Sigma-invariants and tropical geometry

Alex Suciu

Northeastern University

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Tropical varieties

- Let $\mathbb{K} = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series over $\mathbb{C}$.

- A non-zero element of $\mathbb{K}$ has the form $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $c_i \in \mathbb{C}^*$ and $a_1 < a_2 < \cdots$ are rational numbers with a common denominator.

- The (algebraically closed) field $\mathbb{K}$ admits a discrete valuation $v\colon \mathbb{K}^* \to \mathbb{Q}$, given by $v(c(t)) = a_1$.

- Let $v\colon (\mathbb{K}^*)^n \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the n-fold product of the valuation.

- The *tropicalization* of a variety $W \subset (\mathbb{K}^*)^n$, denoted $\mathrm{Trop}(W)$, is the closure of the set $v(W)$ in $\mathbb{R}^n$.

- This is a rational polyhedral complex in $\mathbb{R}^n$. For instance, if W is a curve, then $\mathrm{Trop}(W)$ is a graph with rational edge directions.

- If T be an algebraic subtorus of $(\mathbb{K}^*)^n$, then $\operatorname{Trop}(T)$ is the linear subspace $\operatorname{Hom}(\mathbb{K}^*, T) \otimes \mathbb{R} \subset \operatorname{Hom}(\mathbb{K}^*, (\mathbb{K}^*)^n) \otimes \mathbb{R} = \mathbb{R}^n$.
- Moreover, if $z \in (\mathbb{K}^*)^n$, then $\operatorname{Trop}(z \cdot T) = \operatorname{Trop}(T) + v(z)$.
- For a variety $W \subset (\mathbb{C}^*)^n$, we may define its tropicalization by setting $\operatorname{Trop}(W) = \operatorname{Trop}(W \times_{\mathbb{C}} \mathbb{K})$.
- In this case, the tropicalization is a polyhedral fan in $\mathbb{R}^n$.
- If $W = V(f)$ is a hypersurface, defined by a Laurent polynomial $f \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, then $\operatorname{Trop}(W)$ is the positive-codimensional skeleton of the inner normal fan to the Newton polytope of f.

Exponential tangent cones

- Let $\exp\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$. Given a subvariety $W \subset (\mathbb{C}^*)^n$, put
$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$
- $\tau_1(W)$ depends only on $W_{(1)}$; it is non-empty iff $1 \in W$.
- If $T \cong (\mathbb{C}^*)^r$ is an algebraic subtorus, then $\tau_1(T) = T_1(T) \cong \mathbb{C}^r$
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces.
- Set $\tau_1^{\Bbbk}(W) = \tau_1(W) \cap \Bbbk^n$, for a subfield $\Bbbk \subset \mathbb{C}$.

LEMMA

Let $W \subset (\mathbb{C}^*)^n$ be an algebraic variety. Then $\tau_1^{\mathbb{R}}(W) \subseteq \operatorname{Trop}(W)$.

PROOF.

Every irreducible component of $\tau_1^{\mathbb{R}}(W)$ is of the form $L \otimes_{\mathbb{Q}} \mathbb{R}$, for some linear subspace $L \subset \mathbb{Q}^n$. The complex torus $T := \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$ lies inside W. Thus, $\operatorname{Trop}(T) = L \otimes_{\mathbb{Q}} \mathbb{R}$ lies inside $\operatorname{Trop}(W)$. □

Characteristic varieties

- Let $\mathbb{T}_G := \text{Hom}(G, \mathbb{C}^*)$ be the character group of $G = \pi_1(X)$, also denoted by $\text{Char}(X) := H^1(X, \mathbb{C}^*)$.
- The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \mathbb{T}_G \mid H_i(X, \mathbb{C}_\rho) \neq 0\}.$$

- If X has finite q-skeleton, then $\mathcal{V}^i(X)$ is Zariski closed for all $i \leq q$.
- We may define similarly $\mathcal{V}^i(X, \Bbbk) \subset H^1(X, \Bbbk^*)$ for any field k.
- Let $X^{\text{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\text{ab}}, \mathbb{C})$ as a module over $\mathbb{C}[G_{\text{ab}}]$. Then

$$\bigcup_{i \leq q} \mathcal{V}^i(X) = \bigcup_{i \leq q} V\left(\text{ann}\left(H_i(X^{\text{ab}}, \mathbb{C})\right)\right).$$

The Alexander polynomial

- Let $H = G_{\mathrm{ab}}/\operatorname{tors}(G_{\mathrm{ab}})$ be the maximal torsion-free abelian quotient of $G = \pi_1(X)$.
- $\mathbb{Z}[H]$ is a commutative Noetherian ring and a unique factorization domain.
- Let $q\colon X^H \to X$ be the cover corresponding to the $G \twoheadrightarrow H$.
- Set $A_X := H_1(X^H, q^{-1}(x_0), \mathbb{Z})$, viewed as a $\mathbb{Z}[H]$-module.
- Let $E_1(A_X) \subseteq \mathbb{Z}[H]$ be the ideal of codimension 1 minors in a presentation for A_X.
- $\Delta_X := \gcd(E_1(A_X)) \in \mathbb{Z}[H]$ is the *Alexander polynomial* of X. It only depends on G, so also write it as Δ_G.
- Suppose $I_H^p \cdot (\Delta_G) \subseteq E_1(A_G)$, for some $p \geq 0$. Then

$$\mathcal{V}^1(X) \cap \mathbb{T}_G^0 = \{1\} \cup V(\Delta_G).$$

- Let $\mathrm{Newt}(\Delta_G) \subset H_1(G, \mathbb{R})$ be the Newton polytope of Δ_G.
- Given $\phi \in H^1(G;\mathbb{Z}) \cong \mathrm{Hom}(H, \mathbb{Z})$, its *Alexander norm*, $\|\phi\|_A$, is the length of $\phi(\mathrm{Newt}(\Delta_G))$.
- This defines a semi-norm on $H^1(G, \mathbb{R})$, with unit ball

$$B_A = \{\phi \in H^1(G;\mathbb{R}) \mid \|\phi\|_A \leq 1\}.$$

- If Δ_G is symmetric (i.e., invariant under $t_i \mapsto t_i^{-1}$), then B_A is, up to a scale factor of $1/2$, the polar dual of the Newton polytope of Δ_G,

$$2B_A = \mathrm{Newt}(\Delta_G)^*.$$

Resonance varieties

- Let $A = H^*(X, \mathbb{C})$. For each $a \in A^1$, we have that $a^2 = 0$. Thus, there is a cochain complex
$$(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots.$$
- The *resonance varieties* of X are the homogeneous algebraic sets
$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^i(A, a) \neq 0\}.$$
- Identify $A^1 = H^1(X, \mathbb{C})$ with $\mathbb{C}^n$, where $n = b_1(X)$. The map $\exp\colon H^1(X, \mathbb{C}) \to H^1(X, \mathbb{C}^*)$ has image $\mathbb{T}^0_G = (\mathbb{C}^*)^n$.
- (Dimca–Papadima–S. 2009)
$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X).$$
- (DPS-2009, DP-2014) If X is a q-formal space, then, for all $i \leq q$,
$$\tau_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X).$$

The Bieri–Neumann–Strebel–Renz invariants

- Let G be a finitely generated group, $n = b_1(G) > 0$. Let $S(G) = S^{n-1}$ be the unit sphere in $\mathrm{Hom}(G, \mathbb{R}) = \mathbb{R}^n$.
- (Bieri–Neumann–Strebel 1987)

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathcal{C}_\chi(G) \text{ is connected}\},$$

  where $\mathcal{C}_\chi(G)$ is the induced subgraph of $\mathrm{Cay}(G)$ on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.
- (Bieri–Renz 1988)

$$\Sigma^q(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_q\},$$

  i.e., there is a projective $\mathbb{Z}G_\chi$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq q$. Moreover, $\Sigma^1(G, \mathbb{Z}) = -\Sigma^1(G)$.
- The BNSR-invariants of form a descending chain of open subsets,

$$S(G) \supseteq \Sigma^1(G, \mathbb{Z}) \supseteq \Sigma^2(G, \mathbb{Z}) \supseteq \cdots .$$

- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which G/N is free abelian:

$$N \text{ is of type } \mathrm{FP}_q \iff S(G,N) \subseteq \Sigma^q(G,\mathbb{Z})$$

where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$.

- In particular: $\ker(\chi\colon G \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$.

- More generally, let X be a connected CW-complex with finite q-skeleton, for some $q \geq 1$.

- Let $G = \pi_1(X, x_0)$. For each $\chi \in S(X) := S(G)$, let

$$\widehat{\mathbb{Z}G}_\chi = \Big\{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) \geq c\} \text{ is finite, } \forall c \in \mathbb{R}\Big\}$$

be the Novikov–Sikorav completion of $\mathbb{Z}G$.

- (Farber–Geoghegan–Schütz 2010)

$$\Sigma^q(X,\mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \leq q\}.$$

- (Bieri 2007) If G is FP_k, then $\Sigma^q(G,\mathbb{Z}) = \Sigma^q(K(G,1),\mathbb{Z})$, $\forall q \leq k$.

Novikov–Betti numbers

- Let $\chi \in S(X)$, and set $\Gamma = \text{im}(\chi)$; then $\Gamma \cong \mathbb{Z}^r$, for some $r \geq 1$.
- A Laurent polynomial $p = \sum_\gamma n_\gamma \gamma \in \mathbb{Z}\Gamma$ is χ-*monic* if the greatest element in $\chi(\text{supp}(p))$ is 0, and $n_0 = 1$.
- Let $\mathcal{R}\Gamma_\chi$ be the Novikov ring, i.e., the localization of $\mathbb{Z}\Gamma$ at the multiplicative subset of all χ-monic polynomials ($\mathcal{R}\Gamma_\chi$ is a PID).
- Let $b_i(X, \chi) = \text{rank}_{\mathcal{R}\Gamma_\chi} H_i(X, \mathcal{R}\Gamma_\chi)$ be the Novikov–Betti numbers.

Bounding the Σ-invariants

Theorem (Papadima–S. 2010)

Let X be a connected CW-complex with finite q-skeleton, and let $\chi\colon \pi_1(X) \to \mathbb{R}$ be a non-zero character. Then,

- $-\chi \in \Sigma^q(X,\mathbb{Z}) \implies b_i(X,\chi) = 0,\ \forall i \le q.$
- $\chi \notin \tau_1^{\mathbb{R}}(\mathcal{V}^{\le q}(X)) \iff b_i(X,\chi) = 0,\ \forall i \le q.$

Corollary

$$\Sigma^q(X,\mathbb{Z}) \subseteq S\left(\tau_1^{\mathbb{R}}\left(\mathcal{V}^{\le q}(X)\right)\right)^{\mathrm{c}}$$

Thus, $\Sigma^q(X,\mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.

Tropicalizing the characteristic varieties

- Recall $\mathbb{K} = \mathbb{C}\{\{t\}\}$ comes with a valuation map, $v\colon \mathbb{K}^* \to \mathbb{Q}$.
- Let $\nu_X\colon \operatorname{Char}_{\mathbb{K}}(X) \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the composite

$$H^1(X, \mathbb{K}^*) \xrightarrow{v_*} H^1(X, \mathbb{Q}) \longrightarrow H^1(X, \mathbb{R}).$$

- I.e., if $\rho\colon \pi_1(X) \to \mathbb{K}^*$ is a $\mathbb{K}$-valued character, then the morphism $v \circ \rho\colon \pi_1(X) \to \mathbb{Q}$ defines $\nu_X(\rho) \in H^1(X, \mathbb{Q}) = \mathbb{Q}^n \subset \mathbb{R}^n$.
- Given an algebraic subvariety $W \subset H^1(X, \mathbb{C}^*)$ we define its *tropicalization* as the closure in $H^1(X, \mathbb{R}) \cong \mathbb{R}^n$ of the image of $W \times_{\mathbb{C}} \mathbb{K} \subset H^1(X, \mathbb{K}^*)$ under ν_X,

$$\operatorname{Trop}(W) := \overline{\nu_X(W \times_{\mathbb{C}} \mathbb{K})}.$$

- Applying this definition to the characteristic varieties $\mathcal{V}^i(X)$, and noting that $\mathcal{V}^i(X, \mathbb{K}) = \mathcal{V}^i(X) \times_{\mathbb{C}} \mathbb{K}$, we have that

$$\operatorname{Trop}(\mathcal{V}^i(X)) = \overline{\nu_X\big(\mathcal{V}^i(X, \mathbb{K})\big)}.$$

Proposition

- $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subseteq \operatorname{Trop}(\mathcal{V}^i(X))$, *for all* $i \leq q$.
- *If there is a subtorus* $T \subset \operatorname{Char}^0(X)$ *such that* $T \not\subset \mathcal{V}^i(X)$, *yet* $\rho T \subset \mathcal{V}^i(X)$ *for some* $\rho \in \operatorname{Char}(X)$, *then* $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subsetneq \operatorname{Trop}(\mathcal{V}^i(X))$.

Proposition

Suppose Δ_G *is symmetric and* $I_H^p \cdot (\Delta_G) \subseteq E_1(A_G)$, *for some* $p \geq 0$. *Then* $\operatorname{Trop}\left(\mathcal{V}^1(G) \cap \mathbb{T}_G^0\right)$ *coincides with the positive-codimension skeleton of* $\mathcal{F}(B_A)$, *the face fan of the unit ball in the Alexander norm.*

Theorem (PS-2010)

Let $\rho\colon \pi_1(X) \to \Bbbk^*$ *be a character such that* $\rho \in \mathcal{V}^{\leq q}(X, \Bbbk)$. *Let* $v\colon \Bbbk^* \to \mathbb{R}$ *be the homomorphism defined by a valuation on* $\Bbbk$, *and write* $\chi = v \circ \rho$. *If the homomorphism* $\chi\colon \pi_1(X) \to \mathbb{R}$ *is non-zero, then* $\chi \notin \Sigma^q(X, \mathbb{Z})$.

A tropical bound for the Σ-invariants

THEOREM

$$\Sigma^q(X,\mathbb{Z}) \subseteq S(\operatorname{Trop}(\mathcal{V}^{\leq q}(X)))^{\mathrm{c}} \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)))^{\mathrm{c}}.$$

COROLLARY

If $\mathcal{V}^{\leq q}(X)$ contains one of the connected components of $\operatorname{Char}(X)$, then $\Sigma^q(X,\mathbb{Z}) = \emptyset$.

COROLLARY

$$\Sigma^1(G) \subseteq -S(\operatorname{Trop}(\mathcal{V}^1(G)))^{\mathrm{c}} \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))^{\mathrm{c}}.$$

PROPOSITION

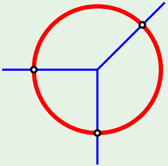
If Δ_G is symmetric and $I_H^p \cdot (\Delta_G) \subseteq E_1(A_G)$, for some $p \geq 0$, then

$$\Sigma^1(G) \subseteq \bigcup_{F \text{ an open facet of } B_A} S(F).$$

Two-generator, one-relator groups

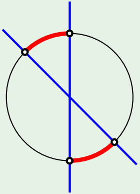
- Let $G = \langle x, y \mid r \rangle$, with $b_1(G) = 2$. K. Brown gave a combinatorial algorithm for computing $\Sigma^1(G)$.

Example



- Let $G = \langle a, b \mid b^2(ab^{-1})^2 a^{-2} \rangle$.
- Then $\Sigma^1(G) = S^1 \setminus \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, -1), (-1, 0)\}$.
- On the other hand, $\Delta_G = 1 + a + b$.
- Thus, $\Sigma^1(G) = -S(\operatorname{Trop}(V(\Delta_G)))^{\mathrm{c}}$, though $\tau_1 \mathcal{V}^1(G) = \{0\}$.

Example



- Let $G = \langle a, b \mid a^2ba^{-1}ba^2ba^{-1}b^{-3}a^{-1}ba^2ba^{-1}ba$ $b^{-1}a^{-2}b^{-1}ab^{-1}a^{-2}b^{-1}ab^3ab^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b \rangle$.
- Then $\Delta_G = (a - 1)(ab - 1)$, and so $S(\operatorname{Trop}(V(\Delta_G)))$ consists of two pairs of points.
- Yet $\Sigma^1(G)$ consists of two open arcs joining those points.

Compact 3-manifolds

- Let M be a compact, connected, orientable 3-manifold with $b_1(M) > 0$.
- A non-zero class $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ is a *fibered* if there exists a fibration $p\colon M \to S^1$ such that the induced map $p_*\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ coincides with ϕ.
- The *Thurston norm* $\|\phi\|_T$ of a class $\phi \in H^1(M;\mathbb{Z})$ is the infimum of $-\chi(\hat{S})$, where S runs though all the properly embedded, oriented surfaces in M dual to ϕ, and $\hat{S}$ denotes the result of discarding all components of S which are disks or spheres.
- Thurston showed that $\| - \|_T$ defines a seminorm on $H^1(M;\mathbb{Z})$, which can be extended to a continuous seminorm on $H^1(M;\mathbb{R})$.
- The unit norm ball, $B_T = \{\phi \in H^1(M;\mathbb{R}) \mid \|\phi\|_T \le 1\}$, is a rational polyhedron with finitely many sides and symmetric in the origin.

- There are facets of B_T, called the *fibered faces* (coming in antipodal pairs), so that a class $\phi \in H^1(M;\mathbb{Z})$ fibers if and only if it lies in the cone over the interior of a fibered face.
- Bieri, Neumann, and Strebel showed that the BNS invariant of $G = \pi_1(M)$ is the projection onto $S(G)$ of the open fibered faces of the Thurston norm ball B_T; in particular, $\Sigma^1(G) = -\Sigma^1(G)$.

PROPOSITION

Let M be a compact, connected, orientable, 3-manifold with empty or toroidal boundary. Set $G = \pi_1(M)$ and assume $b_1(M) \geq 2$. Then

1. $\operatorname{Trop}\left(\mathcal{V}^1(G) \cap \mathbb{T}^0_G\right)$ *is the positive-codimension skeleton of $\mathcal{F}(B_A)$, the face fan of the unit ball in the Alexander norm.*
2. $\Sigma^1(G)$ *is contained in the union of the open cones on the facets of B_A.*

Part (2) is inspired by, and partly recovers a theorem of C. McMullen.

Kähler manifolds

- Let M be a compact Kähler manifold.
- (Deligne–Griffiths–Morgan–Sullivan) M is formal.
- (Beauville, Catanese, Green–Lazarsfeld, Simpson, Arapura, B. Wang) $\mathcal{V}^i(M)$ are finite unions of torsion translates of algebraic subtori of $H^1(M, \mathbb{C}^*)$.

Theorem (Delzant 2010)

$$\Sigma^1(M) = S(M) \setminus \bigcup_\alpha S(f_\alpha^*(H^1(C_\alpha, \mathbb{R}))),$$

where the union is taken over those orbifold fibrations $f_\alpha \colon M \to C_\alpha$ with the property that either $\chi(C_\alpha) < 0$, or $\chi(C_\alpha) = 0$ and f_α has some multiple fiber.

Corollary

$$\Sigma^1(M) = S(\operatorname{Trop}(\mathcal{V}^1(M))^{\mathrm{c}}.$$

Hyperplane arrangements

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an (essential, central) arrangement of hyperplanes in $\mathbb{C}^d$.
- Its complement, $M(\mathcal{A}) \subset (\mathbb{C}^*)^d$, is a smooth, quasi-projective Stein manifold; thus, it has the homotopy type of a finite, d-dimensional CW-complex.
- $H^*(M(\mathcal{A}), \mathbb{Z})$ is the Orlik–Solomon algebra of $L(\mathcal{A})$.
- (Arapura) The characteristic varieties $\mathcal{V}^i(\mathcal{A}) := \mathcal{V}^i(M(\mathcal{A})) \subset (\mathbb{C}^*)^n$. are unions of translated subtori.
- Consequently, $\operatorname{Trop}(\mathcal{V}^i(\mathcal{A})) = -\operatorname{Trop}(\mathcal{V}^i(\mathcal{A}))$.
- (DSY 2016/17) $M(\mathcal{A})$ is an “abelian duality space," and hence its characteristic varieties propagate: $\mathcal{V}^1(\mathcal{A}) \subseteq \mathcal{V}^2(\mathcal{A}) \subseteq \cdots \subseteq \mathcal{V}^d(\mathcal{A})$.
- (Arnol'd, Brieskorn) $M(\mathcal{A})$ is formal. Thus, $\tau_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A})$.

Theorem

$$\Sigma^q(M(\mathcal{A}),\mathbb{Z}) \subseteq S\left(\left(\mathrm{Trop}(\mathcal{V}^q(\mathcal{A}))\right)\right)^{c}, \quad \forall q \le d.$$

Question (MFO Miniworkshop 2007)

Given an arrangement $\mathcal{A}$, do we have

$$\Sigma^1(M(\mathcal{A})) = S(\mathcal{R}^1(\mathcal{A},\mathbb{R}))^{c}? \qquad (\star)$$

Example (Koban–McCammond–Meier 2013)

- Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^n$, defined by $\prod_{1\le i<j\le n}(z_i - z_j) = 0$. Then $M(\mathcal{A}) = \mathrm{Conf}(n,\mathbb{C}) \simeq K(P_n, 1)$.
- Answer to ($\star$) is yes: $\Sigma^1(M(\mathcal{A}))$ is the complement of the union of $\binom{n}{3} + \binom{n}{4}$ planes in $\mathbb{C}^{\binom{n}{2}}$, intersected with the unit sphere.

Example

- Let $\mathcal{A}$ be the "deleted B_3" arrangement, defined by $z_1 z_2(z_1^2 - z_2^2)(z_1^2 - z_2^2)(z_2^2 - z_3^2) = 0$.
- (S. 2002) $\mathcal{V}^1(\mathcal{A})$ contains a (1-dimensional) translated torus $\rho \cdot T$.
- Thus, $\mathrm{Trop}(\rho \cdot T) = \mathrm{Trop}(T)$ is a line in $\mathbb{C}^8$ which is *not* contained in $\mathcal{R}^1(\mathcal{A},\mathbb{R})$. Hence, the answer to ($\star$) is no.

Question (Revised)

$$\Sigma^1(M(\mathcal{A})) = S(\mathrm{Trop}(\mathcal{V}^1(\mathcal{A}))^{\mathrm{c}}? \qquad (\star\star)$$

Reference

Alexander I. Suciu, *Sigma-invariants and tropical varieties*, arXiv:2010.07499.