# HOMOLOGICAL FINITENESS IN THE JOHNSON FILTRATION


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# References

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# FILTRATIONS AND GRADED LIE ALGEBRAS

Let $G$ be a group, with commutator $(x, y) = xyx^{-1}y^{-1}$.
Suppose given a descending filtration

$$G = \Phi^1 \supseteq \Phi^2 \supseteq \cdots \supseteq \Phi^s \supseteq \cdots$$

by subgroups of $G$, satisfying

$$(\Phi^s, \Phi^t) \subseteq \Phi^{s+t}, \quad \forall s, t \geqslant 1.$$

Then $\Phi^s \lhd G$, and $\Phi^s/\Phi^{s+1}$ is abelian. Set

$$\operatorname{gr}_\Phi(G) = \bigoplus_{s \geqslant 1} \Phi^s/\Phi^{s+1}.$$

This is a graded Lie algebra, with bracket $[\ ,\ ]\colon \operatorname{gr}^s_\Phi \times \operatorname{gr}^t_\Phi \to \operatorname{gr}^{s+t}_\Phi$ induced by the group commutator.

Basic example: the *lower central series*, $\Gamma^s = \Gamma^s(G)$, defined as

$$\Gamma^1 = G, \Gamma^2 = G', \dots, \Gamma^{s+1} = (\Gamma^s, G), \dots$$

Then for any filtration $\Phi$ as above, $\Gamma^s \subseteq \Phi^s$; thus, we have a morphism of graded Lie algebras,

$$\iota_\Phi \colon \operatorname{gr}_\Gamma(G) \longrightarrow \operatorname{gr}_\Phi(G) .$$

### EXAMPLE (P. HALL, E. WITT, W. MAGNUS)

Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank $n$. Then:

- $F_n$ is residually nilpotent, i.e., $\bigcap_{s \geqslant 1} \Gamma^s(F_n) = \{1\}$.
- $\operatorname{gr}_\Gamma(F_n)$ is isomorphic to the free Lie algebra $\mathcal{L}_n = \operatorname{Lie}(\mathbb{Z}^n)$.
- $\operatorname{gr}^s_\Gamma(F_n)$ is free abelian, of rank $\frac{1}{s}\sum_{d|s} \mu(d) n^{\frac{s}{d}}$.
- If $n \geqslant 2$, the center of $\mathcal{L}_n$ is trivial.

# AUTOMORPHISM GROUPS

Let $\mathrm{Aut}(G)$ be the group of all automorphisms $\alpha\colon G \to G$, with $\alpha \cdot \beta := \alpha \circ \beta$. The *Andreadakis–Johnson filtration*,

$$\mathrm{Aut}(G) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^s \supseteq \cdots$$

has terms $F^s = F^s(\mathrm{Aut}(G))$ consisting of those automorphisms which act as the identity on the $s$-th nilpotent quotient of $G$:

$$\begin{aligned} F^s &= \ker\big(\mathrm{Aut}(G) \to \mathrm{Aut}(G/\Gamma^{s+1})\big) \\ &= \{\alpha \in \mathrm{Aut}(G) \mid \alpha(x) \cdot x^{-1} \in \Gamma^{s+1},\ \forall x \in G\} \end{aligned}$$

Kaloujnine [1950]: $(F^s, F^t) \subseteq F^{s+t}$.
First term is the *Torelli group*,

$$\mathcal{T}_G = F^1 = \ker\big(\mathrm{Aut}(G) \to \mathrm{Aut}(G_{\mathrm{ab}})\big).$$

By construction, $F^1 = \mathcal{T}_G$ is a normal subgroup of $F^0 = \mathrm{Aut}(G)$. The quotient group,

$$\mathcal{A}(G) = F^0/F^1 = \mathrm{im}(\mathrm{Aut}(G) \to \mathrm{Aut}(G_{\mathrm{ab}}))$$

is the *symmetry group* of $\mathcal{T}_G$; it fits into exact sequence

$$1 \longrightarrow \mathcal{T}_G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathcal{A}(G) \longrightarrow 1 \; .$$

The Torelli group comes endowed with two filtrations:

- The Johnson filtration $\{F^s(\mathcal{T}_G)\}_{s\geqslant 1}$, inherited from $\mathrm{Aut}(G)$.
- The lower central series filtration, $\{\Gamma^s(\mathcal{T}_G)\}$.

The respective associated graded Lie algebras, $\mathrm{gr}_F(\mathcal{T}_G)$ and $\mathrm{gr}_\Gamma(\mathcal{T}_G)$, come endowed with natural actions of $\mathcal{A}(G)$; moreover, the morphism $\iota_F \colon \mathrm{gr}_\Gamma(\mathcal{T}_G) \to \mathrm{gr}_F(\mathcal{T}_G)$ is $\mathcal{A}(G)$-equivariant.

# THE JOHNSON HOMOMORPHISM

Given a graded Lie algebra $\mathfrak{g}$, let

$$\mathrm{Der}^s(\mathfrak{g}) = \{\delta\colon \mathfrak{g}^\bullet \to \mathfrak{g}^{\bullet+s} \text{ linear} \mid \delta[x,y] = [\delta x, y] + [x, \delta y], \forall x, y \in \mathfrak{g}\}.$$

Then $\mathrm{Der}(\mathfrak{g}) = \bigoplus_{s\geqslant 1} \mathrm{Der}^s(\mathfrak{g})$ is a graded Lie algebra, with bracket $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$.

**THEOREM**

*Given a group $G$, there is a monomorphism of graded Lie algebras,*

$$J\colon \mathrm{gr}_F(\mathcal{T}_G) \longrightarrow \mathrm{Der}(\mathrm{gr}_\Gamma(G)) \ ,$$

*given on homogeneous elements $\alpha \in F^s(\mathcal{T}_G)$ and $x \in \Gamma^t(G)$ by*

$$J(\bar{\alpha})(\bar{x}) = \overline{\alpha(x) \cdot x^{-1}}.$$

*Moreover, $J$ is equivariant with respect to the natural actions of $\mathcal{A}(G)$.*

The Johnson homomorphism informs on the Johnson filtration.

**THEOREM**

*Let $G$ be a group. For each $q \geqslant 1$, the following are equivalent:*

1. $J \circ \iota_F \colon \operatorname{gr}_\Gamma^s(\mathcal{T}_G) \to \operatorname{Der}^s(\operatorname{gr}_\Gamma(G))$ *is injective, for all* $s \leqslant q$.
2. $\Gamma^s(\mathcal{T}_G) = F^s(\mathcal{T}_G)$, *for all* $s \leqslant q+1$.

**PROPOSITION**

*Suppose $G$ is residually nilpotent, $\operatorname{gr}_\Gamma(G)$ is centerless, and $J \circ \iota_F \colon \operatorname{gr}_\Gamma^1(\mathcal{T}_G) \to \operatorname{Der}^1(\operatorname{gr}_\Gamma(G))$ is injective. Then $F^2(\mathcal{T}_G) = \mathcal{T}_G'$.*

**PROBLEM**

*Determine the homological finiteness properties of the groups $F^s(\mathcal{T}_G)$. In particular, decide whether $\dim H_1(\mathcal{T}_G', \mathbb{Q}) < \infty$.*

# AN OUTER VERSION

Let $\mathrm{Inn}(G) = \mathrm{im}(\mathrm{Ad}\colon G \to \mathrm{Aut}(G))$, where $\mathrm{Ad}_x\colon G \to G$, $y \mapsto xyx^{-1}$.
Define the *outer* automorphism group of a group $G$ by

$$1 \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \xrightarrow{\pi} \mathrm{Out}(G) \longrightarrow 1 .$$

We then have

- Filtration $\{\widetilde{F}^s\}_{s\geqslant 0}$ on $\mathrm{Out}(G)$: $\widetilde{F}^s := \pi(F^s)$.
- The *outer Torelli group* of $G$: subgroup $\widetilde{\mathcal{T}}_G = \widetilde{F}^1$ of $\mathrm{Out}(G)$.
- Exact sequence: $1 \longrightarrow \widetilde{\mathcal{T}}_G \longrightarrow \mathrm{Out}(G) \longrightarrow \mathcal{A}(G) \longrightarrow 1$ .

### THEOREM

*Suppose* $Z(\mathrm{gr}_\Gamma(G)) = 0$. *Then the Johnson homomorphism induces an* $\mathcal{A}(G)$*-equivariant monomorphism of graded Lie algebras,*

$$\widetilde{J}\colon \mathrm{gr}_{\widetilde{F}}(\widetilde{\mathcal{T}}_G) \longrightarrow \widetilde{\mathrm{Der}}(\mathrm{gr}_\Gamma(G)) ,$$

*where* $\widetilde{\mathrm{Der}}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g}) / \mathrm{im}(\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}))$.

# THE ALEXANDER INVARIANT

- Let $G$ be a group, and $G_{\text{ab}} = G/G'$ its maximal abelian quotient.
- Let $G'' = (G', G')$; then $G/G''$ is the maximal metabelian quotient. Get exact sequence $0 \longrightarrow G'/G'' \longrightarrow G/G'' \longrightarrow G_{\text{ab}} \longrightarrow 0$ .
- Conjugation in $G/G''$ turns the abelian group

$$B(G) := G'/G'' = H_1(G', \mathbb{Z})$$

into a module over $R = \mathbb{Z}G_{\text{ab}}$, called the *Alexander invariant* of $G$.
- Since both $G'$ and $G''$ are characteristic subgroups of $G$, the action of $\text{Aut}(G)$ on $G$ induces an action on $B(G)$. This action need not respect the $R$-module structure. Nevertheless:

PROPOSITION

*The Torelli group $\mathcal{T}_G$ acts $R$-linearly on the Alexander invariant $B(G)$.*

# CHARACTERISTIC VARIETIES

- Let $G$ be a finitely generated group.
- Let $\widehat{G} = \operatorname{Hom}(G, \mathbb{C}^*)$ be its *character group*: an algebraic group, with coordinate ring $\mathbb{C}[G_{\mathrm{ab}}]$.
- The map $\mathrm{ab}\colon G \twoheadrightarrow G_{\mathrm{ab}}$ induces an isomorphism $\widehat{G}_{\mathrm{ab}} \xrightarrow{\simeq} \widehat{G}$.
- $\widehat{G}^{\circ} \cong (\mathbb{C}^*)^n$, where $n = \operatorname{rank} G_{\mathrm{ab}}$.

### DEFINITION

The (first) *characteristic variety* of $G$ is the support of the (complexified) Alexander invariant $B = B(G) \otimes \mathbb{C}$:

$$\mathcal{V}(G) := V(\operatorname{ann} B) \subset \widehat{G}.$$

This variety informs on the Betti numbers of normal subgroups $H \lhd G$ with $G/H$ abelian. In particular (for $H = G'$):

### PROPOSITION

*The set $\mathcal{V}(G)$ is finite if and only if $b_1(G') = \dim_{\mathbb{C}} B(G) \otimes \mathbb{C}$ is finite.*

# RESONANCE VARIETIES

Let $V$ be a finite-dimensional $\mathbb{C}$-vector space, and let $K \subset V \wedge V$ be a subspace.

**DEFINITION**

The *resonance variety* $\mathcal{R} = \mathcal{R}(V, K)$ is the set of elements $a \in V^*$ for which there is an element $b \in V^*$, not proportional to $a$, such that $a \wedge b$ belongs to the orthogonal complement $K^\perp \subseteq V^* \wedge V^*$.

- $\mathcal{R}$ is a conical, Zariski-closed subset of the affine space $V^*$.
- For instance, if $K = 0$ and $\dim V > 1$, then $\mathcal{R} = V^*$.
- At the other extreme, if $K = V \wedge V$, then $\mathcal{R} = 0$.

The resonance variety $\mathcal{R}$ has several other interpretations.

# KOSZUL MODULES

- Let $S = \mathrm{Sym}(V)$ be the symmetric algebra on $V$.
- Let $(S \otimes_{\mathbb{C}} \bigwedge V, \delta)$ be the Koszul resolution, with differential $\delta_p \colon S \otimes_{\mathbb{C}} \bigwedge^p V \to S \otimes_{\mathbb{C}} \bigwedge^{p-1} V$ given by

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \mapsto \sum_{j=1}^{p} (-1)^{j-1} v_{i_j} \otimes (v_{i_1} \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_{i_p}).$$

- Let $\iota \colon K \to V \wedge V$ be the inclusion map.
- The *Koszul module* $\mathcal{B}(V, K)$ is the graded $S$-module presented as

$$S \otimes_{\mathbb{C}} \left(\bigwedge^3 V \oplus K\right) \xrightarrow{\delta_3 + \mathrm{id} \otimes \iota} S \otimes_{\mathbb{C}} \bigwedge^2 V \twoheadrightarrow \mathcal{B}(V, K) .$$

**PROPOSITION**

*The resonance variety $\mathcal{R} = \mathcal{R}(V, K)$ is the support of the Koszul module $\mathcal{B} = \mathcal{B}(V, K)$:*

$$\mathcal{R} = V(\mathrm{ann}(\mathcal{B})) \subset V^*.$$

*In particular, $\mathcal{R} = 0$ if and only if $\dim_{\mathbb{C}} \mathcal{B} < \infty$.*

# COHOMOLOGY JUMP LOCI

- Let $A = A(V, K)$ be the quadratic algebra defined as the quotient of the exterior algebra $E = \bigwedge V^*$ by the ideal generated by $K^\perp \subset V^* \wedge V^* = E^2$.
- Then $\mathcal{R}$ is the set of points $a \in A^1$ where the cochain complex

$$A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2$$

  is not exact (in the middle).
- The graded pieces of the (dual) Koszul module can be reinterpreted in terms of the linear strand in a Tor module:

$$\mathcal{B}_q^* \cong \mathrm{Tor}^E_{q+1}(A, \mathbb{C})_{q+2}$$

# VANISHING RESONANCE

Setting $m = \dim K$, we may view $K$ as a point in the Grassmannian $\mathrm{Gr}_m(V \wedge V)$, and $\mathbb{P}(K^{\perp})$ as a codimension $m$ projective subspace in $\mathbb{P}(V^* \wedge V^*)$.

**LEMMA**

*Let $\mathrm{Gr}_2(V^*) \hookrightarrow \mathbb{P}(V^* \wedge V^*)$ be the Plücker embedding. Then,*

$$\mathcal{R}(V,K) = 0 \Longleftrightarrow \mathbb{P}(K^{\perp}) \cap \mathrm{Gr}_2(V^*) = \emptyset.$$

**THEOREM**

*For any integer $m$ with $0 \leqslant m \leqslant \binom{n}{2}$, where $n = \dim V$, the set*

$$U_{n,m} = \{K \in \mathrm{Gr}_m(V \wedge V) \mid \mathcal{R}(V,K) = 0\}$$

*is Zariski open. Moreover, this set is non-empty if and only if $m \geqslant 2n-3$, in which case there is an integer $q = q(n,m)$ such that $\mathcal{B}_q(V,K) = 0$, for every $K \in U_{n,m}$.*

# RESONANCE VARIETIES OF GROUPS

- The resonance variety of a f.g. group $G$:

$$\mathcal{R}(G) = \mathcal{R}(V, K),$$

where $V^* = H^1(G, \mathbb{C})$ and $K^\perp = \ker(\cup_G \colon V^* \wedge V^* \to H^2(G, \mathbb{C}))$.
- Rationally, every resonance variety arises in this fashion:

**PROPOSITION**

*Let $V$ be a finite-dimensional $\mathbb{C}$-vector space, and let $K \subseteq V \wedge V$ be a linear subspace, defined over $\mathbb{Q}$. Then, there is a finitely presented, commutator-relators group $G$ with $V^* = H^1(G, \mathbb{C})$ and $K^\perp = \ker(\cup_G)$.*

- $\mathcal{R} = \mathcal{R}(G)$ is an approximation to $\mathcal{V} = \mathcal{V}(G)$.

**THEOREM (LIBGOBER, DIMCA–PAPADIMA–S.)**

*Let $\mathrm{TC}_1(\mathcal{V})$ be the tangent cone to $\mathcal{V}$ at $1$, viewed as a subset of $T_1(\widehat{G}) = H^1(G, \mathbb{C})$. Then $\mathrm{TC}_1(\mathcal{V}) \subseteq \mathcal{R}$. Moreover, if $G$ is $1$-formal, then equality holds, and $\mathcal{R}$ is a union of rational subspaces.*

## EXAMPLE (RIGHT-ANGLED ARTIN GROUPS)

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a (finite, simple) graph. The corresponding *right-angled Artin group* is

$$G_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathsf{E} \rangle.$$

- $V = H_1(G_\Gamma, \mathbb{C})$ is the vector space spanned by $\mathsf{V}$.
- $K \subseteq V \wedge V$ is spanned by $\{v \wedge w \mid \{v, w\} \in \mathsf{E}\}$.
- $A = A(V, K)$ is the exterior Stanley–Reisner ring of $\Gamma$.
- $\mathcal{R}(G_\Gamma)$ is the union of all coordinate subspaces $\mathbb{C}^{\mathsf{W}} \subset \mathbb{C}^{\mathsf{V}}$, taken over all $\mathsf{W} \subset \mathsf{V}$ for which the induced graph $\Gamma_{\mathsf{W}}$ is disconnected.
- $\sum_{q\geqslant 0} \dim_{\mathbb{C}}(\mathcal{B}_q) t^{q+2} = Q_\Gamma(t/(1-t))$, where
$$Q_\Gamma(t) = \sum_{k\geqslant 0} \sum_{\mathsf{W}\subset \mathsf{V}:\ |\mathsf{W}|=k} \tilde{b}_0(\Gamma_{\mathsf{W}}) t^k.$$

# ROOTS, WEIGHTS, AND VANISHING RESONANCE

- Let $\mathfrak{g}$ be a complex, semisimple Lie algebra.
- Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set of simple roots $\Delta \subset \mathfrak{h}^*$.
- Let $( \, , \, )$ be the inner product on $\mathfrak{h}^*$ defined by the Killing form.
- Each simple root $\beta \in \Delta$ gives rise to elements $x_\beta, y_\beta \in \mathfrak{g}$ and $h_\beta \in \mathfrak{h}$ which generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.
- Each irreducible representation of $\mathfrak{g}$ is of the form $V(\lambda)$, where $\lambda$ is a dominant weight.
- A non-zero vector $v \in V(\lambda)$ is a maximal vector (of weight $\lambda$) if $x_\beta \cdot v = 0$, for all $\beta \in \Delta$. Such a vector is uniquely determined (up to non-zero scalars), and is denoted by $v_\lambda$.

**LEMMA**

*The representation $V(\lambda) \wedge V(\lambda)$ contains a direct summand isomorphic to $V(2\lambda - \beta)$, for some simple root $\beta$, if and only if $(\lambda, \beta) \neq 0$. When it exists, such a summand is unique.*

### Theorem

*Let $V = V(\lambda)$ be an irreducible $\mathfrak{g}$-module, and let $K \subset V \wedge V$ be a submodule. Let $V^* = V(\lambda^*)$ be the dual module, and let $v_{\lambda*}$ be a maximal vector for $V^*$.*

1. *Suppose there is a root $\beta \in \Delta$ such that $(\lambda^*, \beta) \neq 0$, and suppose the vector $v_{\lambda*} \wedge y_\beta v_{\lambda*}$ (of weight $2\lambda^* - \beta$) belongs to $K^\perp$. Then $\mathcal{R}(V, K) \neq 0$.*
2. *Suppose that $2\lambda^* - \beta$ is not a dominant weight for $K^\perp$, for any simple root $\beta$. Then $\mathcal{R}(V, K) = 0$.*

### Corollary

*$\mathcal{R}(V, K) = 0$ if and only if $2\lambda^* - \beta$ is not a dominant weight for $K^\perp$, for any simple root $\beta$ such that $(\lambda^*, \beta) \neq 0$.*

# THE CASE OF $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

- $\mathfrak{h}^*$ is spanned $t_1$ and $t_2$ (the dual coordinates on the subspace of diagonal $2 \times 2$ complex matrices), subject to $t_1 + t_2 = 0$.
- There is a single simple root, $\beta = t_1 - t_2$.
- The defining representation is $V(\lambda_1)$, where $\lambda_1 = t_1$.
- The irreps are of the form

$$V_n = V(n\lambda_1) = \mathrm{Sym}_n(V(\lambda_1)),$$

  for some $n \geqslant 0$. Moreover, $\dim V_n = n + 1$ and $V_n^* = V_n$.
- The second exterior power of $V_n$ decomposes into irreducibles, according to the Clebsch-Gordan rule:

$$V_n \wedge V_n = \bigoplus_{j \geqslant 0} V_{2n-2-4j}.$$

  These summands occur with multiplicity $1$, and $V_{2n-2}$ is always one of those summands.

**PROPOSITION**

*Let $K$ be an $\mathfrak{sl}_2(\mathbb{C})$-submodule of $V_n \wedge V_n$. TFAE:*

1. *The variety $\mathcal{R}(V_n, K)$ consists only of $0 \in V_n^*$.*
2. *The $\mathbb{C}$-vector space $\mathcal{B}(V_n, K)$ is finite-dimensional.*
3. *The representation $K$ contains $V_{2n-2}$ as a direct summand.*

The $\mathrm{Sym}(V_n)$-modules $W(n) = \mathcal{B}(V_n, V_{2n-2})$ were studied by Weyman and Eisenbud (1990). We strengthen one of their results:

**COROLLARY**

*For any $\mathfrak{sl}_2(\mathbb{C})$-submodule $K \subset V_n \wedge V_n$, the Koszul module $\mathcal{B}(V_n, K)$ is finite-dimensional over $\mathbb{C}$ if and only if $\mathcal{B}(V_n, K)$ is a quotient of $W(n)$.*

Open problem: compute $\mathrm{Hilb}(W(n))$. The vanishing of $W_{n-2}(n)$, for all $n \geqslant 1$, would imply the generic Green Conjecture on free resolutions of canonical curves.

# AUTOMORPHISM GROUPS OF FREE GROUPS

- Identify $(F_n)_{\mathrm{ab}} = \mathbb{Z}^n$, and $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$. The morphism $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ is onto; thus, $\mathcal{A}(F_n) = \mathrm{GL}_n(\mathbb{Z})$.
- Denote the Torelli group by $\mathrm{IA}_n = \mathcal{T}_{F_n}$, and the Johnson–Andreadakis filtration by $J_n^s = F^s(\mathrm{Aut}(F_n))$.
- Magnus [1934]: $\mathrm{IA}_n$ is generated by the automorphisms

$$\alpha_{ij}\colon \begin{cases} x_i \mapsto x_j x_i x_j^{-1} \\ x_\ell \mapsto x_\ell \end{cases} \qquad \alpha_{ijk}\colon \begin{cases} x_i \mapsto x_i \cdot (x_j, x_k) \\ x_\ell \mapsto x_\ell \end{cases}$$

  with $1 \leqslant i \neq j \neq k \leqslant n$.
- Thus, $\mathrm{IA}_1 = \{1\}$ and $\mathrm{IA}_2 = \mathrm{Inn}(F_2) \cong F_2$ are finitely presented.
- Krstić and McCool [1997]: $\mathrm{IA}_3$ is not finitely presentable.
- It is not known whether $\mathrm{IA}_n$ admits a finite presentation for $n \geqslant 4$.

Nevertheless, $\mathrm{IA}_n$ has some interesting finitely presented subgroups:

- The McCool group of "pure symmetric" automorphisms, $\mathrm{P}\Sigma_n$, generated by $\alpha_{ij}$, $1 \leqslant i \neq j \leqslant n$.
- The "upper triangular" McCool group, $\mathrm{P}\Sigma_n^+$, generated by $\alpha_{ij}$, $i > j$. Cohen, Pakianathan, Vershinin, and Wu [2008]: $\mathrm{P}\Sigma_n^+ = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, with extensions by $\mathrm{IA}$-automorphisms.
- The pure braid group, $P_n$, consisting of those automorphisms in $\mathrm{P}\Sigma_n$ that leave the word $x_1 \cdots x_n \in F_n$ invariant. $P_n = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, with extensions by pure braid automorphisms.

# THE TORELLI GROUP OF $F_n$

Let $\mathcal{T}_{F_n} = J_n^1 = IA_n$ be the Torelli group of $F_n$. Recall we have an equivariant $\mathrm{GL}_n(\mathbb{Z})$-homomorphism,

$$J\colon \operatorname{gr}_F(\mathrm{IA}_n) \to \operatorname{Der}(\mathcal{L}_n),$$

In degree $1$, this can be written as

$$J\colon \operatorname{gr}_F^1(\mathrm{IA}_n) \to H^* \otimes (H \wedge H),$$

where $H = (F_n)_{\mathrm{ab}} = \mathbb{Z}^n$, viewed as a $\mathrm{GL}_n(\mathbb{Z})$-module via the defining representation. Composing with $\iota_F$, we get a homomorphism

$$J \circ \iota_F \colon (\mathrm{IA}_n)_{\mathrm{ab}} \longrightarrow H^* \otimes (H \wedge H)\ .$$

**THEOREM (ANDREADAKIS, COHEN–PAKIANATHAN, FARB, KAWAZUMI)**

*For each $n \geqslant 3$, the map $J \circ \iota_F$ is a $\mathrm{GL}_n(\mathbb{Z})$-equivariant isomorphism.*

Thus, $H_1(\mathrm{IA}_n, \mathbb{Z})$ is free abelian, of rank $b_1(IA_n) = n^2(n-1)/2$.

We have a commuting diagram,

$$\begin{array}{ccccccccc}
 & & \mathrm{Inn}(F_n) & \xrightarrow{=} & \mathrm{Inn}(F_n) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathrm{IA}_n & \longrightarrow & \mathrm{Aut}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \mathrm{OA}_n & \longrightarrow & \mathrm{Out}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 1
\end{array}$$

- Thus, $\mathrm{OA}_n = \widetilde{\mathcal{T}}_{F_n}$.
- Write the induced Johnson filtration on $\mathrm{Out}(F_n)$ as $\widetilde{J}_n^s = \pi(J_n^s)$.
- $\mathrm{GL}_n(\mathbb{Z})$ acts on $(\mathrm{OA}_n)_{\mathrm{ab}}$, and the outer Johnson homomorphism defines a $\mathrm{GL}_n(\mathbb{Z})$-equivariant isomorphism

$$\widetilde{J} \circ \iota_{\widetilde{F}} \colon (\mathrm{OA}_n)_{\mathrm{ab}} \xrightarrow{\cong} H^* \otimes (H \wedge H) / H \,.$$

- Moreover, $\widetilde{J}_n^2 = \mathrm{OA}'_n$, and we have an exact sequence

$$1 \longrightarrow F'_n \xrightarrow{\mathrm{Ad}} \mathrm{IA}'_n \longrightarrow \mathrm{OA}'_n \longrightarrow 1 \,.$$

# DEEPER INTO THE JOHNSON FILTRATION

**CONJECTURE (F. COHEN, A. HEAP, A. PETTET 2010)**

*If $n \geqslant 3$, $s \geqslant 2$, and $1 \leqslant i \leqslant n-2$, the cohomology group $H^i(J_n^s, \mathbb{Z})$ is not finitely generated.*

We disprove this conjecture, at least rationally, in the case when $n \geqslant 5$, $s = 2$, and $i = 1$.

**THEOREM**

*If $n \geqslant 5$, then $\dim_{\mathbb{Q}} H^1(J_n^2, \mathbb{Q}) < \infty$.*

To start with, note that $J_n^2 = IA_n'$. Thus, it remains to prove that $b_1(\mathrm{IA}_n') < \infty$, i.e., $(IA_n' / IA_n'') \otimes \mathbb{Q}$ is finite dimensional.

# REPRESENTATIONS OF $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{h}$: the Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, with coordinates $t_1, \dots, t_n$.
- $\Delta = \{t_i - t_{i+1} \mid 1 \leqslant i \leqslant n-1\}$.
- $\lambda_i = t_1 + \cdots + t_i$.
- $V(\lambda)$: the irreducible, finite dimensional representation of $\mathfrak{sl}_n(\mathbb{C})$ with highest weight $\lambda = \sum_{i<n} a_i\lambda_i$, with $a_i \in \mathbb{Z}_{\geqslant 0}$.

Set $H_{\mathbb{C}} = H_1(F_n, \mathbb{C}) = \mathbb{C}^n$, and

$$V^* := H^1(\mathrm{OA}_n, \mathbb{C}) = H_{\mathbb{C}} \otimes (H_{\mathbb{C}}^* \wedge H_{\mathbb{C}}^*)/H_{\mathbb{C}}^*.$$

$$K^{\perp} := \ker\left(\cup \colon V^* \wedge V^* \to H^2(\mathrm{OA}_n, \mathbb{C})\right).$$

**THEOREM (PETTET 2005)**

*Let $n \geqslant 4$. Set $\lambda = \lambda_2 + \lambda_{n-1}$ (so that $\lambda^* = \lambda_1 + \lambda_{n-2}$) and $\mu = \lambda_1 + \lambda_{n-2} + \lambda_{n-1}$. Then $V^* = V(\lambda^*)$ and $K^{\perp} = V(\mu)$, as $\mathfrak{sl}_n(\mathbb{C})$-modules.*

**THEOREM**

*For each $n \geqslant 4$, the resonance variety $\mathcal{R}(\mathrm{OA}_n)$ vanishes.*

**PROOF.**

$2\lambda^* - \mu = t_1 - t_{n-1}$ is not a simple root. Thus, $\mathcal{R}(V, K) = 0$. □

**REMARK**

When $n = 3$, the proof breaks down, since $t_1 - t_2$ is a simple root. In fact, $K^\perp = V^* \wedge V^*$ in this case, and so $\mathcal{R}(V, K) = V^*$.

**COROLLARY**

*For each $n \geqslant 4$, let $V = V(\lambda_2 + \lambda_{n-1})$ and let $K^\perp = V(\lambda_1 + \lambda_{n-2} + \lambda_{n-1}) \subset V^* \wedge V^*$ be the Pettet summand. Then $\dim \mathcal{B}(V, K) < \infty$ and $\dim \operatorname{gr}_q B(\mathrm{OA}_n) \leqslant \dim \mathcal{B}_q(V, K)$, for all $q \geqslant 0$.*

Using now a result of Dimca–Papadima (2013) on the "geometric irreducibility" of representations of arithmetic groups, we obtain:

**THEOREM**

*If $n \geqslant 4$, then $\mathcal{V}(\mathrm{OA}_n)$ is finite, and so $b_1(\mathrm{OA}'_n) < \infty$.*

Finally,

**THEOREM**

*If $n \geqslant 5$, then $b_1(\mathrm{IA}'_n) < \infty$.*

**PROOF.**

The spectral sequence of the extension $1 \to F'_n \to \mathrm{IA}'_n \to \mathrm{OA}'_n \to 1$ gives rise to the exact sequence

$$H_1(F'_n, \mathbb{C})_{\mathrm{IA}'_n} \longrightarrow H_1(\mathrm{IA}'_n, \mathbb{C}) \longrightarrow H_1(\mathrm{OA}'_n, \mathbb{C}) \longrightarrow 0\,.$$

The last term is finite-dimensional for all $n \geqslant 4$, by previous theorem. The first term is finite-dimensional for all $n \geqslant 5$, by nilpotency of the action of $\mathrm{IA}'_n$ on $F'_n/F''_n$.

# TORELLI GROUPS OF SURFACES

- Let $\Sigma_g$ be a Riemann surface of genus $g$, and let $\mathcal{T}_g = \mathcal{T}_{\pi_1(\Sigma_g)}$.
- $\mathcal{T}_1$ is trivial, $\mathcal{T}_2$ is not finitely generated.
- So assume $g \geqslant 3$, in which case $\mathcal{T}_g$ is finitely generated.
- $\mathrm{Out}^+(\pi_1(\Sigma_g)) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is surjective; thus, there is a natural $\mathrm{Sp}_{2g}(\mathbb{Z})$-action on $V = H_1(\mathcal{T}_g, \mathbb{C})$. This action extends to a rational irrep of $\mathrm{Sp}_{2g}(\mathbb{C})$, and thus, of $\mathfrak{sp}_{2g}(\mathbb{C})$.
- Dominant weights: $\lambda_i = t_1 + \cdots + t_i$, for $1 \leqslant i \leqslant g$.
- Let $V^* = H^1(\mathcal{T}_g, \mathbb{C})$, and let $K^\perp = \ker(\cup) \subset V^* \wedge V^*$.
- Hain (1997): $V^* = V(\lambda_3)$ and $K^\perp = V(2\lambda_2) \oplus V(0)$. Moreover, the decomposition of $V^* \wedge V^*$ into irreps is multiplicity-free.

**THEOREM**

$\mathcal{R}(\mathcal{T}_g) = 0$, *for each* $g \geqslant 4$.

**PROOF.**

- Simple roots: $\Delta = \{t_1 - t_2, t_2 - t_3, \dots, t_{g-1} - t_g, 2t_g\}$.
- The only $\beta \in \Delta$ for which $(\lambda_3, \beta) \neq 0$ is $\beta = t_3 - t_4$.
- Clearly, $2\lambda_3 - \beta = \lambda_2 + \lambda_4$ is not a dominant weight for $K^\perp$.
- Hence, $\mathcal{R}(V, K) = 0$.

□

Let $K_g \subset \mathcal{T}_g$ be the "Johnson kernel", i.e., the subgroup generated by Dehn twists about separating curves on $\Sigma_g$. The above result (and some more work) implies the following:

**THEOREM (DIMCA–PAPADIMA 2013)**

$H_1(K_g, \mathbb{C})$ *is finite-dimensional, for each* $g \geqslant 4$.