{\Delta'} L_0 \to B(\mathcal{A}) \to 0,$$

where L_0 is the complementary summand to $K_0' = \oplus_V C_2'(V)$ in C_2, $\pi_0 : C_2 \to L_0$ is the canonical projection, and $\Delta' = \pi_0 \circ \mathbf{\Theta}_2(\mu)^{-1} \circ \mathbf{\Theta}_2(\delta)^{-1} \circ d_3$.

Note that this presentation has only $\binom{n}{2} - b_2(M)$ generators, and $\binom{n}{3}$ relations.

Remark 5.6. For an arbitrary complex arrangement, the map $\mathbf{\Theta}_2(\delta)$ need not be invertible. Thus the simplification of the presentation of the Alexander invariant afforded by the above result may not be available. However, for any arrangement, we obtain an analogous simplified presentation for the I-adic completion, $\widehat{B}(\mathcal{A})$, of the Alexander invariant of $\mathcal{A}$ in Corollary 6.6.

6. Decomposition of the Alexander Invariant

We now relate the Alexander invariant of an arrangement $\mathcal{A}$ to a "combinatorial" Alexander invariant, determined by the intersection lattice of $\mathcal{A}$. For these purposes, we restrict our attention to central arrangements and their generic sections. It is enough to consider an affine arrangement, $\mathcal{A} = \{H_1, \dots, H_n\}$, of n lines in $\mathbb{C}^2$ that is transverse to infinity (that is, no two lines of $\mathcal{A}$ are parallel). Recall that we identify the set of rank two elements in the lattice of $\mathcal{A}$ and the collection of vertex sets of $\mathcal{A}$: $L_2(\mathcal{A}) = \{V_1, \dots, V_s\}$.

6.1. **The Coarse Combinatorial Alexander Invariant.** For each $V \in L_2(\mathcal{A})$, let $\mathcal{A}_V = \{H_{V,1}, \dots, H_{V,n}\}$ denote the arrangement obtained from $\mathcal{A}$ by perturbing the lines so that all lines except those passing through the corresponding vertex v are in general position. Let $\mathcal{A}^{\mathrm{cc}} = \prod_V \mathcal{A}_V$ be the product of these arrangements (see [27]). Define the *coarse combinatorial Alexander invariant* of $\mathcal{A}$ to be the module $B^{\mathrm{cc}}(\mathcal{A}) = B(\mathcal{A}^{\mathrm{cc}}) \otimes_{\mathbb{Z}\mathbb{Z}^{ns}} \mathbb{Z}\mathbb{Z}^n$ induced from the Alexander invariant of $\mathcal{A}^{\mathrm{cc}}$ by the projection $t_{V,j} \mapsto t_j$.

By Remark 4.5, the group of $\mathcal{A}_V$ is isomorphic to the vertex group G_V. Let B_V be the Alexander invariant of G_V, as determined in 4.4. By Theorem 1.9, the coarse combinatorial Alexander invariant of $\mathcal{A}$ decomposes as a direct sum of Alexander invariants of vertex groups:

$$B^{\mathrm{cc}}(\mathcal{A}) = \bigoplus_{V \in L_2(\mathcal{A})} B_V.$$

Our goal in this subsection is to define an epimorphism $\Pi : B \to B^{\mathrm{cc}}$ from the Alexander invariant of the arrangement $\mathcal{A}$ to its "coarse combinatorial approximation."

Notice that the module B_V depends only on the cardinality $|V|$ of the vertex set V. Consequently, the module B^{cc} depends only on the number and multiplicities of the elements of $L_2(\mathcal{A})$. This Λ-module admits a free resolution

$$\cdots \to L_2 \xrightarrow{D_2} L_1 \xrightarrow{D_1} L_0 \to B^{\mathrm{cc}} \to 0$$

obtained by taking the direct sum of the resolutions (4.9): $L_k = \oplus_V C_2(V') \wedge C_k$, $D_k = \oplus_V \Delta^k_V$. (Since $\mathcal{A}$ is assumed to be transverse to infinity, $L_0 = \oplus_V C_2(V')$ is indeed the complementary summand of $K'_0 = \oplus_V C'_2(V)$ in $K_0 = C_2 = \oplus_V C_2(V)$.)

Let $\Psi_{V,\bullet} : C_\bullet \to C_2(V') \wedge C_{\bullet-2}$ be the chain map introduced in (4.10). Define a chain map $\Psi_\bullet : C_\bullet \to L_{\bullet-2}$ by $\Psi_k = \sum_V \Psi_{V,k}$, for $k \geq 2$.

Proposition 6.2. *The image of the composition $\Psi_2 \circ \Phi : \bigoplus_V C_1(V') \to L_0$ is contained in the image of the map $D_1 : L_1 \to L_0$. Therefore, there exists a map $\Gamma : \bigoplus_V C_1(V') \to L_1$ such that $D_1 \circ \Gamma = \Psi_2 \circ \Phi$.*

Proof. Let A^δ_V be a braid monodromy generator of $\mathcal{A}$, where $V = \{i_1, \dots, i_r\}$ and δ is some pure braid. Using the pure braid relations to rewrite δ if necessary, we may assume that this pure braid is a word in the generators $\{A_{r,s} \mid \{r,s\} \not\subset V\}$.

For $j \in V'$, we have $\Psi_2 \circ \Phi(e_j) = \Psi_2 \circ \Theta_2(\delta)(\nabla_V \wedge e_j)$. Since $\delta \in \mathrm{IA}(F_n)$, we have $\mathrm{im}(\Theta_k(\delta) - \mathrm{id}) \subset I \cdot C_k$. Hence, $\Theta_2(\delta)(e_i \wedge e_j) = e_i \wedge e_j + W^\delta_{i,j}$, where $W^\delta_{i,j} = \sum w_{p,q} e_p \wedge e_q$, with $w_{p,q} \in I$. Thus $\Psi_2 \circ \Phi(e_j) = \Psi_2(\nabla_V \wedge e_j) + \sum_{i \in V} t_{V^i} \cdot \Psi_2(W^\delta_{i,j})$. Since $\Psi_2(\nabla_V \wedge e_j) = \pi_{V'} \circ \Theta_2(\mu_V)^{-1}(\Theta(\mu_V)(e_{i_1} \wedge e_j)) = \pi_{V'}(e_{i_1} \wedge e_j) = 0$, it suffices to show that

$$\Psi_2(W^\delta_{i,j}) \in \mathrm{im}(D_1). \tag{6.1}$$

For a vertex set $U \in L_2(\mathcal{A})$, recall the natural projection $\pi_U : C_k \to C_k(U)$, and denote by $I^\perp_U$ the ideal in Λ generated by $\langle 1 - t_k \mid k \notin U \rangle$.

Claim. For each vertex set $U \in L_2(\mathcal{A})$, we have $\pi_U(W^\delta_{i,j}) \in I^\perp_U \cdot C_2(U)$.

Before proving this claim, let us show that it implies (6.1). For $1 \leq p < q \leq n$, let $V(p,q)$ denote the unique vertex set of $\mathcal{A}$ with $p, q \in V(p,q)$. If $w \cdot e_p \wedge e_q$ is a summand of $W^\delta_{i,j}$, write $U = V(p,q)$. Then, by the claim, we have $w \in I^\perp_U$. Now $\Psi_2(e_p \wedge e_q) = \Psi_{U,2}(e_p \wedge e_q) \in C_2(U')$, and it is readily checked that $I^\perp_U \cdot C_2(U') \subset \mathrm{im}(\Delta_U)$. It follows that $\Psi_2(w \cdot e_p \wedge e_q) \in \mathrm{im}(\Delta_U)$.

Thus it suffices to prove the claim. This may be accomplished by induction on the length of the word δ. If $\delta = 1$, then $W^\delta_{i,j} = 0$, and there is nothing to prove. If $\delta = A^{\pm 1}_{r,s}$, a computation shows that $\Theta_2(A_{r,s})(e_i \wedge e_j) = e_i \wedge e_j + W^{r,s}_{i,j}$ and

$\Theta_2(A_{r,s}^{-1})(e_i \wedge e_j) = e_i \wedge e_j - (t_r t_s)^{-1} W_{i,j}^{r,s}$, where

$$W_{i,j}^{r,s} = \begin{cases} t_r(t_s-1)e_i \wedge e_r + t_r(1-t_r)e_i \wedge e_s & \text{if } i<j=r<s,\\ (1-t_j)[(1-t_s)e_i \wedge e_r + (t_r-1)e_i \wedge e_s] & \text{if } i<r<j<s,\\ (t_r-1)e_i \wedge e_s + (1-t_s)e_i \wedge e_r & \text{if } i<r<j=s,\\ (t_r t_s-1)e_r \wedge e_s & \text{if } i=r<j=s,\\ (t_r-1)e_i \wedge e_s + (t_s-1)[e_r \wedge e_i + (t_i-1)e_r \wedge e_s] & \text{if } r<i<j=s,\\ (1-t_i)[(1-t_s)e_r \wedge e_j + (t_r-1)e_s \wedge e_j] & \text{if } r<i<s<j,\\ t_r(t_s-1)e_r \wedge e_j + t_r(1-t_r)e_s \wedge e_j & \text{if } r=i<s<j,\\ (t_r-1)e_s \wedge e_j + (1-t_s)e_r \wedge e_j & \text{if } r<i=s<j,\\ t_r(t_s-1)e_r \wedge e_j + (t_r-1)[t_r e_j \wedge e_s + (1-t_j)e_r \wedge e_s] & \text{if } r=i<j<s,\\ (t_j-1)[(1-t_s)e_r \wedge e_i + (1-t_r)e_i \wedge e_s] & \text{if } r<i<j<s,\\ \quad +(1-t_i)[(1-t_s)e_r \wedge e_j + (1-t_r)e_j \wedge e_s] & \\ 0 & \text{otherwise.} \end{cases}$$

If $\{r,s\} \not\subset V$, it is readily checked that $\pi_U(W_{i,j}^{\delta}) \in I_U^{\perp} \cdot C_2(U)$ for each $U \in L_2(\mathcal{A})$.

In general, write δ as the product of $A_{r,s}^{\pm 1}$ and δ', and assume inductively that $\Theta_2(\delta')(e_i \wedge e_j) = e_i \wedge e_j + W'_{i,j}$ satisfies $\pi_U(W'_{i,j}) \in I_U^{\perp} \cdot C_2(U)$ for each U. Then $\Theta_2(\delta)(e_i \wedge e_j) = \Theta_2(A_{r,s}^{\pm 1})(e_i \wedge e_j) + \Theta_2(A_{r,s}^{\pm 1})(W'_{i,j})$, and by the above, it remains to analyze the latter summand. If $w \cdot e_p \wedge e_q$ is a summand of $W'_{i,j}$, then $w \in I_{V(p,q)}^{\perp}$ by induction. Case-by-case analysis then shows that each summand $x \cdot e_\ell \wedge e_m$ of $\Theta_2(A_{r,s}^{\pm 1})(w \cdot e_p \wedge e_q)$ satisfies $x \in I_{V(\ell,m)}^{\perp}$. This completes the proof of the claim, and hence that of the proposition. □

We can now formulate the main result of this subsection.

Theorem 6.3. *There exists a chain map $\Upsilon_\bullet$ from the presentation $K_\bullet \to B(\mathcal{A})$ to the resolution $L_\bullet \to B^{\mathrm{cc}}(\mathcal{A})$,*

$$\begin{array}{ccccccccc} & & & & K_1 & \xrightarrow{\Delta} & K_0 & \longrightarrow & B & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \Upsilon_1} & & \downarrow{\scriptstyle \Upsilon_0} & & \downarrow{\scriptstyle \Pi} & & \\ \cdots & \longrightarrow & L_2 & \xrightarrow{D_2} & L_1 & \xrightarrow{D_1} & L_0 & \longrightarrow & B^{\mathrm{cc}} & \longrightarrow & 0, \end{array}$$

given by $\Upsilon_0 = \Psi_2$, and $\Upsilon_1(x,y) = \Gamma(x) + \Psi_3(y)$. Furthermore, the resulting map $\Pi : B \to B^{\mathrm{cc}}$ is surjective.

Proof. It is immediate from the above that $\Upsilon_\bullet$ is a chain map. Thus it suffices to show that the map $\Upsilon_0 : K_0 \to L_0$, which by definition equals $\Psi_2 = \sum_V \pi_{V'} \circ \Theta_2(\mu_V)^{-1} : C_2 \to \oplus_V C_2(V')$, is surjective. With respect to the decomposition $K_0 = K_0' \oplus L_0 = (\oplus_V C_2'(V)) \oplus (\oplus_V C_2(V'))$, we have $\Upsilon_0 = \begin{pmatrix} \Upsilon_0' & \mathrm{id}_{L_0} \end{pmatrix}^{\top} : K_0' \oplus L_0 \to L_0$, where $\Upsilon_0'(e_i \wedge x) = (e_i - \nabla_V) \wedge x$, for $i = \min V$ and $x \in C_1(V')$. Thus $\Upsilon_0 : K_0 \to L_0$ is surjective. □

6.4. **Decomposition of the Completion.** Recall that if B is a Λ-module, then $\widehat{B}$ denotes its I-adic completion, and that if $f : A \to B$ is a map of Λ-modules, we write $\widehat{f} : \widehat{A} \to \widehat{B}$ for the extension of f to the completions. The I-adic completion functor takes chain complexes to chain complexes, and chain maps to chain maps.

Theorem 6.5. *The chain map $\widehat{\Upsilon}_\bullet : \widehat{K}_\bullet \to \widehat{L}_\bullet$ induces an isomorphism $\widehat{B} \xrightarrow{\sim} \widehat{B}^{\mathrm{cc}}$ if and only if the map $\widehat{\Psi}_3 : \widehat{C}_3 \to \widehat{L}_1$ is surjective.*

Proof. Consider the mapping cone, $K_\bullet(\widehat{\Upsilon})$, of $\widehat{\Upsilon}_\bullet$, given by

$$\cdots \to \widehat{L}_2 \oplus \widehat{K}_1 \xrightarrow{\partial_2} \widehat{L}_1 \oplus \widehat{K}_0 \xrightarrow{\partial_1} \widehat{L}_0, \tag{6.2}$$

where $\partial_2(x,y) = (\widehat{D}_2(x) - \widehat{\Upsilon}_1(y), \widehat{\Delta}(y))$ and $\partial_1(x,y) = \widehat{D}_1(x) + \widehat{\Upsilon}_0(y)$, and the short exact sequence of chain complexes

$$0 \to \widehat{L}_\bullet \xrightarrow{\iota_\bullet} K_\bullet(\widehat{\Upsilon}) \xrightarrow{\pi_\bullet} \widehat{K}_{\bullet-1} \to 0, \tag{6.3}$$

where $\iota_\bullet$ and $\pi_\bullet$ denote the natural inclusion and projection. Since $\widehat{\Upsilon}_0$ is surjective, we have $H_0(K_\bullet(\widehat{\Upsilon})) = 0$. Also, since $\widehat{L}_\bullet$ is a resolution, $\widetilde{H}_*(\widehat{L}_\bullet) = 0$. Thus the associated long exact sequence in homology reduces to

$$0 \to H_1(K_\bullet(\widehat{\Upsilon})) \to H_0(\widehat{K}_\bullet) \xrightarrow{\Upsilon_*} H_0(\widehat{L}_\bullet) \to 0. \tag{6.4}$$

The map $\widehat{\Upsilon}_* : H_0(\widehat{K}_\bullet) = H_0(\widehat{L}_\bullet)$ identifies canonically with $\widehat{\Pi} : \widehat{B} \to \widehat{B}^{\mathrm{cc}}$. Thus, it suffices to show that $\operatorname{coker}(\widehat{\Psi}_3) = 0$ if and only if $H_1(K_\bullet(\widehat{\Upsilon})) = 0$.

Recall the map $\Phi : \oplus_V C_1(V') \to C_2$ from Theorem 5.3. Recall also (from the proof of Proposition 6.2) that $\Phi|_{C_1(V')} = \Phi_V \mod I$. Thus, $\widehat{\Phi}|_{\widehat{C}_1(V')} = \widehat{\Phi}_V \bmod \mathfrak{m}$. Using the identification $C_1(V') \xrightarrow{\sim} C_2'(V)$, $x \mapsto e_i \wedge x$, where $i = \min V$, and the projection on the first factor $p' : K_0 = K_0' \oplus L_0 \to K_0'$, define $\Phi' := p' \circ \Phi : K_0' \to K_0'$. Since $\widehat{\nabla}_V = \sum_{i \in V} e_i \mod \mathfrak{m}$, the map $\widehat{p}' \circ \widehat{\Phi}_V : \widehat{C}_1(V') \to \widehat{C}_2'(V)$ coincides with the above identification. Hence, $\widehat{\Phi}' = \mathrm{id} \mod \mathfrak{m}$. Consequently, $\widehat{\Phi}'$ is an isomorphism.

We now alter the short exact sequence (6.3). Write $K_1(\widehat{\Upsilon}) = \widehat{L}_1 \oplus \widehat{K}_0' \oplus \widehat{L}_0$ and $K_2(\widehat{\Upsilon}) = \widehat{L}_2 \oplus \widehat{K}_0' \oplus \widehat{C}_3$, and define $\rho \in \operatorname{Aut} K_1(\widehat{\Upsilon})$ and $\psi \in \operatorname{Aut} K_2(\widehat{\Upsilon})$ by

$$\rho(x,y,z) = \left(x + \widehat{\Gamma} \circ \widehat{\Phi'}^{-1}(y), y, D_1(x) + \widehat{\Upsilon}_0'(y) + z\right),$$
$$\psi(x,y,z) = \left(x, \widehat{\Phi'}^{-1}(y) - \widehat{\Phi'}^{-1} \circ p' \circ \widehat{d}_3(z), z\right).$$

Note that the restriction of ψ to $\widehat{L}_2$ is the identity, and let $\bar{\psi}$ denote the restriction of ψ to $\widehat{K}_1 = \widehat{K}_0' \oplus \widehat{C}_3$. We modify the sequence (6.3) as indicated below.

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & \widehat{L}_2 & \xrightarrow{D_2} & \widehat{L}_1 & \xrightarrow{D_1} & \widehat{L}_0 \\
 & & \downarrow{\scriptstyle \psi^{-1}\circ\iota_2} & & \downarrow{\scriptstyle \rho\circ\iota_1} & & \downarrow{\scriptstyle \iota_0=\mathrm{id}} \\
\ldots & \longrightarrow & \widehat{L}_2 \oplus \widehat{K}_1 & \xrightarrow{\rho\circ\partial_2\circ\psi} & \widehat{L}_1 \oplus \widehat{K}_0 & \xrightarrow{\partial_1\circ\rho^{-1}} & \widehat{L}_0 \\
 & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1\circ\rho^{-1}} & & \downarrow \\
 & & \widehat{K}_1 & \xrightarrow{\Delta\circ\bar{\psi}} & \widehat{K}_0 & \longrightarrow & 0
\end{array} \tag{6.5}$$

Since $\bar{\psi} \circ \pi_2 = \pi_2 \circ \psi$, this diagram commutes.

Consider the map $\Xi : \widehat{L}_2 \oplus \widehat{C}_3 \to \widehat{L}_1$ defined by

$$\Xi(x,z) = \widehat{D}_2(x) + \widehat{\Gamma} \circ \widehat{\Phi'}^{-1} \circ p' \circ \widehat{d}_3(z) - \widehat{\Psi}_3(z). \tag{6.6}$$

Computations with the definitions (making use of the fact that $\widehat{\Upsilon}_\bullet$ is a chain map) reveal that $\partial_1 \circ \rho^{-1}(x,y,z) = z$ and $\rho \circ \partial_2 \circ \psi(x,y,z) = (\Xi(x,z), y, 0)$. Thus, Ξ provides a presentation for the module $H_1(K_\bullet(\widehat{\Upsilon}))$, and this module is trivial if and

only if Ξ is surjective. Since $\operatorname{im}(\widehat{D}_2) \subset \mathfrak{m} \cdot \widehat{L}_1$ and $\operatorname{im}(\widehat{\Gamma} \circ \widehat{\Phi'}^{-1} \circ p' \circ \widehat{d}_3) \subset \mathfrak{m} \cdot \widehat{L}_1$, the map Ξ is surjective if and only if $\widehat{\Psi}_3$ is surjective. □

The above proof has several consequences, even in the instance when the map $\widehat{\Psi}_3$ is not surjective, see below and Theorem 7.5. These results hold for an arbitrary arrangement (real or complex, compare 5.3 and 5.5) that is transverse to infinity.

Corollary 6.6. *The I-adic completion $\widehat{B}$ of the Alexander invariant an arrangement $\mathcal{A}$ has a presentation with $\binom{n}{2} - b_2(M(\mathcal{A}))$ generators, and $\binom{n}{3}$ relations.*

Proof. From the commutative diagram (6.5), we have the presentation $\widehat{\Delta} \circ \bar{\psi} : \widehat{K}_1 \to \widehat{K}_0$ for $\widehat{B}$. Let $p'' : K_0 = K_0' \oplus L_0 \to L_0$ denote the projection onto second factor. With respect to the decompositions $\widehat{K}_1 = \widehat{K}_0' \oplus \widehat{C}_3$ and $\widehat{K}_0 = \widehat{K}_0' \oplus \widehat{L}_0$, the map $\widehat{\Delta} \circ \bar{\psi} : \widehat{K}_1 \to \widehat{K}_0$ is given by

$$\widehat{\Delta} \circ \bar{\psi}(x, y) = \left(x, \widehat{p}'' \left(\widehat{\Phi} \circ \widehat{\Phi'}^{-1}(x) - \widehat{\Phi} \circ \widehat{\Phi'}^{-1} \circ \widehat{p}' \circ \widehat{d}_3(y) + \widehat{d}_3(y) \right) \right).$$

Define

$$\widehat{\Delta}^\sharp = \widehat{p}'' \circ \left(\mathrm{id} - \widehat{\Phi} \circ \widehat{\Phi'}^{-1} \circ \widehat{p}' \right) \circ \widehat{d}_3 : \widehat{C}_3 \to \widehat{L}_0, \tag{6.7}$$

and define $\chi \in \operatorname{Aut} \widehat{K}_0$ by $\chi(x, y) = (x, y - \widehat{p}'' \circ \widehat{\Phi} \circ \widehat{\Phi'}^{-1}(x))$. Then $\chi \circ \widehat{\Delta} \circ \bar{\psi}(x, y) = (x, \widehat{\Delta}^\sharp(y))$. Thus, $\widehat{\Delta}^\sharp$ provides a presentation for $\widehat{B}$ with the specified numbers of generators and relations. □

7. Combinatorics and the Chen Groups

In this section, we examine the relationship between the results obtained in the previous sections and the combinatorics of a hyperplane arrangement. We refer to the Chen groups of the group G of an arrangement $\mathcal{A}$ as simply the Chen groups of $\mathcal{A}$.

An invariant of $\mathcal{A}$ is called *combinatorial* if it is determined by the isomorphism type of the lattice $L(\mathcal{A})$. As is well-known from [12], the ranks ϕ_k of the LCS quotients of the group of $\mathcal{A}$ are combinatorially determined. Thus, the ranks of the first three Chen groups of $\mathcal{A}$ are combinatorial. We now describe some explicit combinatorial bounds and formulas for the ranks θ_k of the Chen groups of $\mathcal{A}$.

7.1. **A Bound on Chen Ranks.** Recall that the coarse combinatorial Alexander invariant B^{cc} of $\mathcal{A}$ is the direct sum $\oplus_V B_V$ of the Alexander invariants of the vertex groups G_V, indexed by $V \in L_2(\mathcal{A})$, the rank two elements of $L(\mathcal{A})$. For $k \geq 2$, define the *coarse combinatorial Chen ranks* by

$$\theta_k^{\mathrm{cc}}(\mathcal{A}) = \sum_{V \in L_2(\mathcal{A})} \theta_k(G_V).$$

From Theorem 1.3 and Theorem 6.3, we obtain the following.

Corollary 7.2. *For $k \geq 2$, the ranks of the Chen groups of $\mathcal{A}$ are bounded below by the coarse combinatorial Chen ranks: $\theta_k(\mathcal{A}) \geq \theta_k^{\mathrm{cc}}(\mathcal{A})$.*

To compute these lower bounds explicitly, recall that $G_V \cong F_{r-1} \times \mathbb{Z}^{n-r}$, where $r = |V|$. By Corollary 1.10, the ranks of the Chen groups of G_V are given by

$\theta_k(G_V) = (k-1)\binom{k+r-3}{k}$ for $k \geq 2$. Let c_r denote the number of elements of $L_2(\mathcal{A})$ of multiplicity r, and write $\binom{m}{k} = 0$ if $m < k$. Then,

$$\theta_k^{\mathrm{cc}} = \sum_{V \in L_2(\mathcal{A})} (k-1)\binom{k+|V|-3}{k} = \sum_{r \geq 3} c_r (k-1)\binom{k+r-3}{k},$$

and so θ_k^{cc} is determined by (only) the multiplicities of the elements of $L_2(\mathcal{A})$.

Remark 7.3. The ranks of the lower central series quotients of the group G of $\mathcal{A}$ satisfy analogous lower bounds: $\phi_k(\mathcal{A}) \geq \phi_k^{\mathrm{cc}}(\mathcal{A}) = \sum_{V \in L_2(\mathcal{A})} \phi_k(G_V)$, see [13], Proposition 3.8.

Remark 7.4. The lower bounds for the ranks of the Chen groups of $\mathcal{A}$ may be expressed in terms of the Möbius function $\mu : L(\mathcal{A}) \to \mathbb{Z}$:

$$\theta_k(\mathcal{A}) \geq \sum_{V \in L_2(\mathcal{A})} (k-1)\binom{k+\mu(V)-2}{k}.$$

We now analyze the difference $\theta_k - \theta_k^{\mathrm{cc}}$. Recall that $\theta_k = \operatorname{rank}(\mathfrak{m}^{k-2}\widehat{B}/\mathfrak{m}^{k-1}\widehat{B})$. Checking that the image of the map $\widehat{\Delta}^\sharp : \widehat{C}_3 \to \widehat{L}_0$ defined in (6.7) is contained in $\mathfrak{m} \cdot \widehat{L}_0$, we see that $\theta_2(\mathcal{A}) = \operatorname{rank}(\widehat{B}/\mathfrak{m}\widehat{B}) = \binom{n}{2} - b_2(M(\mathcal{A}))$, and thus $\theta_2(\mathcal{A}) = \theta_2^{\mathrm{cc}}(\mathcal{A})$.

Recall the mapping cone $K_\bullet(\widehat{\Upsilon})$ from (6.2), and set $H = H_1(K_\bullet(\widehat{\Upsilon}))$.

Theorem 7.5. *For $k \geq 3$, the rank of the k^{th} Chen group of $\mathcal{A}$ is given by $\theta_k(\mathcal{A}) = \operatorname{rank}(\mathfrak{m}^{k-3}H/\mathfrak{m}^{k-2}H) + \theta_k^{\mathrm{cc}}(\mathcal{A})$. In particular, $\theta_3(\mathcal{A}) = \operatorname{rank}(\operatorname{coker}\widehat{\Psi}_3) + \theta_3^{\mathrm{cc}}(\mathcal{A})$.*

Proof. Consider the short exact sequence (6.4), rewritten as

$$0 \to H \xrightarrow{\tau} \widehat{B} \xrightarrow{\Pi} \widehat{B}^{\mathrm{cc}} \to 0. \tag{7.1}$$

Altering the commutative diagram (6.5) using the isomorphism $\chi \in \operatorname{Aut}\widehat{K}_0$ defined in the proof of Corollary 6.6, we see that τ is induced by $\chi \circ \pi_1 \circ \rho^{-1} : \widehat{L}_1 \oplus \widehat{K}_0' \oplus \widehat{L}_0 \to \widehat{K}_0$, $(x,y,z) \mapsto (y, z - \widehat{D}_1(x))$. Thus the restriction of this map to $\ker(\partial_1 \circ \rho^{-1}) = \widehat{L}_1 \oplus \widehat{K}_0'$ is given by $(x,y) \mapsto (y, -\widehat{D}_1(x))$. Since $\widehat{B} = \operatorname{coker}(\chi \circ \widehat{\Delta} \circ \bar{\psi})$ and $\chi \circ \widehat{\Delta} \circ \bar{\psi}(y,0) = (y,0)$, the map $\tau : H \to \widehat{B}$ in homology is induced by $\widehat{D}_1 : \widehat{L}_1 \to \widehat{L}_0$. Since $\operatorname{im}(\widehat{D}_1(z)) \subset \mathfrak{m} \cdot \widehat{L}_0$, from the exact sequence (7.1) we have

$$\operatorname{rank}(\mathfrak{m}^\ell \widehat{B}/\mathfrak{m}^{\ell-1}\widehat{B}) = \operatorname{rank}(\mathfrak{m}^{\ell-1}H/\mathfrak{m}^{\ell-2}H) + \operatorname{rank}(\mathfrak{m}^\ell \widehat{B}^{\mathrm{cc}}/\mathfrak{m}^{\ell-1}\widehat{B}^{\mathrm{cc}}).$$

It follows from Theorem 1.3 that the ranks of the Chen groups of $\mathcal{A}$ are as asserted.

In particular, the third Chen group of $\mathcal{A}$ has rank $\theta_3(\mathcal{A}) = \operatorname{rank}(H/\mathfrak{m}H) + \theta_3^{\mathrm{cc}}(\mathcal{A})$. Recall the presentation, $\Xi : \widehat{L}_2 \oplus \widehat{C}_3 \to \widehat{L}_1$, for the module H from (6.6). Using elementary row and column operations, we obtain a presentation $\Xi' : \widehat{\Lambda}^a \to \widehat{\Lambda}^b$ from this with $b = \operatorname{rank}\widehat{L}_1 - \operatorname{rank}\widehat{\Psi}_3$ generators. Checking that $\operatorname{im}(\Xi') \subset \mathfrak{m} \cdot \widehat{\Lambda}^b$, we get $\operatorname{rank}(H/\mathfrak{m}H) = \operatorname{rank}(\operatorname{coker}\widehat{\Psi}_3)$. □

7.6. **Decomposition is Combinatorial.** Let $\hat{\epsilon} : \widehat{\Lambda} \to \mathbb{Z}$ be the augmentation map, which takes a power series to its constant coefficient. If $\widehat{F} = \widehat{\Lambda}^p$ is a free module, denote its image under $\hat{\epsilon}$ by $\overline{F} = \mathbb{Z}^p$, and if $\widehat{f} : \widehat{F} \to \widehat{F}'$ is a $\widehat{\Lambda}$-linear map, denote its image by $\overline{f} : \overline{F} \to \overline{F}'$.

Lemma 7.7. *The rank of $\widehat{f}$ is equal to the rank of $\overline{f}$.*

Proof. Suppose the rank of $\overline{f} : \mathbb{Z}^p \to \mathbb{Z}^q$ is r. Then there are integral matrices $X \in \mathrm{GL}(p,\mathbb{Z})$ and $Y \in \mathrm{GL}(q,\mathbb{Z})$ so that $X \cdot \overline{f} \cdot Y = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r denotes the $r \times r$ identity matrix. By definition, $\widehat{f} = \overline{f} + Z$, where the entries of Z are in $\mathfrak{m}$. Thus, $X \cdot \widehat{f} \cdot Y = X \cdot \overline{f} \cdot Y + X \cdot Z \cdot Y$. Clearly, the rank of $X \cdot \widehat{f} \cdot Y$, and hence that of $\widehat{f}$, is r.

The converse follows from the functoriality of the construction. □

We now show that the rank of the map $\widehat{\Psi}_3 : \widehat{C}_3 \to \widehat{L}_1$ is combinatorially determined. Thus, the criterion for decomposition of the I-adic completion of the Alexander invariant of Theorem 6.5—the surjectivity of $\widehat{\Psi}_3$—is combinatorial as well. By the lemma, it suffices to show that the rank of $\overline{\Psi}_3 : \overline{C}_3 \to \overline{L}_1$ is combinatorially determined.

For this, let $\mathcal{A}$ and $\mathcal{A}^*$ be lattice-isomorphic arrangements of n lines in $\mathbb{C}^2$ (which are transverse to the line at infinity). Let $\mathcal{W}$ be a braided wiring diagram associated to $\mathcal{A}$, and let $\overline{\Psi}_3 : \overline{C}_3 \to \overline{L}_1 = \bigoplus_V \overline{C}_2(V') \wedge \overline{C}_1$ be the map defined by the vertex sets $\{V_1, \dots, V_s\}$ of $\mathcal{W}$. Choose arbitrary orderings of the hyperplanes and rank two lattice elements of $\mathcal{A}^*$, and denote the elements of $L_2(\mathcal{A}^*)$ by $\{U_1, \dots, U_s\}$. Then formally construct the map $\overline{\Psi}^*_3 = \sum_U \overline{\Psi}_{U,3} : \overline{C}_3 \to \overline{L}^*_1 = \bigoplus_U \overline{C}_2(U') \wedge \overline{C}_1$ using (4.10), the Magnus embedding, and the augmentation map $\hat{\epsilon}$.

Since $\mathcal{A}$ and $\mathcal{A}^*$ are lattice-isomorphic, there are permutations $\omega \in \Sigma_n$ and $\nu \in \Sigma_s$ so that $\omega(V_k) = U_{\nu(k)}$ for each k, $1 \le k \le s$. The permutation ω induces an isomorphism $\omega_k : C_k \to C_k$ defined by $\omega_k(e_J) = e_{\omega(J)}$. The map $\overline{\Psi}_3$ is combinatorially determined in the sense of the following.

Proposition 7.8. *There is an isomorphism $\overline{\xi} : \overline{L}_1 \to \overline{L}^*_1$ so that $\overline{\xi} \circ \overline{\Psi}_3 = \overline{\Psi}^*_3 \circ \overline{\omega}_3$.*

Proof. Let V be a vertex set of $\mathcal{A}$, and $U = \omega(V)$ be the corresponding vertex set of $\mathcal{A}^*$. Define a map $\xi^U_V : C_2(V') \wedge C_1 \to C_2(U') \wedge C_1$ by

$$\xi^U_V = (\pi_{U'} \wedge \mathrm{id}) \circ \Theta_2(\mu_U)^{-1} \circ \omega_3 \circ \Theta_2(\mu_V) \circ (\iota_{V'} \wedge \mathrm{id}). \tag{7.2}$$

Clearly, ξ^U_V is an isomorphism, with inverse ξ^V_U. Moreover, $\xi^U_V \circ \Psi_{V,3} = \Psi_{U,3} \circ \omega_3$. The collection $\{\xi^V_U\}$ defines a map $\xi : L_1 \to L^*_1$, which yields the desired isomorphism $\overline{\xi}$. □

Combining these results with those of the previous section, we obtain

Theorem 7.9. *The rank of the third Chen group of the arrangement $\mathcal{A}$ is given by the combinatorial formula $\theta_3(\mathcal{A}) = \mathrm{rank}(\mathrm{coker}\, \overline{\Psi}_3) + \theta^{\mathrm{cc}}_3(\mathcal{A})$.*

Furthermore, if the map $\overline{\Psi}_3 : \overline{C}_3 \to \overline{L}_1$ is surjective, then the I-adic completion of the Alexander invariant of $\mathcal{A}$ decomposes as a direct sum: $\widehat{B} \cong \widehat{B}^{\mathrm{cc}} = \oplus_V \widehat{B}_V$, and the ranks of the Chen groups of $\mathcal{A}$ are given by $\theta_k(\mathcal{A}) = \theta^{\mathrm{cc}}_k(\mathcal{A})$ for all $k \ge 2$.

Remark 7.10. If $\mathcal{A}$ is an arrangement for which the map $\overline{\Psi}_3 : \overline{C}_3 \to \overline{L}_1$ is *not* surjective, the ranks $\theta_k(\mathcal{A})$ of the Chen groups of $\mathcal{A}$ for $k \ge 4$ may be computed using the Groebner basis algorithm described in 1.2. Alternatively, in light of Theorem 7.5, one can apply this algorithm to the presentation (6.6) of the module H (or the smaller presentation described in the proof of Theorem 7.5) to determine $\mathrm{rank}(\mathfrak{m}^{k-3}H/\mathfrak{m}^{k-2}H) = \theta_k(\mathcal{A}) - \theta^{\mathrm{cc}}_k(\mathcal{A})$.

8. Examples

In this section, we illustrate the results of the previous sections by means of several explicit examples. We order the hyperplanes of an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ in the order indicated by the defining polynomial $Q(\mathcal{A}) = \prod_{k=1}^{n} \ell_k$ (so $H_k = \ker \ell_k$).

Example 8.1. Consider the central 3-arrangement $\mathcal{A}$ with defining polynomial $Q = xyz(y+z)(x-z)(2x+y)$. Randell [28] noted that this arrangement is not $K(\pi,1)$, and that there is no aspherical arrangement with the same lattice in ranks one and two. Arvola [1] further showed that the group of this arrangement is not of type FL.

The rank two elements of the lattice of $\mathcal{A}$ are

$$\{\{1,2,6\},\{1,3,5\},\{2,3,4\},\{1,4\},\{2,5\},\{4,5\},\{3,6\},\{4,6\},\{5,6\}\}.$$

It is readily checked that the map $\overline{\Psi}_3 : \mathbb{Z}^{20} \to \mathbb{Z}^{12}$ is surjective. By Theorem 7.9, $\widehat{B} \cong \widehat{B}^{\mathrm{cc}} = \widehat{B}_{\{1,2,6\}} \oplus \widehat{B}_{\{1,3,5\}} \oplus \widehat{B}_{\{2,3,4\}}$. It follows that the ranks of the Chen groups of $\mathcal{A}$ are $\theta_1 = 6$ and $\theta_k = 3(k-1)$ for $k \geq 2$. Notice that these ranks coincide with those of the Chen groups of a direct product of three free groups on two generators, though clearly $G \ncong F_2 \times F_2 \times F_2$.

Using Theorem 5.5 and elementary row operations, one can show that the Alexander invariant itself decomposes as a direct sum, $B \cong B^{\mathrm{cc}} = B_{\{1,2,6\}} \oplus B_{\{1,3,5\}} \oplus B_{\{2,3,4\}}$.

Example 8.2. The braid arrangement $\mathcal{A}_4$ is the smallest arrangement for which the completion of the Alexander invariant does not decompose. The polynomial $Q = xyz(x-y)(x-z)(y-z)$ defines a central 3-arrangement whose complement is homotopy equivalent to that of $\mathcal{A}_4$. The rank two elements of $L(\mathcal{A}_4)$ (the partition lattice) are

$$\{\{1,2,4\},\{1,3,5\},\{2,3,6\},\{3,4\},\{2,5\},\{4,5,6\},\{1,6\}\}.$$

The map $\overline{\Psi}_3 : \mathbb{Z}^{20} \to \mathbb{Z}^{16}$ is *not* surjective. Thus $\widehat{B}(\mathcal{A}_4)$ does not decompose. A basis for $\operatorname{coker} \overline{\Psi}_3$ is given by the two elements

$$\kappa_1 = e_{\{2,4\}} \wedge (e_6 - e_3) + e_{\{3,5\}} \wedge (e_4 - e_6) + e_{\{3,6\}} \wedge (e_1 - e_4) + e_{\{5,6\}} \wedge (e_3 - e_1),$$
$$\kappa_2 = e_{\{2,4\}} \wedge (e_6 - e_5) + e_{\{3,5\}} \wedge (e_2 - e_6) + e_{\{3,6\}} \wedge (e_1 - e_5) + e_{\{5,6\}} \wedge (e_2 - e_1).$$

Since $\operatorname{rank} \overline{\Psi}_3 = 14$, we have $\theta_3(\mathcal{A}_4) = 10$.

Remark 8.3. Note that the rank of the third Chen group of $\mathcal{A}_4$ is equal to that of the product arrangement defined by $xy(y-x)z(z-x)(z-2x)$. In general, by the LCS formula [15], the ranks of the lower central series quotients of the pure braid group P_n are equal to those of the direct product of free groups $\Pi_n = F_{n-1} \times \cdots \times F_1$. These groups are distinguished by their Chen groups. For $k \geq 4$, we have $\theta_k(\Pi_n) = (k-1)\binom{n+k-2}{k+1}$, by Example 1.11, and $\theta_k(P_n) = (k-1)\binom{n+1}{4}$, by the main result of [5]. Thus, $\theta_k(P_n) \neq \theta_k(\Pi_n)$ for $n \geq 4$, and the groups P_n and Π_n are not isomorphic.

Remark 8.4. Example 8.2 provides an easy means for detecting when the completion of the Alexander invariant of an arrangement $\mathcal{A}$ does not decompose. If $\mathcal{S} \subset \mathcal{A}$ is a subarrangement which is lattice-isomorphic to the braid arrangement $\mathcal{A}_4$, one

can use the above elements of $\operatorname{coker}\overline{\Psi}_3(\mathcal{A}_4)$ and maps of the form (7.2) to generate non-trivial elements of $\operatorname{coker}\overline{\Psi}_3(\mathcal{A})$.

It is interesting to note that (the matroid of) such an arrangement $\mathcal{A}$ has "non-local decomposable relations," see [14].

Example 8.5 (Diamond)**.** Let $\mathcal{D}$ be the central 3-arrangement with defining polynomial $Q(\mathcal{D}) = x(x+y+z)(x+y-z)y(x-y-z)(x-y+z)z$. This is a free, simplicial arrangement for which the LCS formula does not hold, and the Orlik-Solomon algebra is not quadratic, see [12], [16], [30]. The rank two elements of $L(\mathcal{D})$ are

$$\{\{3,4,5\},\{1,2,5\},\{1,4\},\{1,3,6\},\{2,4,6\},\{1,7\},\{2,3,7\},\{4,7\},\{5,6,7\}\}.$$

This arrangement has three distinct subarrangements lattice-isomorphic to $\mathcal{A}_4$. One such subarrangement is $\mathcal{S} = \mathcal{D} \setminus \{H_1\}$. Define $\omega : [6] \hookrightarrow [7]$ by $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 7$, $5 \mapsto 6$, $6 \mapsto 5$. The map ω gives rise to a lattice-isomorphism $L(\mathcal{A}_4) \xrightarrow{\sim} L(\mathcal{S})$. Define $\overline{\xi} : \bigoplus_{V\in L_2(\mathcal{A}_4)} \overline{C}_2(V') \wedge \overline{C}_1 \to \bigoplus_{U\in L_2(\mathcal{D})} \overline{C}_2(U') \wedge \overline{C}_1$ as in (7.2): $\overline{\xi}(e_{\{5,6\}} \wedge e_j) = -e_{\{6,7\}} \wedge \overline{\omega}_1(e_j)$ and $\overline{\xi}(e_K \wedge e_j) = \overline{\omega}_3(e_K \wedge e_j)$ for $K \neq \{5,6\}$. Recall the elements κ_1 and κ_2 from Example 8.2, and let $\kappa_i' = \overline{\xi}(\kappa_i)$. Then

$$\kappa_1' = e_{\{3,7\}} \wedge (e_5 - e_4) + e_{\{4,6\}} \wedge (e_7 - e_5) + e_{\{4,5\}} \wedge (e_2 - e_7) + e_{\{6,7\}} \wedge (e_2 - e_4),$$
$$\kappa_2' = e_{\{3,7\}} \wedge (e_5 - e_6) + e_{\{4,6\}} \wedge (e_3 - e_5) + e_{\{4,5\}} \wedge (e_2 - e_6) + e_{\{6,7\}} \wedge (e_2 - e_3)$$

are in the cokernel of $\overline{\Psi}_3 : \mathbb{Z}^{35} \to \mathbb{Z}^{30}$. We obtain 6 distinct elements of $\operatorname{coker}\overline{\Psi}_3$ in this way. However, there is a relation among them. We have $\operatorname{rank}\overline{\Psi}_3 = 25$, and $\theta_3(\mathcal{D}) = 17$.

The ranks of the higher Chen groups may be found via the Groebner basis algorithm of Theorem 1.5. By Example 1.12, we can simplify the computation by working with the decone of $\mathcal{D}$ defined by $Q(\mathcal{D})|_{z=1}$. Rotating this arrangement counterclockwise to insure that first coordinate projection is generic, we obtain a wiring diagram $\mathcal{W} = \{\{3,4,5\},\{1,2,5\},\{1,4\},\{1,3,6\},\{2,4,6\}\}$. The image of the associated braid monodromy $\alpha : F_5 \to P_6$ is generated by $\{A_{3,4,5},\ A_{1,2,5},\ A_{1,4}^{A_{3,4}},\ A_{1,3,6},\ A_{2,4,6}^{A_{3,4}A_{3,6}}\}$. From the presentation $\Lambda^{20} \xrightarrow{\Delta} \Lambda^6 \to B \to 0$ provided by Theorem 5.5, we find $\theta_k(\mathcal{D}) = 9(k-1)$, for $k \geq 4$.

We have found a number of other arrangements for which the completion of the Alexander invariant does not decompose as a direct sum. For example, for the Coxeter arrangement of type B_3, we have $\overline{C}_3 = \mathbb{Z}^{84}$, $\overline{L}_1 = \mathbb{Z}^{85}$, and $\overline{\Psi}_3 : \overline{C}_3 \to \overline{L}_1$ is obviously not surjective. (This arrangement also has (many) subarrangements lattice-isomorphic to $\mathcal{A}_4$.) More subtle examples include the following.

Example 8.6 (MacLane)**.** The polynomials

$$Q^{\pm} = xy(y-x)z(z-x-\omega^2 y)(z+\omega y)(z-x)(z+\omega^2 x+\omega y)$$

where $\omega = (-1 \pm \sqrt{-3})/2$, define complex conjugate realizations $\mathcal{A}^{\pm}$ of the MacLane matroid (the 8_3 configuration). These arrangements were used by Rybnikov [29] in his construction of lattice-isomorphic arrangements with distinct fundamental groups. Rybnikov's arrangements are not distinguished by their Chen groups.

Complex conjugation induces an isomorphism of the groups of $\mathcal{A}^+$ and $\mathcal{A}^-$, and thus an isomorphism of the Alexander invariants, $B^+ \cong B^-$. Neither of these arrangements has a subarrangement lattice-isomorphic to the braid arrangement

$\mathcal{A}_4$. Nevertheless, the maps $\overline{\Psi}_3^{\pm}$ are not surjective, and the modules $\widehat{B}^{\pm}$ do not decompose.

The hyperplane $H = \{7x - 5y + z = 1\}$ is generic with respect to both these arrangements. Moreover, the projection $\mathrm{pr} : \mathbb{C}^2 \to \mathbb{C}$ defined by $\mathrm{pr}(x, y) = 13x - 4y$ is generic with respect to both affine 2-arrangements $H \cap \mathcal{A}^{\pm}$. Changing coordinates accordingly, we obtain braided wiring diagrams $\mathcal{W}^{\pm} = \{V_1, \beta_1^{\pm}, V_2, \beta_2^{\pm}, \dots, \beta_{11}^{\pm}, V_{12}\}$ with vertex sets and intermediary braids given by:

$$
\begin{array}{llll}
V_1 = \{3,4\}, & \beta_1^{\pm} = 1, & V_7 = \{2,5,7\}, & \beta_7^{\pm} = \sigma_3^{\mp 1}, \\
V_2 = \{3,5,6\}, & \beta_2^{\pm} = 1, & V_8 = \{6,7\}, & \beta_8^{\pm} = \sigma_4^{\pm 1}\sigma_3^{\pm 1}, \\
V_3 = \{3,7,8\}, & \beta_3^{\pm} = 1, & V_9 = \{1,6,8\}, & \beta_9^{\pm} = \sigma_1^{\mp 1}, \\
V_4 = \{2,4,6\}, & \beta_4^{\pm} = \sigma_4^{\mp 1}\sigma_3^{\mp 1}, & V_{10} = \{1,4,7\}, & \beta_{10}^{\pm} = \sigma_2^{\mp 1}, \\
V_5 = \{2,8\}, & \beta_5^{\pm} = 1, & V_{11} = \{1,5\}, & \beta_{11}^{\pm} = \sigma_4^{\pm 1}, \\
V_6 = \{4,5,8\}, & \beta_6^{\pm} = \sigma_2^{\pm 1}\sigma_5^{\pm 1}, & V_{12} = \{1,2,3\}. &
\end{array}
$$

An argument as in [7] shows that the braid monodromies associated to $\mathcal{W}^+$ and $\mathcal{W}^-$ are equivalent, but not braid equivalent. Calculations with these monodromies reveal that $\theta_3^{\mathrm{cc}}(\mathcal{A}^{\pm}) = 16$, $\theta_3(\mathcal{A}^{\pm}) = 21$, and $\theta_k(\mathcal{A}^{\pm}) = \theta_k^{\mathrm{cc}}(\mathcal{A}^{\pm}) = 8(k-1)$ for $k \geq 4$. Thus the failure of $\overline{\Psi}_3$ to be surjective is detected only by the third Chen group.

Example 8.7 (9_3 Configurations)**.** The relationship between the (completion of the) Alexander invariant and the combinatorics of an arrangement appears to be quite delicate. As an illustration, consider the arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by

$$Q(\mathcal{P}_1) = xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(2x-5y+z),$$
$$Q(\mathcal{P}_2) = xyz(x+y)(y+z)(x+3z)(x+2y+z)(x+2y+3z)(2x+3y+3z).$$

The arrangement $\mathcal{P}_1$ is a realization of the Pappus configuration $(9_3)_1$, while $\mathcal{P}_2$ is a realization of the configuration $(9_3)_2$. Note that neither of these arrangements has a subarrangement lattice-isomorphic to the braid arrangement.

The combinatorial distinction between these arrangements (resp., their underlying matroids) is detected by the maps $\overline{\Psi}_3(\mathcal{P}_k) : \mathbb{Z}^{84} \to \mathbb{Z}^{63}$. The map $\overline{\Psi}_3(\mathcal{P}_2)$ is surjective, and consequently the module $\widehat{B}(\mathcal{P}_2)$ decomposes as a direct sum. Thus, $\theta_k(\mathcal{P}_2) = 9(k-1)$ for $k \geq 2$. However, the map $\overline{\Psi}_3(\mathcal{P}_1)$ is not surjective, and $\widehat{B}(\mathcal{P}_1)$ does not decompose. A calculation shows that $\theta_2(\mathcal{P}_1) = 9$, and $\theta_k(\mathcal{P}_1) = 10(k-1)$ for $k \geq 3$.

It was conjectured in [5] that, for k sufficiently large, one has $\theta_k(\mathcal{A}) = \theta_k^{\mathrm{cc}}(\mathcal{A}) + (k-1)\beta(\mathcal{A})$, where $\beta(\mathcal{A})$ is the number of subarrangements of $\mathcal{A}$ that are lattice-isomorphic to $\mathcal{A}_4$. The arrangement $\mathcal{P}_1$ has $\beta = 0$ and $\theta_k^{\mathrm{cc}} = 9(k-1)$, and hence provides a counterexample to that conjecture.

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