Hyperplane arrangements and Milnor fibrations

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Geometry and Topology Seminar

Western University, London, ON

June 1, 2015

HYPERPLANE ARRANGEMENTS

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^\ell$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$.
- The Boolean arrangement $\mathcal{B}_n$
  - $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0,1\}^n$.
  - $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.
- The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type A_{n-1})
  - $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1,\dots,n\}$.
  - $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for P_n, the pure braid group on n strings).

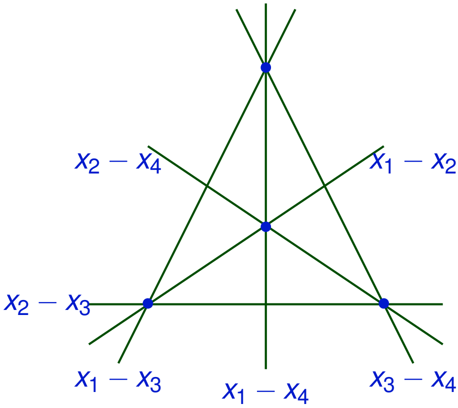


FIGURE : A planar slice of the braid arrangement $\mathcal{A}_4$

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i\colon \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$. Define an injective linear map

$$\iota\colon \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota\colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Hence, $M(\mathcal{A}) = \iota(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n$, a "very affine" subvariety of $(\mathbb{C}^*)^n$, and thus, a Stein manifold.
- Therefore, $M = M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension ℓ.
- In fact, M has a minimal cell structure (Dimca–Papadima, Randell, Salvetti, Adiprasito,...). Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.

- The Betti numbers are given by
$$\sum_{q=0}^{\ell} b_q(M)t^q = \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$
where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function, defined recursively by $\mu(\mathbb{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y\supsetneq X} \mu(Y)$.
- Let $E = \bigwedge(\mathcal{A})$ be the exterior algebra on degree 1 classes e_H dual to the meridians, and set $e_{\mathcal{B}} = \prod_{H\in\mathcal{B}} e_H$ for each $\mathcal{B} \subset \mathcal{A}$.
- Define a differential $\partial\colon E \to E$ of degree -1, starting from $\partial(e_H) = 1$, and extending to E by the graded Leibniz rule.
- The cohomology ring $H^*(M,\mathbb{Z})$ is isomorphic to the Orlik–Solomon algebra $A = E/I$, where I is the ideal generated by $\partial(\prod_{H\in\mathcal{B}} e_H)$, for all $\mathcal{B} \subset \mathcal{A}$ such that $\operatorname{codim}(\bigcap_{H\in\mathcal{B}} H) < |\mathcal{B}|$.
- The space M is formal: the de Rham algebra $(\Omega^*_{\mathrm{dR}}(M), d)$ is quasi-isomorphic to $(A\otimes\mathbb{R}, d = 0)$.

COHOMOLOGY JUMP LOCI

- Let X be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.
- Let $\Bbbk$ be an algebraically closed field, and let $\operatorname{Hom}(\pi, \Bbbk^*)$ be the affine algebraic group of $\Bbbk$-valued, multiplicative characters on π.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}^q_s(X, \Bbbk) = \{\rho \in \operatorname{Hom}(\pi, \Bbbk^*) \mid \dim_{\Bbbk} H_q(X, \Bbbk_\rho) \geqslant s\}.$$

  Here, $\Bbbk_\rho$ is the local system defined by ρ, i.e, $\Bbbk$ viewed as a $\Bbbk\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \Bbbk_\rho) = H_i(C_*(\widetilde{X}, \Bbbk) \otimes_{\Bbbk\pi} \Bbbk_\rho)$.

- These loci are Zariski closed subsets of the character group.

- Let $A = H^*(X, \Bbbk)$. If $\operatorname{char} \Bbbk = 2$, assume that $H_1(X, \mathbb{Z})$ has no 2-torsion. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- Thus, we get a cochain complex

$$(A, \cdot a)\colon \; A^0 \xrightarrow{\;a\;} A^1 \xrightarrow{\;a\;} A^2 \longrightarrow \cdots .$$

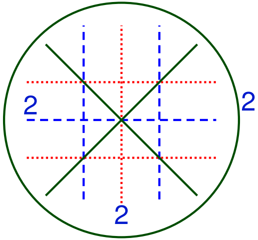
- The *resonance varieties* of X are the jump loci for the cohomology of these complexes,

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\}.$$

- These loci are *homogeneous* subvarieties of the affine space $A^1 = H^1(X, \Bbbk)$.
- In particular, $a \in A^1$ belongs to $\mathcal{R}^1_1(X, \Bbbk)$ iff there is $b \in A^1$ not proportional to a, such that $a \cup b = 0$ in A^2.

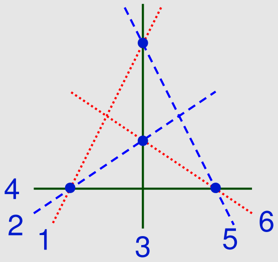
JUMP LOCI OF ARRANGEMENTS

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $\mathbb{C}^3$, and identify $H^1(M(\mathcal{A}), \Bbbk) = \Bbbk^n$, with basis dual to the meridians.
- The resonance varieties $\mathcal{R}^1_s(\mathcal{A}, \Bbbk) := \mathcal{R}^1_s(M(\mathcal{A}), \Bbbk) \subset \Bbbk^n$ lie in the hyperplane $\{x \in \Bbbk^n \mid x_1 + \cdots + x_n = 0\}$.
- $\mathcal{R}^1(\mathcal{A}) = \mathcal{R}^1_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $\mathbb{C}^n$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}^1_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$.



- Each flat $X \in L_2(\mathcal{A})$ of multiplicity $k \geqslant 3$ gives rise to a *local* component of $\mathcal{R}^1(\mathcal{A})$, of dimension $k - 1$.
- More generally, every k*-multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of dimension $k - 1$, and all components of $\mathcal{R}^1(\mathcal{A})$ arise in this way.
- The resonance varieties $\mathcal{R}^1(\mathcal{A}, \Bbbk)$ can be more complicated, e.g., they may have non-linear components.

EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 components coming from the triple points, and one component from the above 3-net:

$$\begin{aligned} L_{124} &= \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\}, \\ L_{135} &= \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\}, \\ L_{236} &= \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\}, \\ L_{456} &= \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\}, \\ L &= \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}. \end{aligned}$$

- Let $\text{Hom}(\pi_1(M), \mathbb{k}^*) = (\mathbb{k}^*)^n$ be the character torus.
- The characteristic variety $\mathcal{V}^1(\mathcal{A}, \mathbb{k}) := \mathcal{V}^1_1(M(\mathcal{A}), \mathbb{k}) \subset (\mathbb{k}^*)^n$ lies in the substorus $\{t \in (\mathbb{k}^*)^n \mid t_1 \cdots t_n = 1\}$.
- $\mathcal{V}^1(\mathcal{A}) = \mathcal{V}^1(\mathcal{A}, \mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}^1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}^1(\mathcal{A})$.
- All components of $\mathcal{V}^1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).
- In general, though, there are translated subtori in $\mathcal{V}^1(\mathcal{A})$.

PROPAGATION OF JUMP LOCI

THEOREM (DENHAM, S., YUZVINSKY 2014)

Let $\mathcal{A}$ be a central, essential hyperplane arrangement in $\mathbb{C}^n$ with complement $M = M(\mathcal{A})$. Suppose $A = \mathbb{Z}[\pi]$ or $A = \mathbb{Z}[\pi_{ab}]$. Then $H^p(M, A) = 0$ for all $p \neq n$, and $H^n(M, A)$ is a free abelian group.

COROLLARY

1. $M = M(\mathcal{A})$ *is a duality space of dimension* n *(due to Davis, Januszkiewicz, Okun 2011).*
2. M *is an abelian duality space of dimension* n*.*
3. *The characteristic and resonance varieties of* $\mathcal{A}$ *propagate:*

$$\mathcal{V}_1^1(M, \mathbb{C}) \subseteq \cdots \subseteq \mathcal{V}_1^n(M, \mathbb{C})$$

$$\mathcal{R}_1^1(M, \mathbb{C}) \subseteq \cdots \subseteq \mathcal{R}_1^n(M, \mathbb{C})$$

MILNOR FIBRATIONS OF ARRANGEMENTS

- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel H.
- For each choice of multiplicities $m = (m_H)_{H \in \mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H^{m_H},$$

  a homogeneous polynomial of degree $N = \sum_{H \in \mathcal{A}} m_H$.
- The map $Q_m \colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.
- $F_m(\mathcal{A})$ has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, and of dimension $\ell - 1$.
- The *(geometric) monodromy* is the diffeomorphism

$$h\colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi \mathrm{i}/N} z.$$

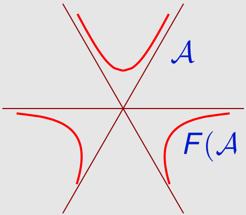
- If all $m_H = 1$, the polynomial $Q = Q_m(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A}) = F_m(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.
- In general, $F(\mathcal{A})$ is not formal, and it does not admit a minimal cell structure.

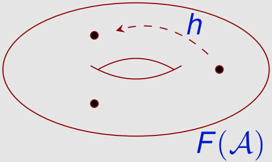
EXAMPLE

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then $M(\mathcal{A}) = \mathbb{C}^*$, $Q_m(\mathcal{A}) = z^m$, and $F_m(\mathcal{A}) = m$-roots of 1.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:





More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

- Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential arrangement. The inclusion $\iota\colon M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{array}{ccccc} F_m(\mathcal{A}) & \longrightarrow & M(\mathcal{A}) & \xrightarrow{Q_m(\mathcal{A})} & \mathbb{C}^* \\ \downarrow & & \downarrow{\scriptstyle \iota} & & \| \\ F_m(\mathcal{B}_n) & \longrightarrow & M(\mathcal{B}_n) & \xrightarrow{Q_m(\mathcal{B}_n)} & \mathbb{C}^* \end{array}$$

- Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$

HOMOLOGY OF THE MILNOR FIBER

- Assume $\gcd(m) = 1$. Then $F_m(\mathcal{A})$ is the regular $\mathbb{Z}_N$-cover of $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))$ defined by the homomorphism

$$\delta_m \colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N, \quad x_H \mapsto m_H \bmod N$$

- Let $\widehat{\delta_m} \colon \operatorname{Hom}(\mathbb{Z}_N, \Bbbk^*) \to \operatorname{Hom}(\pi_1(U(\mathcal{A})), \Bbbk^*)$. If $\operatorname{char}(\Bbbk) \nmid N$, then

$$\dim_{\Bbbk} H_q(F_m(\mathcal{A}), \Bbbk) = \sum_{s \geqslant 1} \left| \mathcal{V}^q_s(U(\mathcal{A}), \Bbbk) \cap \operatorname{im}(\widehat{\delta_m}) \right| .$$

- This gives a formula for the characteristic polynomial

$$\Delta^{\Bbbk}_q(t) = \det(t \cdot \mathrm{id} - h_*)$$

  of the algebraic monodromy, $h_* \colon H_q(F(\mathcal{A}), \Bbbk) \to H_q(F(\mathcal{A}), \Bbbk)$, in terms of the characteristic varieties of $U(\mathcal{A})$ and multiplicities m.

- Let $\Delta = \Delta_1^{\mathbb{C}}$, and write

$$\Delta(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})}, \qquad (\star)$$

where $\Phi_d(t)$ is the d-th cyclotomic polynomial, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.

- Question: Is $\Delta(t)$ determined by $L_{\leqslant 2}(\mathcal{A})$? Equivalently, are the integers $e_d(\mathcal{A})$ determined by $L_{\leqslant 2}(\mathcal{A})$?
- Not all divisors of n appear in $(\star)$. For instance, if $d \nmid |\mathcal{A}_X|$, for some $X \in L_2(\mathcal{A})$, then $e_d(\mathcal{A}) = 0$.
- In particular, if $L_2(\mathcal{A})$ has only flats of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

MODULAR RESONANCE

- Let $A = H^*(M(\mathcal{A}), \Bbbk)$, where $\operatorname{char}(\Bbbk) = p > 0$.
- Let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$ be the "diagonal" vector, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

- Clearly, $\beta_p(\mathcal{A})$ depends only on $L_{\leqslant 2}(\mathcal{A})$ and p. Moreover, $0 \leqslant \beta_p(\mathcal{A}) \leqslant |\mathcal{A}| - 2$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $s \geqslant 1$.

THEOREM

If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k - 2$.

Combinatorial determination of $b_1(F(\mathcal{A}))$

Theorem (Papadima–S. 2014)

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$ with $r > 1$. Then:

1. $\beta_3(\mathcal{A}) \neq 0 \Leftrightarrow \mathcal{A}$ *admits a* 3*-net* $\Leftrightarrow \mathcal{A}$ *admits a reduced* 3*-multinet.*
2. $\beta_3(\mathcal{A}) \leqslant 2$.
3. $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$.

Corollary (PS)

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1} \cdot (t^2+t+1)^{\beta_3(\mathcal{A})}.$$

In particular, $b_1(F(\mathcal{A}))$ is combinatorially determined.

Similarly, if $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$, then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2$$

CONJECTURE (PS)

Let $\mathcal{A}$ be an arrangement of rank at least 3. Then $e_{p^s}(\mathcal{A}) = 0$, for all primes p and integers $s \geqslant 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$

That is,

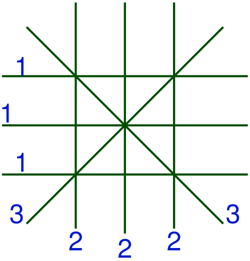
$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

This conjecture has been verified for several classes of arrangements. including complex reflection arrangements and certain types of complexified real arrangements.

TORSION IN HOMOLOGY

THEOREM (COHEN–DENHAM–S. 2003)

For every prime $p \geqslant 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion.



Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2y(x^2-y^2)^3(x^2-z^2)^2(y^2-z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

These examples may be reinterpreted and generalized, as follows.

THEOREM (DENHAM–S. 2014)

Suppose $\mathcal{A}$ admits a 'pointed' multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

This torsion is explained by the fact that the geometry of $\mathcal{V}^1(\mathcal{A}', \Bbbk)$ varies with $\operatorname{char}(\Bbbk)$.

To produce p-torsion in the homology of $F(\mathcal{A})$, we use a 'polarization' construction: $(\mathcal{A}, m) \rightsquigarrow \mathcal{A} \,\|\, m$, an arrangement of $N = \sum_{H\in\mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank}\mathcal{A} + |\{H \in \mathcal{A} \colon m_H \geqslant 2\}|$.

THEOREM (DS)

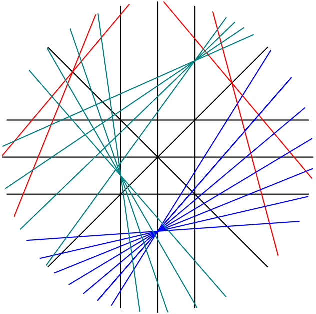
Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}' \| m'$ and $q = 1 + \big|\{K \in \mathcal{A}' : m'_K \geqslant 3\}\big|$.

Noite: The Milnor fiber $F(\mathcal{B})$ does not admit a minimal cell structure.

COROLLARY (DS)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.



Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$

$$((x-y)^2-w_2^2)((x+y)^2-w_3^2)((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).

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